

Path-path size multipartite Ramsey numbers

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Abstract. Let $j \geq 2$ be a natural number. For graphs G and H , the size multipartite Ramsey number $m_j(G, H)$ is the smallest natural number t such that any 2-coloring by red and blue on the edges of $K_{j \times t}$ necessarily forces a red G or a blue H as subgraph. Let P_n be a path on n vertices. In this note, we determine the exact value of the size multipartite Ramsey number $m_j(P_4, P_n)$ for $n \geq 2$.

1 Introduction

In this paper, we use $K_{j \times k}$ to denote a complete multipartite graph with j partite sets, each of size k . Following the notion introduced in [1], we define the size multipartite Ramsey number as follows. Let $j \geq 2$ be a natural number. For two graphs G and H , the *size multipartite Ramsey number* $m_j(G, H)$ is the smallest natural number t such that any red-blue coloring of the edges of $K_{j \times t}$ necessarily forces a red G or a blue H as a subgraph.

There are only few results known so far on the size multipartite Ramsey numbers $m_j(G, H)$. Some related to this work are mentioned in the following. Finding the value of $m_2(P_s, P_t)$ was considered by Gyárfás and Lehel [4], and by Faudree and Schelp [3]. Hattingh and Henning [6] determined the exact values of $m_2(P_m, S_n)$ and the upper bound on the size multipartite Ramsey numbers for a combination of complete graphs, namely $m_2(a, b) \leq m_2(b, a - 1) + m_2(a, b - 1) + 1$ with $a, b \geq 2$ [5], where $m_2(x, y) = m_2(K_{x,x}, K_{y,y})$.

For all $j \geq 2$, Day *et al.* [2], and Burger and van Vuuren [1] determined the exact values of $m_j(K_{2 \times 2}, H)$ where $H \cong K_{2 \times 2}$ or $K_{3 \times 1}$. Recently, Syafrizal Sy *et al.* [7, 8] determined the exact values of $m_j(P_s, G)$ where $s = 2$ or 3,

and G is a path P_n , a cycle C_n , a wheel W_n , a star S_n , or a fan F_n . In this paper, we consider the size multipartite Ramsey numbers of paths versus paths, $m_j(P_4, P_n)$ for $n \geq 2$.

2 Main Results

To show Theorem 1, we use the following lemma.

Lemma 1. *Let $G \cong K_{2 \times n}$ be a bipartite graph with $n \geq 2$. Suppose $G = G_1 \oplus G_2$ with G_1 consists of at least three components. If G_1 contains the largest component H where $|V(H)| \leq |V(G \setminus H)|$ then G_2 contains a path on $2n$ vertices.*

Proof. Without loss of generality, it suffices to prove for the case where G_1 consists of exactly three components, each constitutes a complete bipartite graph. Let K_{a_1, b_1} , K_{a_2, b_2} , and K_{a_3, b_3} be the components of G_1 with $a_1 + b_1 \geq a_2 + b_2 \geq a_3 + b_3$ and $a_1 + b_1 \leq \sum_{i=2}^3 (a_i + b_i)$. We assume here that $a_1 \leq b_1$. We consider two cases:

Case 1. $a_1 \leq b_2$.

We show first that $a_1 + a_3 \geq b_2$. Suppose that $a_1 + a_3 < b_2$. Noting that $a_1 + a_2 + a_3 \geq b_1 + b_2$ then we have $a_2 > b_1$. As a consequence, we obtain $a_2 + b_2 > a_1 + b_1$. A contradiction to the fact that K_{a_1, b_1} is the largest. Thus we must have $a_1 + a_3 \geq b_2$.

Next, we will show that $a_1 + a_3 \leq b_1 + b_2$. Suppose that $a_1 + a_3 > b_1 + b_2$. Recall that G is balanced. Thus $a_2 < b_3$. Moreover, the assumption that $a_1 \leq b_1$ implies $a_3 > b_2$. As a consequence, we obtains $a_3 + b_3 > a_2 + b_2$. This contradicts the order of $|K_{a_i, b_i}|$. Hence $a_1 + a_3 \leq b_1 + b_2$.

Let $A_i = V_1 \cap V(K_{a_i, b_i})$ and $B_i = V_2 \cap V(K_{a_i, b_i})$ be the partite subsets of G , with $i = 1, 2, 3$.

Now we construct a path as follows. Starting at a vertex in A_1 , we obtain the longest path P^1 by using vertices in A_1 and B_2 . Since $a_1 \leq b_2$, P^1 saturates all vertices in A_1 and terminates at a vertex in B_2 . Extend P^1 by using the remaining vertices in B_2 and vertices in A_3 . Since $a_1 + a_3 \geq b_2$, this extension, call P^2 , terminates at a vertex of A_3 , after saturating the remaining vertices in B_2 . We continue extending P^2 by using the rest of A_3 and B_1 . The resulting path, call P^3 , ends at vertex in B_1 , since $a_1 + a_3 \leq b_1 + b_2$. P^3 can be lengthened into P^4 by using the rest of B_1 and A_2 . P^4 must terminate at vertex in A_2 , since G is balanced. Observe that at this stages the unsaturated vertices are the remaining of A_2 and B_3 . Recalling G is

balanced, P^4 can be extended into a final path saturating all vertices in G .

Case 2. $a_1 > b_2$.

We show first that $a_1 \leq b_2 + b_3$. Suppose $a_1 > b_2 + b_3$, then we obtain $a_2 + a_3 < b_1$. As a consequence, $a_1 + b_1 > (a_2 + b_2) + (a_3 + b_3)$. A contradiction to the assumption $a_1 + b_1 \leq (a_2 + b_2) + (a_3 + b_3)$. Therefore, we must have $a_1 \leq b_2 + b_3$.

Next, we will show that $a_1 + a_2 \geq b_2 + b_3$. Supposing that $a_1 + a_2 < b_2 + b_3$, we obtain $a_3 > b_1 \geq a_1 > b_2$. This implies $b_2 + a_2 < a_1 + a_2 < b_2 + b_3 < a_3 + b_3$. This contradicts the order of $|K_{a_i, b_i}|$. Therefore, we must have $a_1 + a_2 \geq b_2 + b_3$.

Using a similar procedure as in Case 1, we can construct a blue path using all the vertices of G . In this case the path has the initial vertex in A_1 and the terminal vertex B_1 . \square

Theorem 1. For integers $j, n \geq 2$,

$$m_j(P_4, P_n) = \begin{cases} \lceil \frac{4}{j} \rceil, & \text{for } j \geq 2 \text{ and } n = 2 \text{ or } 3, \\ \lceil \frac{n}{2} \rceil + 1, & \text{for } j = 2 \text{ and } n \geq 4, \\ \lceil \frac{n+1}{j} \rceil, & \text{for } j \geq 3 \text{ and } n \geq 4. \end{cases}$$

Proof. By [7], we obtain $m_j(P_4, P_n) = \lceil \frac{4}{j} \rceil$ for $j \geq 2$ and $n = 2$ or 3 . To prove multipartite Ramsey numbers for combination of other j and n , we consider two cases as follows.

Case 1. $j = 2$ and $n \geq 4$.

Let $t = \lceil \frac{n}{2} \rceil + 1$. Consider $G_1 \cong K_{2 \times (t-1)}$. Take a subgraph $2K_{1, t-2}$ in G_1 , and color the edges of $2K_{1, t-2}$ by red. The remaining edges of G_1 are colored by blue. Clearly, G_1 contains neither red P_4 nor blue P_n . Therefore, $m_2(P_4, P_n) \geq t$.

Now, we will show that $m_2(P_4, P_n) \leq t$. Consider $F_1 \cong K_{2 \times t}$. Suppose all edges of F_1 are arbitrarily colored by red and blue such that F_1 contains no red P_4 . If all edges of F_1 are blue then trivially F_1 contains blue P_n and the proof is complete. Therefore, we may assume F_1 contains at least one red component. Since F_1 contains no red P_4 then each red component in F_1 will form a star. To show that F_1 contains a blue P_n , we consider two possibilities.

Subcase 1.1 F_1 contains either red star $K_{1, t}$ or red star $2K_{1, t-1}$. In this

case, let $V_1 = \{a_1, a_2, \dots, a_t\}$ and $V_2 = \{b_1, b_2, \dots, b_t\}$ be the partite sets of F_1 . Without loss of generality, if F_1 contains red $K_{1,t}$, let $a_1 \in V_1$ be the center of $K_{1,t}$ then we have red path $P := b_1 a_2 b_2 a_3 b_3 \dots a_{t-1} b_{t-1} a_t b_t$ in F_1 . Next, if F_1 contain two red star $2K_{1,t-1}$, let $a_1 \in V_1$ and $b_t \in V_2$ be the centers of $2K_{1,t-1}$ then we have red path $Q := b_1 a_2 b_2 a_3 b_3 \dots a_{t-1} b_{t-1} a_t$ in F_1 . Observe that in both cases, we have a blue path P_n with at least n vertices in F_1 . Therefore, $m_j(P_4, P_n) \leq t$.

Subcase 1.2 F_1 contains neither red star $K_{1,t}$ nor red star $2K_{1,t-1}$. As a consequence, F_1 contain red largest component with at most t vertices. By Lemma 1, F_1 contains a blue path P_n with $n = 2t$ vertices. Therefore, $m_j(P_4, P_n) \leq t$.

Case 2. $j \geq 3$ and $n \geq 4$.

Let $s = \lceil \frac{n+1}{j} \rceil$. Consider $G_2 \cong K_{j \times (s-1)}$. Take a subgraph $K_{1, (j-1)(s-1)}$ in G_2 , and color its edges by red. The remaining edges of G_2 are colored by blue. Clearly, G_2 contains no red P_4 . Since $j(s-1) - 1 < n$ then G_2 contains no blue P_n . Therefore, $m_j(P_4, P_n) \geq s$.

Now, we will prove that $m_j(P_4, P_n) \leq s$. Consider $F_2 \cong K_{j \times s}$. Suppose all edges of F_2 be arbitrarily colored by red or blue, such that F_2 contains no red P_4 . If all edges of F_2 are blue then F_2 trivially contains a blue P_n and the proof is complete. Now, assume that F_2 contains at least one red component. Since F_2 contains no red P_4 then all these red components form stars or cycles C_3 . Now, let V_1, V_2, \dots, V_j be the partite sets of F_2 . To show that F_2 contains a blue P_n , consider the following two subcases.

Subcase 2.1 j is even.

Partition the partite sets of F_2 into two new partite sets, namely $X_1 = V_1 \cup V_3 \cup \dots \cup V_{j-1}$ and $X_2 = V_2 \cup V_4 \cup \dots \cup V_j$. Then, let $X_1 = \{u_1, u_2, \dots, u_k\}$ and $X_2 = \{v_1, v_2, \dots, v_k\}$ with $k = \frac{j}{2}s$ (see Fig. 1.(a)). Let $E' = \{uv \in E(F_2) | u \in X_1 \text{ and } v \in X_2\}$. Consider $F_2[E'] \subseteq F_2$. To show that F_2 contains a blue P_n , we suffices show that $F_2[E']$ contains a blue P_n . We consider three possibilities.

i). If $F_2[E']$ contains a red star $K_{1,k}$. Without loss of generality, let $u_1 \in V_1$ be the center vertex of red $K_{1,k}$ in X_1 . If $u_1 u_k \notin E(F_2)$ then we obtain a blue path $u_1 u_k v_k v_1 u_2 v_2 u_3 v_3 \dots u_{k-1} v_{k-1}$ with $2k$ vertices in F_2 . The other side, if $u_1 u_k \in E(F_2)$ the we obtain a blue path $v_1 u_2 v_2 u_3 v_3 \dots v_{k-1} u_k$ with $2k - 1$ vertices in F_2 . Therefore, F_2 contains a blue path with at least n vertices (see Fig. 1.(b)).

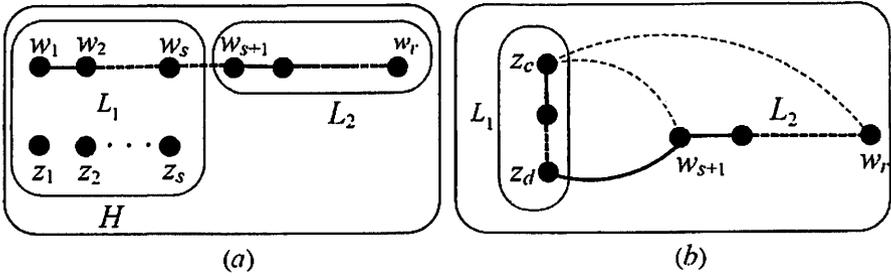


Fig. 1. (a) New partition. (b) $F_2[E'] \supset K_{1,k}$. (c) $F_2[E'] \supset 2K_{1,k-1}$.

ii). If $F_2[E']$ contain two red stars $2K_{1,k-1}$. Without loss of generality, let $u_1 \in V_1$ and $v_k \in V_j$ be the vertices of center of two red star $2K_{1,k-1}$. Since F_2 contains no a red P_4 , then we obtain a blue path $v_k v_1 u_2 v_2 u_3 v_3 \dots u_{k-1} v_{k-1} u_k u_1$ with $2k$ vertices in F_2 , since $u_1 u_k \notin E(F_2)$ and $v_1 v_k \notin E(F_2)$. Thus, we have a blue path with at least $n = js - 1$ vertices in F_2 (see Fig. 1(c)).

iii). If $F_2[E']$ contains no neither red $K_{1,k}$ nor red $2K_{1,k-1}$ then $F_2[E']$ contains red largest component with at most s vertices. By Lemma 1, $F_2[E']$ contains a blue path with js vertices. Therefore, F_2 contains a blue path with $n = js$ vertices.

Subcase 2.2 j is odd.

By using the method in Subcase 2.1, we have two new partite sets, namely $Y_1 = V_1 \cup V_3 \cup \dots \cup V_{j-2}$ and $Y_2 = V_2 \cup V_4 \cup \dots \cup V_{j-1}$ with $|Y_1| = |Y_2| = k$. Let F_2^* be a subgraph of F_2 induced by Y_1 and Y_2 . Since the numbers of the partite sets in F_2^* is even, then by using the method in Subcase 2.1, we get a blue path $L = w_1 w_2 \dots w_s w_{s+1} \dots w_r$ in F_2^* with at least $r = s(j - 1) - 1$ vertices.

Now, let H be a subgraph of F_2 induced by $\{w_1, w_2, \dots, w_s\}$ and $V_j = \{z_1, z_2, \dots, z_s\}$ (see Fig. 2(a)). If H contains a red star $K_{1,s}$ or two red star $2K_{1,s-1}$, then by using the method in Subcase 1.1, we obtain a red path L_1 with at least $2s - 1$ vertices. Let $L_2 = w_{s+1} \dots w_r$. If there is a blue edge connecting a blue path L_1 and a blue path L_2 , then F_2 contains a blue path with at least $n = js - 2$ vertices. Conversely, let $z_c w_{s+1}$ and $z_c w_r$ be the red edges. Since F_2 contains no red P_4 , then $z_d w_{s+1}$ is a blue edge (see Fig. 2(b)). Thus, F_2 contains a blue path with at least $n = js - 2$ vertices.

If F_2^* contains neither red star $K_{1,k}$ nor two red star $2K_{1,k-1}$, then F_2^*

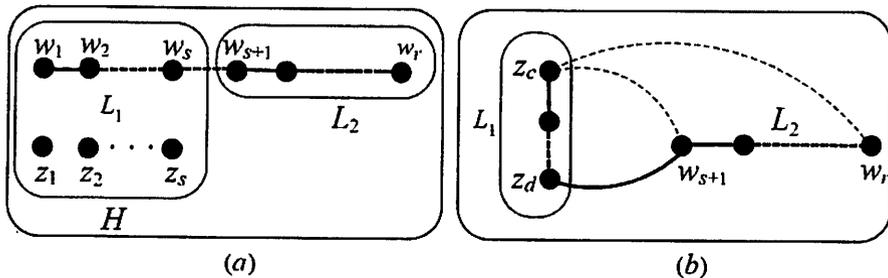


Fig. 2. Illustration of Subcase 2.2.

contains a red largest component with at most s vertices. Thus, by Lemma 1, F_2^* contains a blue path with js vertices. Therefore, F_2 contains a blue path with at least $n = js$ vertices. \square

Corollary 1. For integers $j \geq 2$ and $n \geq 3$,

$$m_j(P_4, C_n) = \begin{cases} \infty & \text{for } j = 2 \text{ and } n \text{ is odd,} \\ \frac{n}{2} + 1 & \text{for } j = 2 \text{ and even } n \geq 4, \\ \lceil \frac{n+1}{j} \rceil & \text{for } j, n \geq 3. \end{cases}$$

Proof. If $G \cong K_{2 \times t}$, then G contains no odd cycle for any natural number t . This implies, $m_2(P_4, C_n) = \infty$ for odd $n \geq 3$. Next for even $n \geq 2$, let aP_b be the blue path obtained in the proof of Theorem 2 with ab is a blue edge. This blue path is consist of at least n vertices. Since ab is a blue edge, then by connecting a and b , we have a blue cycle C_n with at least n vertices. \square

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