The Ramsey number of a certain forest respect to a small wheel

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Abstract. In this paper, we determine the Ramsey number for disjoint union of graphs versus a graph H, especially $R(\bigcup_{i=1}^{k} l_i S_{n_i}(1,1), W_6)$, $R(\bigcup_{i=1}^{k} l_i S_{n_i}(1,2), W_6)$ and $R(\bigcup_{i=1}^{k} l_i S_{n_i}, W_4)$.

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1 Introduction

For given two graphs G and H, a graph F is called a (G,H)-*good graph* if F contains no G and $\overline{F}$ contains no H. Furthermore, any (G,H)-good graph on n vertices will be denoted by (G,H,n)-good graph. The *Ramsey number* $R(G,H)$ is defined as the smallest natural number n such that no (G,H,n)-good graph exists.

A nice survey for an attractive applications of various branches of Ramsey theory in harmonic analysis, metric spaces, ergodic theory, computational geometry, probabilistic, and information theory on dual source codes can be seen in Rosta [7].

We consider finite undirected graphs without loops and multiple edges. Let $G(V,E)$ be a graph, the notation V and E stand for the vertex set and edge set of the graph G, respectively. A graph $H(V',E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. For $A \subseteq V$, $G[A]$ represents the *subgraph induced* by A in G.

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We denote by T_n a tree on n vertices, S_n a star on n vertices, K_n complete graph on n vertices, and C_n a cycle on n vertices. The notation W_n is a wheel on $n+1$ vertices that consists of a cycle C_n with one additional vertex being adjacent to all vertices of C_n. $S_n(l,m)$ is a tree of order n obtained from star $S_{n-(lxm)}$ by subdividing each of l chosen edge m times. $S_n(l)$ is a tree of order n obtained from stars S_l and S_{n-l} by adding an edge joining the centers of them. *Cocktail-Party-Graph*, H_n, is a graph obtained from a complete graph of $2n$ vertices by removing n independent edges.

The Ramsey numbers of trees isomorphic to one of $S_n(1,1)$ or $S_n(1,2)$ versus a wheel on seven vertices have been studied by Chen et al. in [8]. It was shown that $R(S_n(1,1), W_6) = 2n$ for $n \geq 4$ and $R(S_n(1,2), W_6) = 2n$ for $n \equiv 0 (\bmod 3) \geq 6$. Baskoro et al. [2] have proved that $R(S_n, W_4) = 2n-1$ for odd $n \geq 5$; $R(S_n, W_4) = 2n+1$ for even $n \geq 4$. Other results concerning graph Ramsey numbers can be seen in [5].

Let G_i be a graph with vertex set V_i and edge set E_i, $i = 1, 2, ..., k$. The *union* $F := \bigcup_{i=1}^{k} G_i$ has the vertex set $V = \bigcup_{i=1}^{k} V_i$ and the edge set $E = \bigcup_{i=1}^{k} E_i$. If $G_i := G$ for every i then we denote by kG. Furthermore, if G_i is a tree for every i then graph F is called a *forest*. We use F_s to denote a *star forest*, i.e. all the components of F is a star.

The Ramsey numbers for disjoint union of graphs have been intensively studied in [1, 3, 4]. In 2006, Baskoro et al. [1] gave the Ramsey number for k copies of star versus a wheel on five vertices. The result is as follows.

Theorem A. *For $n \geq 3$,*

$$R(kS_n, W_4) = \begin{cases} (k+1)n, & \text{if } n \text{ is even and } k \geq 2, \\ (k+1)n - 1, & \text{if } n \text{ is odd and } k \geq 1. \end{cases}$$

Recently, Hasmawati et al. [4] have proved the following theorem.

Theorem B. *For connected graphs G and H, and $k \geq 1$. Then, $R(kG, H) \leq R(G, H) + (k-1)|G|$.*

A connected graph G is called *H-good* if $R(G, H) = (|G| - 1)(\chi(H) - 1) + s(H)$, where $\chi(H)$ denotes the chromatic number and $s(H)$ the chromatic surplus of H, i.e. the minimum cardinality of a color class taken over all proper $\chi(H)$ colorings of H. In the case $H = K_m$, the trees T_n are known to be K_m-good, see [6]. By using this terminology, Bielak [3] gave the exact Ramsey numbers for disjoint union graphs versus a graph H in more general situation. The result is cited below.

Theorem C. *Let H be a graph with $\chi(H) = m$ and $s(H) = 1$. If G is a graph with H-good components and $c(G) = n$ then $R(G, H) = \max_{(1 \leq j \leq n)}\{(j-$*

$1)(m-2)+\sum_{i=j}^{n} ik_i\}$, *where* k_i *is the number of components of order* i *in* G.

Observe that $R(S_n(1,1),W_6)$ and $R(S_n(1,2),W_6)$ do not satisfy the Chvátal-Harary bound. Evidently, for even $n \geq 4$, $R(S_n,W_4)$ does not holds as well. In contrast to Theorem C, where the Ramsey number for each component of disjoint union of graphs respect to a graph H is same as the Chvátal-Harary bound, we will determine $R(\bigcup_{i=1}^{k} l_i S_{n_i}(1,1),W_6)$, $R(\bigcup_{i=1}^{k} l_i S_{n_i}(1,2),W_6)$ and $R(\bigcup_{i=1}^{k} l_i S_{n_i},W_4)$.

2 The Main Results

Our main results are as follows.

Theorem 1. *For* $1 \leq i \leq k$, *let* $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 4$ *and* $F := \bigcup_{i=1}^{k} l_i S_{n_i}(1,1)$ *with* $l_i \geq 1$. *Then,*

$$R(F,W_6) = \max_{1\leq i\leq k} \left\{ n_i + \sum_{j=i}^{k} l_j n_j \right\}. \tag{1}$$

Theorem 2. *For* $1 \leq i \leq k$, *let* $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 6$ *and* $F' := \bigcup_{i=1}^{k} l_i S_{n_i}(1,2)$ *with* $l_i \geq 1$. *If* $n_i \equiv 0 (\mathrm{mod}\, 3)$ *for every* i *then*

$$R(F',W_6) = \max_{1\leq i\leq k} \left\{ n_i + \sum_{j=i}^{k} l_j n_j \right\}. \tag{2}$$

Theorem 3. *For* $1 \leq i \leq k$, *let* $n_k \geq n_{k-1} \geq ... \geq n_1 \geq 4$ *and* $F_s := \bigcup_{i=1}^{k} l_i S_{n_i}$ *with* $l_i \geq 1$. *If* n_i *is even for every* i *then*

$$R(F_s,W_4) = \max \left\{ \max_{1\leq i\leq k} \left\{ n_i + \sum_{j=i}^{k} l_j n_j \right\}, 2n_k + 1 \right\}. \tag{3}$$

First of all, we prove the following lemmas.

Lemma 1. *For* $n \geq 4$ *and* $k \geq 1$, $R(kS_n(1,1),W_6) = (k+1)n$.

Proof. Consider graph $G \simeq K_{kn-1} \cup \overline{H}$, where $H \simeq C_n$ for $n \neq 6$ and $H \simeq 2C_3$ for $n = 6$. It is not difficult to verify that G is a $(kS_n(1,1),W_6)$-good graph on $(k+1)n-1$ vertices. Thus, $R(kS_n(1,1),W_6) \geq (k+1)n$. The upper bound is proved by using Theorem B and the result of Chen et al. □

Lemma 2. *For* $n \equiv 0 (\mathrm{mod}\, 3) \geq 6$ *and* $k \geq 1$, $R(kS_n(1,2),W_6) = (k+1)n$.

Proof. Consider graph $G' \simeq K_{kn-1} \cup K_{3,3,\ldots,3}$, where $K_{3,3,\ldots,3}$ is a balanced complete $(\frac{n}{3})$-partite graph of order $n \equiv 0(\bmod 3) \geq 6$. A routine procedure, we can observe that G' is a $(kS_n(1,2), W_6)$-good graph on $(k+1)n-1$ vertices. Thus, $R(kS_n(1,2), W_6) \geq (k+1)n$. Again, we will get the reverse inequality by using Theorem B and the result of Chen et al. □

Lemma 3. *Let G be a graph of order $2n$ with even $n \geq 4$. If G contains no S_n and $\overline{G}$ contains no W_4 then $\delta_G(x) = n-2$, for all x in $V(G)$.*

Proof. Since $G \not\supseteq S_n$ then $\delta_G(x) \leq n-2$, for all $x \in V(G)$. We shall show that $\delta_G(x) \geq n-2$ for all $x \in V(G)$. By a contrary, assume that there exists a vertex $x \in V(G)$ with $\delta_G(x) \leq n-3$. Choose $u \in V(G)$ with $xu \notin E(G)$. Let $N[x] = \{w : xw \in E(G)\} \cup \{x\}$ and $N(x) = N[x] \setminus \{x\}$. Write $P = N[x] \cup N[u]$, $Q = V(G) \setminus P$, $R = N(x) \cap N(u)$, and $X = P \setminus \{x, u\}$. Clearly, $0 \leq |R| \leq n-3$, $2 \leq |P| \leq 2n-3$. Since $|N[x] \cup N[u]| = |N[x]| + |N[u]| - |N(x) \cap N(u)|$ then $2n-2 \geq |Q| \geq 3 + |R|$. Let $F := G[Q]$. Since $\overline{G} \not\supseteq W_4$ then $\delta_F(q) \geq |Q|-2$, for all $q \in V(F)$.

Next, define $E(X \setminus R, Q) = \{yq : y \in X \setminus R, q \in Q\}$. Suppose that there exists a vertex $y \in X \setminus R$ not adjacent to at least two vertices of Q, say q_1 and q_2, then $\overline{G}$ contains a wheel $W_4 = \{u, x, q_1, q_2, y\}$ with x or u as a hub, a contradiction. Thus, every vertex y in $X \setminus R$ is adjacent to at least $|Q|-1$ vertices in Q. Consequently, $|E(X \setminus R, Q)| \geq |X \setminus R| \cdot (|Q|-1)$. On the other hand, since $\delta_G(q) \leq n-2$ and $\delta_F(q) \geq |Q|-2$, for all $q \in V(F)$ then every vertex q in Q is incident with at most $(n-2) - \delta_F(q) \leq (n-2) - (|Q|-2) = (n-|Q|)$ edges from $X \setminus R$. Thus $|E(X \setminus R, Q)| \leq |Q| \cdot (n-|Q|)$. Finally, we have $|X \setminus R| \cdot (|Q|-1) \leq |E(X \setminus R, Q)| \leq |Q| \cdot (n_i - |Q|)$.

Now, consider $|X \setminus R| \cdot (|Q|-1) = |X \setminus R| \cdot |Q| - |X \setminus R|$. Since $|X \setminus R| = 2n-3-|Q|$ then $|X \setminus R| \cdot (|Q|-1) = |Q| \cdot (n-|Q|) + (n-2) \cdot (|Q|-2) - 1$. Write $k = (n-2) \cdot (|Q|-2) - 1$. Since $|Q| \geq 3 + |R|$ and $n \geq 4$ then it can be verified that $k > 0$. Thus $|X \setminus R| \cdot (|Q|-1) > |Q| \cdot (n-|Q|)$, which is impossible. This implies $\delta_G(x) \geq n-2$, for all $x \in V(G)$. We concludes that $\delta_G(x) = n-2$, for all $u \in V(G)$. □

Now by the lemmas stated above and the idea of the proof of Theorem C presented in [3] we are ready to prove theorems 1, 2 and 3.

Proof of Theorem 1. For $1 \leq i \leq k$, let F_i be the subgraph of F consisting of all the components with at least n_i vertices, $F_i = \bigcup_{j=i}^{k} l_j S_{n_j}(1,1)$. Suppose that the maximum of $\max_{(1 \leq i \leq k)}\{n_i + \sum_{j=i}^{k} l_j n_j\}$ is achieved for i_0. Write $b_0 = \sum_{j=i_0}^{k} l_j n_j$ and $t = n_{i_0} + b_0$. Consider graph $G \simeq K_{b_0-1} \cup \overline{H}$, where $H \simeq C_{n_{i_0}}$ for $n_{i_0} \neq 6$ and $H \simeq 2C_3$ for $n_{i_0} = 6$. Since $n_i \geq 4$ for any i then G does not contains any subgraph F_{i_0} and its complement contains no W_6. Therefore, G is a (F, W_6)-good graph on $t-1$ vertices. Thus

$R(F, W_6) \geq t$. The reverse inequality is proved by the following reason. Let K be a graph on t vertices and suppose that $\overline{K}$ contains no W_6. Consider the subgraph $F_{k-r} = \bigcup_{j=k-r}^{k} l_j S_{n_j}(1,1)$, for $0 \leq r \leq k-1$. Moreover, $F_i = F_{k-r}$ for $i = k-r$ and $0 \leq r \leq k-1$. We will show that K contains F_1. We prove this by induction on r. For $r = 0$, we have $F_k = l_k S_{n_k}(1,1)$. Note that $t \geq (l_k+1)n_k$. By Lemma 1, K contains $l_k S_{n_k}(1,1)$ and hence $K \supseteq F_k$. Let us state the inductive hypothesis: K contains F_{k-r} for all $r \leq k-2$. By induction on r, K contains $F_{k-r} = F_2$ for $r = k-2$. Note that F_2 has $\sum_{j=2}^{k} l_j n_j$ vertices. Write $C = V(K) \backslash V(F_2)$, then $|C| = t - \sum_{j=2}^{k} l_j n_j$. By definition of t, we get $t \geq n_i + \sum_{j=i}^{k} l_j n_j$, for $1 \leq i \leq k$. Therefore, $t - \sum_{j=2}^{k} l_j n_j \geq (l_1+1)n_1$. Again by Lemma 1, we have that $K[C]$ contains $l_1 S_{n_1}(1,1)$. Thus $K \supseteq F_1$. This concludes that K contains F. □

Proof of Theorem 2. For $1 \leq i \leq k$, let $F_i' = \bigcup_{j=i}^{k} l_j S_{n_j}(1,2)$. Suppose that the maximum of $\max_{(1 \leq i \leq k)}\{n_i + \sum_{j=i}^{k} l_j n_j\}$ is achieved for i_0'. Write $b_0' = \sum_{j=i_0'}^{k} l_j n_j$ and $t' = n_{i_0'} + b_0'$. Consider graph $G' \simeq K_{b_0'-1} \cup K_{3,3,\ldots,3}$, where $K_{3,3,\ldots,3}$ is a balanced complete $(\frac{n_{i_0'}}{3})$-partite graph of order $n_{i_0'} \equiv 0 (\mathrm{mod}\, 3)$. Since $n_{i_0'} \equiv 0 (\mathrm{mod}\, 3) \geq 6$ then G' does not contains any subgraph $F_{i_0'}'$ and its complement contains no W_6. Therefore, G' is a (F', W_6)-good graph on $t'-1$ vertices. Thus $R(F', W_6) \geq t'$. We can prove the reverse inequality by using Lemma 2 and similar reason with the proof of Theorem 1. □

Proof of Theorem 3. For $1 \leq i \leq k$, let $G_i = \bigcup_{j=i}^{k} l_j S_{n_j}$. Suppose that the maximum of $\max_{(1 \leq i \leq k)}\{n_i + \sum_{j=i}^{k} l_j n_j\}$ is achieved for i_0''. Write $b_0'' = \sum_{j=i_0''}^{k} l_j n_j$ and $t_0 = n_{i_0''} + b_0''$. Let t_1 be a maximum of $\{t_0, 2n_k+1\}$. The graphs $L_1 \simeq (H_{(\frac{-2+b_0''}{2})} + K_1) \cup H_{(\frac{n_{i_0''}}{2})}$ and $\overline{L}_2 \simeq \frac{n_k}{2}K_2 + \frac{n_k}{2}K_2$ give $R(F_s, W_4) \geq \max\{t_0, 2n_k+1\} = t_1$, where $H_{(\frac{n_{i_0''}}{2})}$ is a Cocktail-Party Graph on $n_{i_0''}$ vertices.

The reverse inequality is proved by the following reason. Let U be a graph of order t_1 and suppose that $\overline{U}$ contains no W_4. Consider the subgraph $G_{k-r} = \bigcup_{j=k-r}^{k} l_j S_{n_j}$, for $0 \leq r \leq k-1$. Moreover, $G_i = G_{k-r}$ for $i = k-r$ and $0 \leq r \leq k-1$. We will show that U contains G_1. We prove this by induction on r. For $r = 0$, we have $G_k = l_k S_{n_k}$. Since $t_1 \geq (l_k+1)n_k$ and $t_1 \geq 2n_k+1$ then Theorem A gives that U contains $l_k S_{n_k}$ when $l_k \geq 2$. Meanwhile, it is well known that U contains S_{n_k}, when $l_k = 1$ and $t_1 \geq 2n_k+1$. Therefore $U \supseteq G_k$. Let us state the inductive hypothesis: U contains G_{k-r} for all $r \leq k-2$. By induction hypothesis, U contains G_2. Note that G_2 has $y = \sum_{j=2}^{k} l_j n_j$ vertices. Write $B = V(U) \backslash V(G_2)$ and $T = U[B]$, then $|T| = t_1 - y$. By definition of t_1, we get $t_1 \geq n_i + \sum_{j=i}^{k} l_j n_j$ and hence $t_1 - y \geq (l_1 + 1)n_1$. If $l_1 \geq 2$ then Theorem A guarantees that T contains $l_1 S_{n_1}$. Therefore, $U \supseteq G_1$.

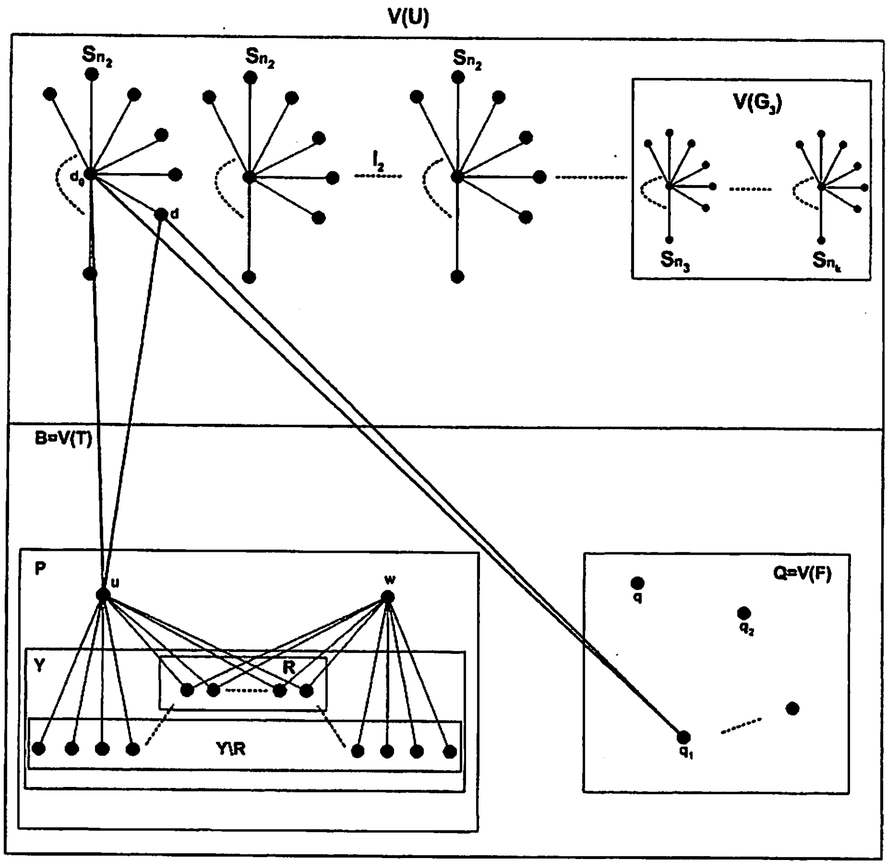

Fig. 1. The illustration of the construction of S'_{n_2} and S_{n_1} in U

Now, consider $l_1 = 1$. This implies that $t_1 - y \geq 2n_1$. If $t_1 - y \geq 2n_1 + 1$ then T contains S_{n_1}. So, $U \supseteq G_1$. Next, consider $t_1 - y = 2n_1$. We will show that T contains S_{n_1}. If there exists a vertex $u \in V(T)$ with $\delta_T(u) \geq (n_1 - 1)$ then we also have $U \supseteq G_1$.

Next, consider $\delta_T(u) \leq n_1 - 2$, for all $u \in V(T)$. Since $\overline{T} \not\supseteq W_4$ then Lemma 3 ensures that $\delta_T(u) = n_1 - 2$, for all $u \in V(T)$. Now, choose $w \in V(T)$ with $uw \notin E(T)$. Let $P = N[u] \cup N[w]$, $Q = V(T) \setminus P$, $R =$

$N(u) \cap N(w)$, and $Y = P \setminus \{u, w\}$. Clearly, $0 \le |R| \le n_1 - 2$, $n_1 \le |P| \le 2n_1 - 2$, and $|Q| = 2 + |R|$. Let $F := T[Q]$. We shall show that F is not a complete graph. By a contrary, suppose that F is a complete graph. Define $E(Y \setminus R, Q) = \{yv : y \in Y \setminus R, q \in Q\}$. Since $\overline{T} \not\supseteq W_4$ then every vertex y in $Y \setminus R$ is adjacent to at least $|Q| - 1$ vertices in Q. This implies that $|E(Y \setminus R, Q)| \ge |Y \setminus R| \cdot (|Q| - 1)$. On the other hand, since $\delta_T(u) = n_1 - 2$, for all $u \in V(T)$ and F is a complete graph then every vertex q in $V(F)$ is incident with exactly $(n_1 - 2) - \delta_F(q) = (n_1 - 2) - (|Q| - 1) = (n_1 - 1 - |Q|)$ edges from $Y \setminus R$. This means that $|E(Y \setminus R, Q)| = |Q| \cdot (n_1 - 1 - |Q|)$. Finally, $|Y \setminus R| \cdot (|Q| - 1) \le |E(Y \setminus R, Q)| = |Q| \cdot (n_1 - 1 - |Q|)$.

Now, consider $|Y \setminus R| \cdot (|Q| - 1) = |Y \setminus R| \cdot |Q| - |Y \setminus R|$. Since $|Y \setminus R| = 2n_1 - 2 - |Q|$ then $|Y \setminus R| \cdot (|Q| - 1) = |Q| \cdot (n_1 - 1 - |Q|) + n_1 \cdot (|Q| - 2) + 2$. Let $h = n_1 \cdot (|Q| - 2) + 2$. Since $|Q| = 2 + |R|$ then it can be verified that $h > 0$. Thus $|Y \setminus R| \cdot (|Q| - 1) > |Q| \cdot (n_1 - 1 - |Q|)$, which leads to a contradiction. Therefore, F is not a complete graph. Now, choose two vertices in $V(F)$ which are not adjacent, say q_1 and q_2. Write $Z = \{u, w\} \cup \{q_1, q_2\}$. Clearly, Z is an independent set.

Without loss of generality, let $D = V(S_{n_2}) \subset V(l_2 S_{n_2}) \subset V(G_2)$ and d_0 be a center of star S_{n_2}. Suppose that there exists a vertex d in D adjacent to at most one vertex in Z call w, then $\{d, q_1, q_2, w, u\}$ will induce a W_4 in $\overline{U}$, with a hub u, a contradiction. Thus every vertex d in D is adjacent to at least two vertices in Z. Without loss of generality, let d_0 and d in D be adjacent to q_1 and u. Since every vertex in Z has degree $n_1 - 2$ then we have two new stars, namely S'_{n_2} and S_{n_1}, where $V(S'_{n_2}) = \{q_1\} \cup V(S_{n_2}) \setminus \{d\}$ with d_0 as the center and $V(S_{n_1}) = N[u] \cup \{d\}$ with u as the center, see Fig. 1. So, we have $T \supseteq S_{n_1}$ and hence $U \supseteq G_1$. This completes the proof. □

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