On the total vertex-irregular strength of a disjoint union of t copies of a path

Nurdin[1,2], A.N.M. Salman[2], N.N. Gaos[2], E.T. Baskoro[2]

[1] Mathematics Department,
Faculty of Mathematics and Natural Sciences,
Hasanuddin University (Universitas Hasanuddin),
Jl. Perintis Kemerdekaan Km 10 Tamalanrea, Makassar, Indonesia

[2] Combinatorial Mathematics Research Group,
Faculty of Mathematics and Natural Sciences
Institut Teknologi Bandung (ITB)
Jalan Ganesha No. 10 Bandung 40132, Indonesia

Emails: nurdin1701@gmail.com, {msalman, ebaskoro, nana}@math.itb.ac.id

Abstract. For a simple graph G with the vertex set V and the edge set E, a labelling $\lambda : V \cup E \to \{1, 2, 3, \ldots, k\}$ is called a vertex-irregular total k-labelling of G if for any two different vertices x and y in V we have $wt(x) \neq wt(y)$ where $wt(x) = \lambda(x) + \sum_{xy \in E} \lambda(xy)$. The total vertex-irregular strength, denoted by $tvs(G)$, is the smallest positive integer k for which G has a vertex-irregular total k-labelling. In this paper, we determine the total vertex-irregular strength of a disjoint union of t copies of a path, denoted by tP_n. We prove that for any $t \geq 2$,

$$tvs(tP_n) = \begin{cases} t & \text{for } n = 1, \\ t+1 & \text{for } 2 \leq n \leq 3, \\ \lceil \frac{nt+1}{3} \rceil & \text{for } n \geq 4. \end{cases}$$

Keywords: *path, total k-labeling, total vertex-irregular strength*

1 Introduction

A *labeling* of a graph is a mapping that carries graph elements to the numbers (usually to the positive or non-negative integers). A labeling of a graph is called a *vertex labeling*, an *edge labeling*, or a *total labeling*, if the domain of the mapping is the vertex set, the edge set, or the union of the vertex set and the edge set, respectively.

In this paper, we consider a total labeling. For a graph $G = (V, E)$, the *weight* of a vertex x under a total labeling λ is

$$wt(x) = \lambda(x) + \sum_{xy \in E} \lambda(xy).$$

A total labeling $\lambda : V \cup E \rightarrow \{1, \ 2, ..., k\}$ is called a *vertex-irregular total k-labeling* of G if every two distinct vertices x and y in V satisfies $wt(x) \neq wt(y)$. The *total vertex-irregular strength* of G, denoted by $tvs(G)$, is the minimum positive integer k for which G has a vertex-irregular total k-labeling.

The notion of a total vertex-irregular strength was introduced by Bača *et al.* [2]. They studied this notion and derived a lower bound and an upper bound of the total vertex-irregular strength of any tree T with no vertices of degree 2. Recently, Nurdin *et al.* have determined the total vertex-irregular strengths of some other types of trees (see [3] and [4]).

There are not many graphs of which their total vertex-irregular strengths are known. Bača *et al.* [2] have determined the total vertex-irregular strengths of some classes of graphs, namely cycles, stars, paths, and prisms. Wijaya *et al.* [5] have determined the total vertex-irregular strengths of complete bipartite graphs. Besides that, Wijaya and Slamin [6] have determined the total vertex-irregular strengths of wheels, fans, and suns. Recently, Ahmad, Nurdin, and Baskoro [1] have determined the total vertex-irregular strengths of Halin graph.

In this paper, we determine the total vertex-irregular strength of a disjoint union of t isomorphic copies of a P_n.

2 Main Result

In this section, we determine the total vertex-irregular strength of the t copies of a path P_n for any $t \geq 2$ and $n \geq 1$.

Theorem 1. *Let tP_n be the t copies of a path on n vertices. For $t \geq 2$,*

$$tvs(tP_n) = \begin{cases} t & \text{for } n = 1, \\ t+1 & \text{for } 2 \leq n \leq 3, \\ \lceil \frac{nt+1}{3} \rceil & \text{for } n \geq 4. \end{cases}$$

Proof Let the vertex set of tP_n be

$$V(tP_n) = \{x_{i,j} | 1 \leq i \leq t \ \text{ and } \ 1 \leq j \leq n\}$$

and the edge set of tP_n be

$$E(tP_n) = \{e_{i,j} = x_{i,j}x_{i,j+1} | 1 \leq i \leq t \ \text{ and } \ 1 \leq j \leq n-1\}.$$

We divide the proof into three cases.

Case 1 $n = 1$.
Since the number of vertices in tP_1 is t and their degree is 0, the smallest and the largest weights of the vertices are at least 1 and t, respectively. Since the weight of a vertex is the label of the vertex, $tvs(tP_1) \geq t$. For an upper bound, define a total t-labeling λ of tP_1 as $\lambda(x_{i,1}) = i$ for $1 \leq i \leq t$. It is easy to see that the largest label is t and the weights of all vertices of tP_1 are distinct. So, $tvs(tP_1) \leq t$.

Case 2 $n = 2$ or $n = 3$.
Since the number of vertices with degree 1 of tP_n is $2t$, the smallest and the largest weights of the vertices are at least 2 and $2t + 1$, respectively. Since the weight of a vertex is the sum of two positive integers, $tvs(tP_n) \geq \lceil (2t+1)/2 \rceil = t + 1$. For an upper bound, define a total $t + 1$-labeling λ of tP_n in the following ways.
For $n = 2$,

$$\lambda(x_{i,j}) = \begin{cases} i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ 1+i & \text{for } 1 \leq i \leq t \text{ and } j = 2, \end{cases}$$

$$\lambda(e_{i,1}) = i \text{ for } 1 \leq i \leq t,$$

for $n = 3$,

$$\lambda(x_{i,j}) = \begin{cases} 1 & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ t+1 & \text{for } 1 \leq i \leq t \text{ and } j = 2, \\ 1+i & \text{for } 1 \leq i \leq t \text{ and } j = 3, \end{cases}$$

$$\lambda(e_{i,j}) = \begin{cases} i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ t & \text{for } 1 \leq i \leq t \text{ and } j = 2. \end{cases}$$

We can check that the weight of $x_{i,j}$ is
for $n = 2$,

$$wt(x_{i,j}) = \begin{cases} 2i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ 2i+1 & \text{for } 1 \leq i \leq t \text{ and } j = 2, \end{cases}$$

and for $n = 3$,

$$wt(x_{i,j}) = \begin{cases} 1+i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ 2t+1+i & \text{for } 1 \leq i \leq t \text{ and } j = 2, \\ t+1+i & \text{for } 1 \leq i \leq t \text{ and } j = 3. \end{cases}$$

It is easy to see that the largest value of λ is $t + 1$ and the weights of all vertices of tP_n are distinct. So, $tvs(tP_n) \leq t + 1$.

Case 3 $n \geq 4$.
Let $k = \lceil (nt+1)/3 \rceil$. Since the number of vertices of tP_n is nt and the minimum degree and the maximum degree of them are 1 and 2, respectively, the smallest and the largest weights of the vertices are at least 2 and $nt+1$, respectively. Since the weight of a vertex of degree 2 is the number of three positive integer, $tvs(tP_n) \geq k$.

To proof that $tvs(tP_n) \leq k$, we consider two subcases below.

Subcase 3.1 $n = 4$ or $n = 5$.
Define a total k-labeling λ of tP_n in the following ways.
For $n = 4$,

$$\lambda(x_{i,j}) = \begin{cases} 1 & \text{for either } 1 \leq i \leq t \text{ and } j = 1 \text{ or} \\ & 1 \leq i \leq k-t \text{ and } j = 4, \\ 2t+1-k & \text{for either } 1 \leq i \leq t \text{ and } j = 2 \text{ or} \\ & 1 \leq i \leq k-t \text{ and } j = 3, \\ 3t+1+i-2k & \text{for } k-t < i \leq t \text{ and } j = 3, \\ 1+t-k+i & \text{for } k-t < i \leq t \text{ and } j = 4, \end{cases}$$

for $n = 5$,

$$\lambda(x_{i,j}) = \begin{cases} 1 & \text{for either } 1 \leq i \leq t \text{ and } j = 1 \\ & \text{or } 1 \leq i \leq k-t \text{ and } j = 5, \\ 2t+1-k & \text{for either } 1 \leq i \leq t \text{ and } j = 2 \\ & \text{or } 1 \leq i \leq k-t \text{ and } j = 4, \\ 4t+1-2k+i & \text{for } 1 \leq i \leq t \text{ and } j = 3, \\ 3t+1+i-2k & \text{for } k-t < i \leq t \text{ and } j = 4, \\ 1+t-k+i & \text{for } k-t < i \leq t \text{ and } j = 5, \end{cases}$$

and for $n = 4$ or $n = 5$,

$$\lambda(e_{i,j}) = \begin{cases} i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ t+i & \text{for } 1 \leq i \leq k-t \text{ and } j = n-1, \\ k & \text{for else } i \text{ and } j. \end{cases}$$

It is easy to check that the largest value of λ is k. We can check that the weights of the vertices of tP_4 are

$$wt(x_{i,j}) = \begin{cases} 1+i & \text{for } 1 \leq i \leq t \text{ and } j = 1, \\ t+1+i & \text{for } 1 \leq i \leq t \text{ and } j = 4, \\ 2t+1+i & \text{for } 1 \leq i \leq t \text{ and } j = 2, \\ 3t+1+i & \text{for } 1 \leq i \leq t \text{ and } j = 3, \end{cases}$$

and the weights of the vertices of tP_5 are

$$wt(x_{i,j}) = \begin{cases} 1+i & \text{for } 1 \le i \le t \text{ and } j=1, \\ t+1+i & \text{for } 1 \le i \le t \text{ and } j=5, \\ 2t+1+i & \text{for } 1 \le i \le t \text{ and } j=2, \\ 3t+1+i & \text{for } 1 \le i \le t \text{ and } j=4, \\ 4t+1+i & \text{for } 1 \le i \le t \text{ and } j=3. \end{cases}$$

Hence, we can see that the weights of all vertices of tP_n are distinct. So, $tvs(tP_n) \le k$.

Subcase 3.2 $n \ge 6$.
Let $a = \lfloor n/2 \rfloor$ and for $1 \le i \le t$, define a total k-labeling λ of tP_n in the following ways:

$$\lambda(e_{i,j}) = \begin{cases} i & \text{for } j=1 \\ \min\{2\lfloor j/2 \rfloor t, k\} & \text{for } 2 \le j < a \\ k & \text{for } j=a \\ \min\{(n-j)t, k\} & \text{for } a+1 \le j \le n-2 \\ t+i & \text{for } j=n-1. \end{cases}$$

and

$$\lambda(x_{i,j}) = \begin{cases} 1 & \text{for } j=1,n \\ 2(j-1)t+1+i-\lambda(e_{i,j-1})-\lambda(e_{i,j}) & \text{for } 2 \le j \le a \\ (n-1)t+1+i-\lambda(e_{i,j-1})-\lambda(e_{i,j}) & \text{for } j=a+1 \\ (2(n-j)+1)t+1+i-\lambda(e_{i,j-1})-\lambda(e_{i,j}) & \text{for } a+2 \le j \le n-1. \end{cases}$$

It is easy to check that the largest label is k, and the weights of tP_n is

$$wt(x_{i,j}) = \begin{cases} 2(j-1)t+1+i & \text{for } 1 \le i \le t \text{ and } 1 \le j \le a, \\ (2(n-j)+1)\,t+1+i & \text{for } 1 \le i \le t \text{ and } a+2 \le j \le n, \\ (n-1)t+1+i & \text{for } 1 \le i \le t \text{ and } j=a+1. \end{cases}$$

Hence, we can see that the weights of all vertices of tP_n are distinct.

Therefore, we conclude that $tvs(tP_n) \le k$. □

For illustration, we show a total vertex-irregular 16-labeling of $5P_9$ in Figure 1.

References

1. A. Ahmad, Nurdin, E.T. Baskoro, On total irregularity strength of generalized Halin graph, (submitted).

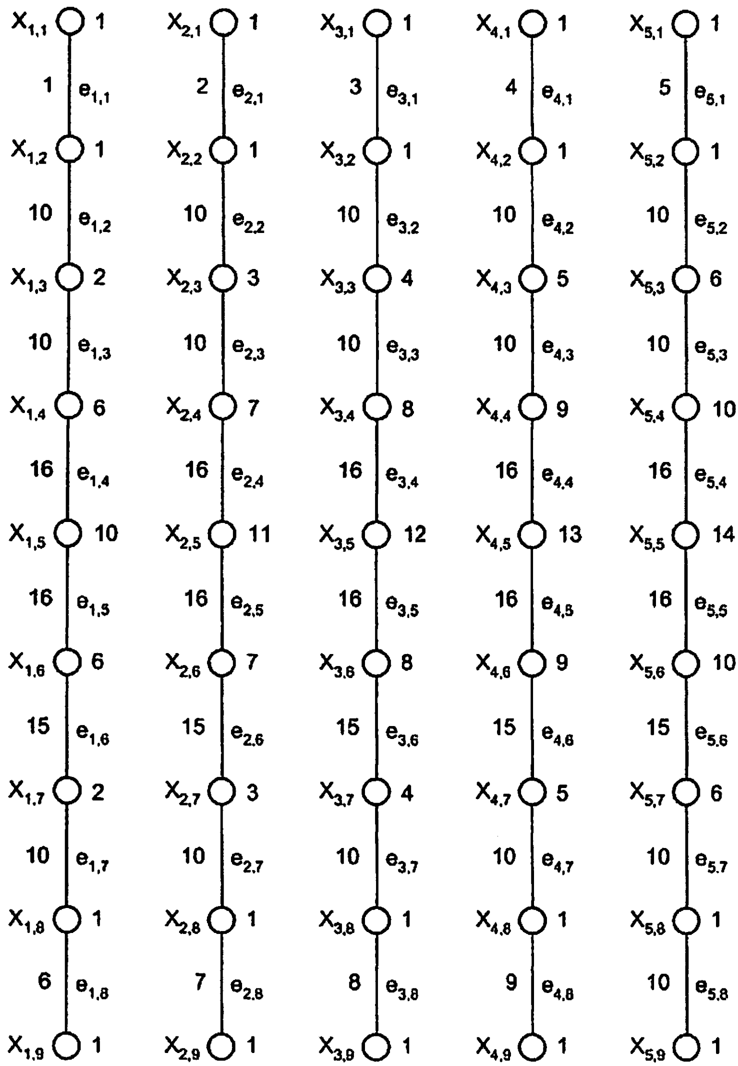

Fig. 1. A total vertex-irregular 16-labelling of $5P_9$

2. M. Bača, S. Jendrol′, M. Miller, J. Ryan, On irregular total labellings, *Discrete Math.* **307:11-12** (2007) 1378-1388.
3. Nurdin, E.T. Baskoro, A.N.M. Salman, N.N. Gaos, On total vertex-irregular labellings for several types of trees, to appears in *Utilitas Mathematica.*
4. Nurdin, E.T. Baskoro, A.N.M. Salman, N.N. Gaos, On total vertex-irregular labellings for trees, (submitted)
5. K. Wijaya, Slamin, Surahmat, S. Jendrol′, Total vertex irregular labeling of complete bipartite graphs, *J. Combin. Math. Combin. Comput.* **55** (2005), 129-136.
6. K. Wijaya, Slamin, Total vertex irregular labeling of wheels, fans, suns and friendship graphs, *J. Combin. Math. Combin. Comput.* **65** (2008), 103-112.