Notes on exponents of asymmetric two-colored digraphs

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Abstract. In this paper we discuss an upper bound for exponents of loopless asymmetric two-colored digraphs. If D is an asymmetric primitive two-colored digraph on n vertices, we show that $\exp(D) \leq 3n^2 + 2n - 2$. For an asymmetric two-colored digraph D which contains a primitive two-colored cycle of length $s \leq n$, we show its exponent is at most $(s^2-1)/2 + (s+1)(n-s)$. We characterize such two-colored digraphs whose exponents equal $(s^2-1)/2 + (s+1)(n-s)$ and show that the largest exponent of asymmetric two-colored digraph lies on the interval $[(n^2-1)/2, 3n^2+2n-2]$ when n is odd, or $[n^2/2, 3n^2+2n-2]$ otherwise.

Keywords: Two-colored digraphs, asymmetric, primitive, 2-exponents

1 Introduction

A two-colored digraph D is a digraph such that each of its arcs is colored by either red or blue but not both. A (u,v)-walk of length n in a two-colored digraph is a sequence of arcs of the form $(u = v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n = v)$, while an $(h,k)^T$-walk from u to v in a two-colored digraph D is a (u,v)-walk consisting of h red arcs and k blue arcs. For a walk w in D we denote $r(w)$ to be the number of red arcs in w and $b(w)$ to be the number of blue arcs in w. The vector $\begin{bmatrix} r(w) \\ b(w) \end{bmatrix}$ is the composition of the walk w. A (u,v)-walk is closed whenever $u = v$ and is open otherwise. A (u,v)-path is a (u,v)-walk without repeated vertices except possibly $u = v$. A cycle is a closed path. A two-colored digraph D is symmetric provided that the arc (u,v) is in D whenever the arc (v,u) is in D. An *asymmetric* two-colored digraph is a symmetric two-colored digraph such that each of its 2-cycles is a $(1,1)^T$-cycle i.e. each of 2-cycle in D is a bichromatic cycle.

A two-colored digraph D is *strongly connected* whenever for each pair of vertices u and v in D there is a (u,v)-walk in D. A two-colored digraph D is *primitive* provided that there are nonnegative integers h and k such that for each pair of vertices u and v there exists an $(h,k)^T$-walk from u to v. The smallest positive integer $h+k$ over all such nonnegative integers h and k is the *exponent* of D. We denote $\exp(D)$ for the exponent of D. Primitive two-colored digraphs and their corresponding exponents have been extensively studied (see [1, 3–5]).

Shader and Suwilo [3] show that the largest exponent of primitive two-colored digraphs on n vertices lies on the interval $(n^3-5n^2)/2, (3n^3+2n^2-2n)/2]$. Shao and Gao [5] consider upper bound on exponents of asymmetric looples two-colored digraphs. They show that for an asymmetric looples two-colored digraph on n vertices, $\exp(D) \leq 6n^2 - n$. In this paper, we improve the Shao and Gao's bound and show that $\exp(D) \leq 3n^2 + 2n - 2$. We then explore a class of asymmetric two-colored digraphs having exponents of order of magnitude $O(n^2)$, and show that the largest exponent of asymmetric primitive two-colored digraph lies on the interval $[(n^2-1)/2, 3n^2+2n-2]$ when n is odd or lies on the interval $[n^2/2, 3n^2+2n-2]$ when n is even.

2 Preliminaries

Let D be a strongly connected two-colored digraph and let $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$ be the set of all cycles in D. Define a cycle matrix of D to be a 2 by t matrix

$$M = \begin{bmatrix} r(\gamma_1) & r(\gamma_2) & \cdots & r(\gamma_t) \\ b(\gamma_1) & r(\gamma_2) & \cdots & r(\gamma_t) \end{bmatrix},$$

that is M is a matrix such that its ith column is the composition of the ith cycle γ_i, $i = 1, 2, \ldots, t$. If the rank of M is 1, the content of M is defined to be 0, and the content of M is defined to be the greatest common divisors of the 2 by 2 minors of M, otherwise. The following result, due to Fornasini and Valcher [1], gives algebraic characterization of a primitive two-colored digraph.

Theorem 1. *Let D be a strongly connected two-colored digraph with at least one arc of each color. Let M be a cycle matrix of D. The two-colored digraph D is primitive if and only if the content of M is* 1.

In the following lemma, we show that for a primitive two-colored digraph with cycle matrix M and for any integer vector $\mathbf{b}$ the system of diophantine equation $M\mathbf{x} = \mathbf{b}$ has integer solution. For $i = 1, 2, \ldots, m$, let $\mathbf{e}_i$ be the ith standard basis vector of $\mathbb{R}^m$.

Lemma 1. *Let A be an m by n integer matrix with content* 1. *Then the equation $A\mathbf{x} = \mathbf{e}_i$ has integer solution.*

Proof. Since the content of A is 1, the Smith Normal Form (see [2]) implies there exist unimodular matrices U and V such that $UAV = \begin{bmatrix} I_m & O \end{bmatrix}$. Therefore, $AV = \begin{bmatrix} U^{-1} & O \end{bmatrix}$. Let P be the n by n unimodular matrix $P = \begin{bmatrix} U & O \\ O & I \end{bmatrix}$. Then $AVP = \begin{bmatrix} I_m & O \end{bmatrix}$. Hence, there is a unimodular matrix $Q = VP$ such that $AQ = \begin{bmatrix} I_m & O \end{bmatrix}$. This implies the solution to $A\mathbf{x} = \mathbf{e}_i$ is $Q(:,i)$ where $Q(:,i)$ represents the ith column of Q. ■

The following lemma, see Lemma 1 of [3], shows that if for every path p_{uv} the equation $M\mathbf{x} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}$ has an integer solution, then D is primitive.

Lemma 2. *Let D be a two-colored digraph with cycle matrix M. For each pair of vertices u and v let p_{uv} be a path from u to v in D and let w be a closed walk in D that passes through all vertices in D. Suppose $\mathbf{z} \geq 0$ such that the system $M\mathbf{x} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}$ has an integer solution $\mathbf{x}_{uv}$ with $\mathbf{z} \geq \mathbf{x}_{uv}$. Then D is primitive and $\exp(D) \leq h + k$ with h and k satisfy*

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z}.$$

The following theorem, see Theorem 7 of [3], is necessary for our results.

Theorem 2. *Let $M = \begin{bmatrix} N & P \end{bmatrix}$ be an m by n integer matrix where N is a square integer matrix of order m with $\det(N) \neq 0$ and P is an m by $n-m$ integer matrix. Let $\mathbf{b}$ be integer vector such that the equation $\begin{bmatrix} N & P \end{bmatrix} \mathbf{z} = \mathbf{b}$ has integer solution. Then there are integer vectors $\mathbf{x}$ and $\mathbf{y}$ such that $N\mathbf{x} + P\mathbf{y} = \mathbf{b}$, $\mathbf{y} \geq 0$ and $\mathbf{1}^T\mathbf{y} \leq |\det(N)| - 1$.*

We note that if we replace $\mathbf{b}$ by $-\mathbf{b}$ in Theorem 2, then we have a solution to $N\mathbf{x} + P\mathbf{y} = -\mathbf{b}$ with $y_i \geq 0$ for $i = 1, 2, \ldots, n-m$ and $\mathbf{1}^T\mathbf{y} \leq |\det(N)| - 1$. Thus there exists an integer solution $\mathbf{z} = \begin{bmatrix} -\mathbf{x} \\ -\mathbf{y} \end{bmatrix}$ to $\begin{bmatrix} N & P \end{bmatrix} \mathbf{z} = \mathbf{b}$ such that $y_i \leq 0$ for $i = 1, 2, \ldots, n-m$ and $|\mathbf{1}^T\mathbf{y}| \leq |\det(N)| - 1$.

Using Theorem 2, Shao and Gao [5] show the following result.

Theorem 3. *Let D be an asymmetric primitive two-colored digraph on n vertices. Then $\exp(D) \leq 6n^2 - n$.*

3 An Improved Upper Bound

In this section, we improve the bound $\exp(D) \leq 6n^2 - n$ for exponents of loopless asymmetric primitive two-colored digraph on n vertices to $\exp(D) \leq 3n^2 + 2n - 2$. Let D be an asymmetric primitive two-colored digraph. By an underlying graph G of D we mean a graph obtained from D by replacing arcs (u, v) and (v, u) in D by an edge in G. Since D is strongly connected, the underlying graph G of D is connected. A *double direction spanning tree* T of D is a spanning subdigraph of D such that the underlying graph of T is a spanning tree in G. Notice that using a double direction spanning tree T of a symmetric two-colored digraph D, we can construct a closed walk w in D that uses each arc in T exactly once. Such walk w that passes each vertex in D is of length $2(n-1)$, moreover the walk w is an $(n-1, n-1)^T$-walk. For the rest of the paper we assume D to be a loopless asymmetric two-colored digraph.

Theorem 4. *Let D be an asymmetric primitive two-colored digraph on n vertices. Then* $\exp(D) \leq 3n^2 + 2(n-1)$

Proof. Let w be the closed walk that passes each arc in the double direction spanning tree T of D. Then w is a closed walk in D that passes through each vertex in D and $(r(w) + b(w)) = 2(n-1)$. Since D is asymmetric, then D has cycle with composition $\begin{bmatrix}1\\1\end{bmatrix}$. Since D is primitive, then D has directed cycle of odd length. Hence the cycle matrix M of D has two columns $\begin{bmatrix}e\\f\end{bmatrix}$ and $\begin{bmatrix}f\\e\end{bmatrix}$ for some nonnegative integers e and f with $e \neq f$. Without loss of generality we assume that $e < f$. Now we can write $M = \begin{bmatrix}N & P\end{bmatrix}$ with

$$N = \begin{bmatrix}N_1 & N_2\end{bmatrix} = \begin{bmatrix}1 & e\\1 & f\end{bmatrix}.$$

Let $\delta = \det(N)$. Then $\delta > 0$. For each pair of vertices u and v in D, we show that there is an $(h, k)^T$-walk from u to v with

$$\begin{bmatrix}h\\k\end{bmatrix} = \begin{bmatrix}r(w)\\b(w)\end{bmatrix} + M\mathbf{z},$$

where $\mathbf{z} = (n^2, n, 0, \ldots, 0)^T$.

For any vertices u and v in D, let p_{uv} be a path from u to v and let $\mathbf{b} = \begin{bmatrix}r(p_{uv})\\b(p_{uv})\end{bmatrix}$. Then $\begin{bmatrix}r(p_{uv})\\b(p_{uv})\end{bmatrix} \leq \begin{bmatrix}n\\n\end{bmatrix}$. Since D is primitive, the content of M is 1. Lemma 1 implies the equation $M\mathbf{z} = \mathbf{b}$ has integer solution. It follows from Theorem 2 and the comment immediately after that, there are integer vectors $\mathbf{x}$ and $\mathbf{y}$ such that $N\mathbf{x} + P\mathbf{y} = \mathbf{b}$ with $\mathbf{y} \leq 0$ and $|\mathbf{1}^T\mathbf{y}| \leq \delta - 1$. By Cramer's rule, we have

$$\mathbf{x} = \frac{1}{\delta}\begin{bmatrix} \det\left[\mathbf{b}\ \ N_2\right] - \sum_{j=1}^{t-2} y_j \det\left[P_j\ \ N_2\right] \\ \det\left[N_1\ \ \mathbf{b}\right] - \sum_{j=1}^{t-2} y_j \det\left[N_1\ \ P_j\right] \end{bmatrix}$$

where P_j is the jth column of P. Notice that $\det\left[\mathbf{b}\ \ N_2\right] \leq n^2$, $\det\left[N_1\ \ \mathbf{b}\right] \leq n$, $\det\left[P_j\ \ N_2\right] \leq n^2$, and $\det\left[N_1\ \ P_j\right] \leq n$. This and Theorem 2 imply

$$\begin{aligned} \mathbf{x} &\leq \frac{1}{\delta}\begin{bmatrix} n^2 - \sum_{j=1}^{t-2} y_j n^2 \\ n - \sum_{j=1}^{t-2} y_j n \end{bmatrix} = \frac{1}{\delta}\begin{bmatrix} n^2 + \sum_{j=1}^{t-2} |y_j| n^2 \\ n + \sum_{j=1}^{t-2} |y_j| n \end{bmatrix} \\ &\leq \frac{1}{\delta}\begin{bmatrix} n^2 + (\delta - 1)n^2 \\ n + (\delta - 1)n \end{bmatrix} = \begin{bmatrix} n^2 \\ n \end{bmatrix}. \end{aligned}$$

We now can write

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z_1} + M\mathbf{z_2}$$

where

$$\mathbf{z_1} = (n^2 - x_1, n - x_2, -y_1, -y_2, \ldots, -y_{t-2})^T$$

and

$$\mathbf{z_2} = (x_1, x_2, y_1, y_2, \ldots, y_{t-2})^T.$$

Notice that $M\mathbf{z_2} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}$, hence

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} r(w) \\ b(w) \end{bmatrix} + M\mathbf{z_1} + \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix}.$$

This implies for every pair of vertices u and v in D, there is an (h,k)-walk from u to v, namely the walk that starts at u, follows the walk w and returns to u, along the way moves $(n^2 - x_1)$ times around the cycle γ_1, $(n - x_2)$ times around the cycle γ_2,$(-y_{j-2})$ times around the cycle γ_j for $j = 3, 4, \ldots, t$, and finally moves to the vertex v along the path p_{uv}.

Since $r(\gamma_i) + b(\gamma_i) < n$ for every cycle γ_i, $i = 1, 2, \ldots, t$, we now conclude that

$$h + k \leq 2(n-1) + \left[2\ n\ n\ \cdots\ n\right]\left[n^2\ n\ 0\ \cdots 0\right]^T. \tag{1}$$

Hence $h+k \leq 2(n-1)+2n^2+n^2 = 3n^2+2(n-1)$. We now can conclude that $\exp(D) \leq 3n^2+2(n-1)$. ∎

As a direct consequence of Theorem 4, we have the following result for upper bound of exponent of asymmetric two-colored digraph with smallest cycle of length $s \geq 3$.

Corollary 1. *Let D be an asymmetric primitive two-colored digraph on n vertices. Let γ' be a smallest cycle of odd length $s \geq 3$. Then* $\exp(D) \leq 2n^2+sn+2(n-1)$.

Proof. Let w be the closed walk that passes double direction spanning tree T and let γ' be a smallest cycle in D of length $s = r(\gamma')+b(\gamma') \geq 3$. Since $\begin{bmatrix} r(w) \\ b(w) \end{bmatrix} = \begin{bmatrix} n-1 \\ n-1 \end{bmatrix}$, it is easy to verify from (1) that for each pair of vertices u and v in D there is an $(h,k)^T$-walk from u to v with h and k defined by

$$\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} n^2+(r(\gamma')+1)n-1 \\ n^2+(b(\gamma')+1)n-1 \end{bmatrix}.$$

Therefore $\exp(D) \leq 2n^2+sn+2(n-1)$. ∎

4 Exponents of asymmetric two-colored digraphs containing primitive cycles

Notice that Corollary 1 implies that if D is two-colored digraph containing a primitive cycle, then $\exp(D) \leq 2n^2+(s+2)n-2$ where s is the length of a smallest primitive cycle in D with $s \geq 3$. In this section, we will improve the upper bound for the exponent of such two-colored digraph. We first discuss the structure of the cycle matrix of such two-colored digraphs.

Lemma 3. *Let D be an asymmetric two-colored digraph containing a primitive two-colored cycle of length s. Then D is primitive if and only if the cycle matrix M has a submatrix of the form*

$$M_C = \begin{bmatrix} (s-1)/2 & (s+1)/2 & 1 \\ (s+1)/2 & (s-1)/2 & 1 \end{bmatrix}.$$

Proof. Let C be a double direction primitive two-colored cycle in D of length s. Then the cycle matrix M of D has a submatrix of the form

$$M_C = \begin{bmatrix} r(\gamma_1) & r(\gamma_2) & r(\gamma_3) \\ b(\gamma_1) & b(\gamma_2) & b(\gamma_3) \end{bmatrix} = \begin{bmatrix} a & s-a & 1 \\ s-a & a & 1 \end{bmatrix}.$$

Since C is primitive, then D is primitive and the length s is odd. Hence by Theorem 1 we have that D is primitive if and only if $\gcd(2as-s^2, s-$

$2a, 2a-s) = 1$. This implies $s-2a = \pm 1$, and hence $a = (s\pm 1)/2$. Hence D is primitive if and only if the cycle matrix M has a submatrix of the form

$$M_C = \begin{bmatrix} (s-1)/2 & (s+1)/2 & 1 \\ (s+1)/2 & (s-1)/2 & 1 \end{bmatrix}.$$

The lemma has been proved. ■

We note that since D is symmetric and every 2-cycle in D is a $(1,1)^T$-cycle, then every $(h,k)^T$-walk in D can be extended to an $(h+t, k+t)^T$-walk for all positive integers $t \geq 1$. Since D is strongly connected, for each pair of vertices u and v in D, there is an $(e,e)^T$-walk form u to v for some positive integer e. Let p_{uv} be a path from u to v, if $r(p_{uv}) > b(p_{uv})$, then an $(e,e)^T$-walk w_{uv} is as follows.

- If p_{uv} has vertices in common with the cycle C, then the walk that starts at u, follows the path p_{uv} to v, along the way moves $r(p_{uv})-b(p_{uv})$ times around the cycle γ_1 is an $(e,e)^T$-walk from u to v. The composition of w_{uv} is

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}.$$

- If all paths p_{uv} have no vertices in common with the cycle C, then the walk that starts at u, moves to some vertex c in C, moves $r(p_{uv})-b(p_{uv})$ times around the cycle γ_1 and finally moves to v is an $(e,e)^T$-walk from u to v. The composition of w_{uv} is

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}.$$

  where $\ell = \min\{d(u,c), d(v,c)\}$ with $d(u,c)$ is the distance from u to v.

Similar argument works when $r(p_{uv}) < b(p_{uv})$.

Using this facts, we have the following upper bound for exponents of asymmetric two-colored digraphs containing primitive cycle.

Lemma 4. *Let D be a strongly connected asymmetric two-colored digraph on n vertices containing a primitive two-colored cycle of length s where $3 \leq s \leq n$. Then* $\exp(D) \leq (s^2-1)/2 + (s+1)(n-s)$.

Proof. Let C be a primitive two-colored cycle in D of length s. We note that since C is primitive, s is odd and D is primitive. By Lemma 3 the cycle matrix of D has a submatrix of the form

$$M_C = \begin{bmatrix} (s-1)/2 & (s+1)/2 & 1 \\ (s+1)/2 & (s-1)/2 & 1 \end{bmatrix}.$$

Let γ_1 be the cycle in D with the composition $((s-1)/2, (s+1)/2)^T$ and let γ_2 be the cycle in D with the composition $((s+1)/2, (s-1)/2)^T$. We show that for each pair of vertices u and v in D there is a $(t,t)^T$-walk from u to v with $t = (s^2-1)/4 + (s+1)(n-s)/2$. Since D is asymmetric it suffices to show that for each pair of vertices u and v in D there is an $(e,e)^T$-walk from u to v, where $e \leq t$.

Notice that since D is asymmetric, for each vertex u in D there is a closed $(1,1)^T$-walk form u to itself. Now let u and v be distinct vertices in D and let p_{uv} be a directed path from u to v. Since $t = ((s-1)/2+(n-s))(s+1)/2$, for every pair of vertices u and v in D, $r(p_{uv}) + b(p_{uv}) \leq t$. If $r(p_{uv}) = b(p_{uv})$, then there is a $(e,e)^T$-walk from u to v with $e \leq t$. Hence we let $r(p_{uv}) \neq b(p_{uv})$ and without loss of generality we assume that $r(p_{uv}) > b(p_{uv})$. We consider two cases. They are the case when there is a path p_{uv} that has vertices in common with C and the case where all paths p_{uv} have no vertex in common with C.

Case 1. Directed path p_{uv} has vertices in common with the cycle C

We construct an $(e,e)^T$-walk from u to v as follows. We start at u and follow the path p_{uv} to v, along the way we move $(r(p_{uv}) - b(p_{uv}))$ times around the cycle γ_1. Notice that in this case we can choose a directed path p_{uv} such that $\ell(p_{uv}) = (r(p_{uv}) + b(p_{uv})) \leq (s-1)/2 + (n-s)$. This implies the composition of the walk w_{uv} is

$$\begin{aligned}
\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} &= \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \frac{1}{2}(r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1) \\ (s+1) \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} 2b(p_{uv}) + (r(p_{uv}) - b(p_{uv}))(s+1) \\ 2b(p_{uv}) + (r(p_{uv}) - b(p_{uv}))(s+1) \end{bmatrix} \\
&\leq (r(p_{uv}) + b(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix}.
\end{aligned}$$

Since $(r(p_{uv}) + b(p_{uv})) \leq (s-1)/2 + (n-s)$, then

$$\begin{aligned}
\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} &\leq ((s-1)/2 + (n-s)) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \\
&= \begin{bmatrix} (s^2-1)/4 + (s+1)(n-s)/2 \\ (s^2-1)/4 + (s+1)(n-s)/2 \end{bmatrix}.
\end{aligned}$$

Case 2. All paths p_{uv} have no vertices in common with the cycle C

Let p_{zc} be a directed path from some vertex z in D but not in C to some vertex c in the cycle C. We consider two cases, they are the case when the path p_{uv} lies on some path p_{zc} and when p_{uv} does not lie on any path p_{zc}.

Suppose the path p_{uv} lies on some path p_{zc}. If $\ell = \ell(p_{uc}) < \ell(p_{vc})$, then the walk that starts at u, moves to the cycle C along the path p_{uc} and moves $(r(p_{uv}) - b(p_{uv}))$ times around the directed cycle γ_1, then moves back to vertex u along the path p_{cu} and finally moves to the vertex v along the path p_{uv} is an $(e,e)^T$-walk w_{uv} for some positive integer $e \geq 1$. If $\ell = \ell(p_{vc}) < \ell(p_{uc})$, then the walk that starts at u, goes to v along the path p_{uv}, goes to c along the path p_{vc}, moves $r(p_{uv}) - b(p_{uv}))$ times around the cycle γ_1, and ends at v along the path p_{cv} is an $(e,e)^T$-walk w_{uv} for some positive integer $e \geq 1$. Notice that on both cases we have $(r(p_{uv}) + b(p_{uv}) + \ell) \leq n - s$. The composition of the walk w_{uv} is

$$\begin{aligned}
\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} &= \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} \\
&\leq \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) + b(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \\
&\leq \ell \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} + (r(p_{uv}) + b(p_{uv})) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \\
&= (r(p_{uv}) + b(p_{uv}) + \ell) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix}.
\end{aligned}$$

Since $(r(p_{uv}) + b(p_{uv}) + \ell) \leq n - s$, we find that

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} \leq (n-s) \begin{bmatrix} (s+1)/2 \\ (s+1)/2 \end{bmatrix} \leq \begin{bmatrix} (s^2-1)/4 + (s+1)(n-s)/2 \\ (s^2-1)/4 + (s+1)(n-s)/2 \end{bmatrix}.$$

We now assume that the path p_{uv} does not lie on any path p_{zc}. Since D is strongly connected, we can choose a vertex y in the path p_{uv} such that there is a path p_{yc} for some vertex c in the cycle C. Consider the walk w_{uv} that starts at u, goes to y along the path p_{uy}, then goes to c along the path p_{yc}, moves $(r(p_{uv}) - b(p_{uv})$ times around the cycle γ_1 and moves back to c, then moves to v along the path p_{cy} and p_{yv} is an $(e,e)^T$-walk from u to v. The composition of w_{uv} is of the form

$$\begin{bmatrix} r(w_{uv}) \\ b(w_{uv}) \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + \ell \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{uv}) - b(p_{uv}) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}$$

where $\ell = \ell(p_{yc})$. Since $(r(p_{uv}) + b(p_{uv}) + \ell) \leq n - s$, as in the previous case one can show that $e \leq t$.

Now using a $(1,1)^T$-cycle we can extend the $(e,e)^T$-walk w_{uv} to a $(t,t)^T$-walk w_{uv} with $t = (s^2-1)/4 + (s+1)(n-s)/2$. Since for every pair of vertices u and v in D there is a $(t,t)^T$-walk from u to v with $t = (s^2 - 1)/4 + (s+1)(n-s)/2$, then we have $\exp_2(D) \leq (s^2-1)/2 + (s+1)(n-s)$. ■

Let k be a positive integer such that $k \leq n-s$, where $s \geq 3$ is odd. Let D be a strongly connected symmetric digraph consisting of paths

$P_1 : s \to 1 \to 2 \to \cdots \to (s-3)/2 \to (s-1)/2)$,
$P_2 : (s-1)/2 \to (s+1)/2) \to (s+3)/2 \to \cdots \to s-1 \to s$,
$P_3 : s \to s-1 \to s-2 \to \cdots \to (s+1)/2 \to (s-1)/2$
$P_4 : (s-1)/2 \to (s-3)/2 \to (s-5)/2 \to \cdots \to 2 \to 1 \to s$
$P_5 : s \to s+1 \to s+2 \to \cdots \to n-k-1 \to n-k$,
$P_6 : n-k \to n-k-1 \to n-k-2 \to \cdots \to s+1 \to s$,
$P_7 : (s-1)/2 \to n-k+1 \to n-k+2 \to \cdots \to n-1 \to n$, and
$P_8 : n \to n-1 \to n-2 \to \cdots \to n-k+1 \to (s-1)/2$.

Notice that path P_1 together with path P_2 is a directed cycle of length s. Similarly, path P_3 together with path P_4 is a directed cycle of length s. We call such digraph to be an $(n, s, n-s-k, k)$-*superlollipop*. We define an $(n, s, n-s)$-*lollipop* to be an $(n, s, n-s, 0)$-superlollipop. As a consequence of Lemma 4 we have bound on exponents of a primitive asymmetric two-colored $(n, s, n-s-k, k)$-superlollipops.

Corollary 2. *Let $s \geq 3$ be odd and let D be an asymmetric primitive two-colored $(n, s, n-s-k, k)$-superlollipop. Then* $\exp(D) \leq (s^2-1)/2 + (s+1)(n-s)$.

Proof. We note that an asymmetric $(n, s, n-s-k, k)$-superlollipop has cycles of length 2 or s. Hence D is primitive if and only if s is odd. This implies D is primitive if and only if the double-directed cycle in D is primitive. Lemma 4 implies that $\exp(D) \leq (s^2-1)/2 + (s+1)(n-s)$. ∎

As a consequence of Corollary 2 we have an upper bound for exponents of primitive asymmetric cycles of length n.

Corollary 3. *Let D be an asymmetric primitive two-colored directed cycle of odd length $n \geq 3$. Then* $\exp(D) \leq (n^2-1)/2$.

Proof. We note that an asymmetric two-colored double direction n-cycle can be thought of as an asymmetric two-colored $(n, s, n-s, 0)$-superlollipop with $s = n$. Hence $\exp(D) \leq (n^2-1)/2$. ∎

The following result gives an upper bound for exponents of asymmetric two-colored digraphs containing a primitive two-colored double direction cycle in term of n, the number of vertices in D.

Theorem 5. *Let D be a strongly connected asymmetric two-colored digraph on n vertices containing a primitive two-colored cycle. Then*

$$\exp(D) \leq \begin{cases} (n^2-1)/2, & \text{if } n \text{ is odd} \\ n^2/2, & \text{if } n \text{ is even.} \end{cases}$$

Proof. From Lemma 4 we have that $\exp(D) \leq (s^2-1)/2 + (n-s)(s+1)$. Notice that $f(s) = (s^2-1)/2 + (s+1)(n-s)$ has global optima at $s = n-1$. Since $f(s)$ is quadratic and s is odd, $f(s)$ has global optima at $s = n-1$ when n is even and has global optima at $s = n$ or $s = n-2$ when n is odd. This implies $\exp(D) \leq (n^2-1)/2$ if n is odd, and $\exp(D) \leq n^2/2$ otherwise. ■

Notice that for a primitive asymmetric two-colored digraph containing a primitive two-colored cycle of length s, Lemma 4 guarantees that $\exp(D) \leq (s^2-1)/2 + (s+1)(n-s)$. To find a better bound, the proof of Theorem 4.3 implies that the primitive two-colored cycle should be chosen with the smallest length. Hence we have the following result.

Theorem 6. *Let D be a strongly connected asymmetric two-colored digraph on n vertices containing primitive two-colored cycles. Let s be the length of the smallest primitive two-colored cycle in D. Then* $\exp(D) \leq (s^2-1)/2 + (s+1)(n-s)$.

We now discuss necessary and sufficient conditions for an asymmetric two-colored digraphs containing primitive two-colored cycles to have exponent satisfy the upper bound $(s^2-1)/2 + (s+1)(n-s)$.

Theorem 7. *Let D be an asymmetric two-colored digraph on n vertices containing a primitive double direction cycle C of odd length s. The* $\exp(D) = (s^2-1)/2 + (s+1)(n-s)$ *if and only if D has distinct vertices u_0 and v_0 such that the shortest directed (u_0, v_0)-path is a monochromatic path of length $(s-1)/2 + (n-s)$.*

Proof. Let u_0 and v_0 be distinct vertices in D such that the shortest directed path $p_{u_0v_0}$ from u_0 to v_0 is a red path of length $q = (s-1)/2 + (n-s)$. Then the shortest path $p_{v_0u_0}$ from v_0 to u_0 is a blue path of length q. Let $w_{u_0v_0}$ and $w_{v_0u_0}$ be respectively the directed walk from u_0 to v_0 and from v_0 to u_0 that have the same composition. We show that

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} = \begin{bmatrix} r(w_{v_0u_0}) \\ b(w_{v_0u_0}) \end{bmatrix} \geq \begin{bmatrix} (s^2-1)/4 + (s+1)(n-2)/2 \\ (s^2-1)/4 + (s+1)(n-2)/2 \end{bmatrix}.$$

Since there are only $(n-s)$ vertices not on the cycle C, the walks $w_{u_0v_0}$ and $w_{v_0u_0}$ must pass through the cycle C. This implies there are two red u_0v_0-paths, one is of length q and the other is of length $q+1$. Hence the compositions of the walk $w_{u_0v_0}$ are of the forms

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} = \begin{bmatrix} q \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + a_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + a_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \quad (2)$$

for some nonnegative integers a_1, a_2, a_3 and $\delta \in \{0,1\}$. The compositions of the walks $w_{v_0u_0}$ are of the form

$$\begin{bmatrix} r(w_{v_0u_0}) \\ b(w_{v_0u_0}) \end{bmatrix} = \begin{bmatrix} 0 \\ q \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + c_3 \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix} \quad (3)$$

for some nonnegative integers c_1, c_2, c_3 and $\epsilon \in \{0,1\}$. Notice that when $\delta = 0$, the walk $w_{u_0v_0}$ uses the red (u_0, v_0)-path of length q. Otherwise, the walk $w_{u_0v_0}$ uses the red (u_0, v_0)-path of length $q+1$. Similar argument applies for walk $w_{v_0u_0}$.

Since the walks $w_{u_0v_0}$ and $w_{v_0u_0}$ have the same compositions, if $\delta = \epsilon = 0$ then from (2) and (3) we have

$$\begin{bmatrix} q \\ -q \end{bmatrix} = (c_1 - a_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_2 - a_2) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} + (c_3 - a_3) \begin{bmatrix} (s+1)/2 \\ (s-1)/2 \end{bmatrix}. \quad (4)$$

Subtracting the second component from the first component on (4) we have

$$(a_2 - c_2) + (c_3 - a_3) = q.$$

This implies $a_2 \geq q$ or $c_3 \geq q$. It is not hard to see the following results. If $\epsilon = 1$ and $\delta = 0$ or if $\epsilon = 0$ and $\delta = 1$, then $a_2 \geq q$ or $c_3 \geq q$. Finally if $\epsilon = \delta = 1$, then $a_2 \geq q+1$ and $c_3 \geq q+1$. These imply that

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} \geq \begin{bmatrix} (s^2-1)/4 + (s+1)(n-s)/2 \\ (s^2-1)/4 + (s+1)(n-s)/2 \end{bmatrix}.$$

This result and Lemma 4 imply that $\exp(D) = (s^2-1)/2 + (s+1)(n-s)$.

Now assume that $\exp(D) = (s^2-1)/2 + (n-s)(s+1)$, we show that there is a pair of vertices u_0 and v_0 in D such that the shortest directed path from u_0 to v_0 is a monochromatic directed path of length $(n-s)+(s-1)/2$. The proof of Lemma 4 shows that the exponent can be achieved by $(t,t)^T$-walks with $t = (s^2-1)/4 + (n-s)(s+1)/2$. That is, for each pair of vertices u and v in D there is a $(t,t)^T$-walk from u to v. We claim that there is a pair of vertices u_0 and v_0 in D such that the shortest $(t,t)^T$-walk from u_0 to v_0 has composition of the form

$$\begin{bmatrix} r(w_{u_0v_0}) \\ b(w_{u_0v_0}) \end{bmatrix} = \begin{bmatrix} r(p_{u_0v_0}) \\ b(p_{u_0v_0}) \end{bmatrix} + (r(p_{u_0v_0}) - b(p_{u_0v_0})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix}. \quad (5)$$

Let u' and v' be vertices in D. Then the $(t,t)^T$-walk from u' to v' has composition of the form

$$\begin{bmatrix} r(w_{u'v'}) \\ b(w_{u'v'}) \end{bmatrix} = \begin{bmatrix} r(p_{u'v'}) \\ b(p_{u'v'}) \end{bmatrix} + \ell_{u'v'} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (r(p_{u'v'}) - b(p_{u'v'})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}$$

for some nonnegative integer $\ell_{u'v'}$. Define

$$\begin{bmatrix} e' \\ e' \end{bmatrix} = \begin{bmatrix} r(p_{u'v'}) \\ b(p_{u'v'}) \end{bmatrix} + (r(p_{u'v'}) - b(p_{u'v'})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}.$$

If for all pair of vertices u and v in D, the $(e,e)^T$-walk from u to v

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} \leq \begin{bmatrix} e' \\ e' \end{bmatrix},$$

then for each pair of vertices u and v in D there is an (e,e)-walk from u to v with $e = e'$. Hence in this case we can choose $u_0 = u'$ and $v_0 = v'$.

Conversely, there is a pair of vertices u and v such that the shortest $(e,e)^T$-walk from u to v has the property that

$$\begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix} > \begin{bmatrix} e' \\ e' \end{bmatrix}.$$

Let u'' and v'' be a pair of such vertices with the property that the walk $w_{u''v''}$ is the longest $(e,e)^T$-walk. Then for each pair of vertices u and v in D there is an (e,e)-walk from u to v with

$$\begin{bmatrix} e \\ e \end{bmatrix} = \begin{bmatrix} r(p_{uv}) \\ b(p_{uv}) \end{bmatrix} + (r(p_{uv}) - b(p_{uv})) \begin{bmatrix} (s-1)/2 \\ (s+1)/2 \end{bmatrix}.$$

Therefore, we can set $u_0 = u''$ and $v_0 = v''$.

Now from (5) we have

$$b(p_{u_0v_0}) + (r(p_{u_0v_0}) - b(p_{u_0v_0}))(s+1)/2 = (s^2-1)/4 + (n-s)(s+1)/2$$

or

$$\frac{2b(p_{u_0v_0})}{s+1} = (s-1)/2 + (n-s) + b(p_{u_0v_0}) - r(p_{u_0v_0}). \qquad (6)$$

Since the right hand side of (6) is an integer, we conclude that $b(p_{u_0v_0}) = 0$. This implies that $r(p_{u_0v_0}) = (s-1)/2 + (n-s)$. Hence there is a pair of vertices u_0 and v_0 such that the shortest path from u_0 to v_0 is a monochromatic path of length $(s-1)/2 + (n-s)$. ■

Example 1. Let D be an $(n, s, n-s-k, k)$-superlollipop. Color the paths P_6, P_1, P_3 and P_7 with red and color the other arcs with blue. Then D is primitive and the path consisting of paths P_6, P_1, and P_7 is a monochromatic path of length $(n-s) + (s-1)/2$. Theorem 7 guarantees that $\exp(D) = (s^2-1)/2 + (n-s)(s+1)$. We notice that the upper bound $(n^2-1)/2$ in Theorem 5 is achieved by an n-cycle or an $(n, n-2, 1, 1)$-superlollipop. While the upper bound $n^2/2$ is achieved by an $(n, n-1, 1)$-lollipop.

Now combining Theorem 4, Theorem 6, and Example 1 we have the following result on the largest 2-exponent of asymmetric two-colored digraphs.

Theorem 8. *Let D be an asymmetric primitive two-colored digraph on n vertices. The largest 2-exponent of two-colored digraphs D lies on the interval $[(n^2-1)/2, 3n^2+2n-2]$ when n is odd, or lies on the interval $[n^2/2, 3n^2+2n-2]$ otherwise.*

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