On (a, d)-H-antimagic coverings of graphs

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Abstract. A simple graph $G = (V(G), E(G))$ admits an *H-covering*, if every edge in $E(G)$ belongs to a subgraph of G that is isomorphic to H. An *(a, d)-H-antimagic total labeling* of G is a bijective function $\xi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, |V(G)| + |E(G)|\}$ such that for all subgraphs H' isomorphic to H, the H-weights $w(H') = \sum_{v \in V(H')} \xi(v) + \sum_{e \in E(H')} \xi(e)$ constitute an arithmetic progression $a, a + d, a + 2d, \ldots, a + (t-1)d$ where a and d are positive integers and t is the number of subgraphs of G isomorphic to H. Additionally, the labeling ξ is called a *super (a, d)-H-antimagic total labeling*, if $\xi(V(G)) = \{1, 2, \ldots, |V(G)|\}$.

In this paper, we introduce the notion of *(a, d)-H-antimagic total labeling* and study some basic properties of such labeling. We provide an example of a family of graphs obtaining the labelings, that is providing (a, d)-cycle-antimagic labelings of fans.

Keywords: (a, d)-H-antimagic total labeling, cycle, fan, H-covering

1 Introduction

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be simple and finite graphs. Let $|V(G)| = v_G, |E(G)| = e_G, |V(H)| = v_H, |E(H)| = e_H$. An edge-covering of G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that any edge of $E(G)$ belongs to at least one of the subgraphs H_j's, $1 \leq j \leq k$. If the H_j are isomorphic to a given graph H, then G admits an H-covering.

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Suppose that G admits an H-covering. Gutiérrez and Lladó [3] defined an H-magic labeling which is a generalization of Kotzig and Rosa's edge-magic total labeling [4]. A bijective function

$$f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, v_G + e_G\}$$

is called an *H-magic labeling* of G if there exists a positive integer c such that for each subgraph H' isomorphic to H satisfies

$$f(H') = \sum_{v \epsilon V(H')} f(v) + \sum_{e \epsilon E(H')} f(e) = c.$$

In this case, we say that G is *H-magic*. When $f(V(G)) = \{1, 2, \ldots, v_G\}$, we say that G is *H-supermagic.*

Lladó and Moragas [5] proved that a wheel W_n, a prism $C_n \times K_2$, a book $K_{1,n} \times K_2$, and a windmill $W(r, k)$ are *C_h-magic*. Maryati, Baskoro, Salman [6] and Salman, Ngurah, Izzati [9] proved that some families of trees are *P_h-supermagic*. Recently, Ngurah, Salman, and Susilowati [8] introduced the dual of an *H-(super)-magic labeling.*

On the other hand, in 2000, Simanjuntak, Miller and Bertault [10] introduced an *(a, d)-edge-antimagic total labeling* of G which is defined as a bijective function $\xi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, v_G + e_G\}$ so that the set of edge-weights $\{g(u) + g(uv) + g(v) | \; uv \in E(G)\}$ is equal to the set $\{a, a + d, a + 2d, \ldots, a + (e_G - 1)d\}$ for some two positive integers a and d. An (a, d)-edge-antimagic total labeling ξ is called *super* if the vertex labels are the smallest possible labels. Several results for the labeling have been provided, see for example [1], [2], and [7].

In this paper, we introduce a new labeling which utilizes both concepts, namely H-covering and (a, d)-antimagic. As an example of family of graphs obtaining the labelings, we study (a, d)-C_h-antimagic labelings of fans for some a, d, and h.

2 (a, d)-H-antimagic total labelings

In this section we define an *(a, d)-H-antimagic total labeling* and provide some basic properties of the labeling. Let G and H be graphs. An *(a, d)-H-antimagic total labeling* of G is a bijective function $\xi : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, v_G + e_G\}$ such that for all subgraphs H' isomorphic to H, the set of *H-weights*

$$w(H') = \sum_{v\epsilon V(H')} \xi(v) + \sum_{e\epsilon E(H')} \xi(e)$$

constitutes an arithmetic progression with a difference d, namely $a, a + d, \ldots, a + (t-1)d$, where a and d are some positive integers and t is the number of subgraphs isomorphic to H. In this case we say that G is (a, d)-*H-antimagic*. When $\xi(V(G)) = \{1, 2, \ldots, v_G\}$, we say that ξ is a *super* (a, d)-*H-antimagic total labeling* and G is *super* (a, d)-*H-antimagic*.

Note that if $d = 0$, then the labeling becomes the *H-magic labeling*. The next theorem give the largest possible value for a difference d.

Theorem 1. *Let G and H be graphs and t is the number of subgraphs of G isomorphic to H. If G is (a, d)-H-antimagic and $t \geq 2$, then*

$$d \leq \frac{(v_H + e_H)[v_G + e_G - v_H - e_H]}{t-1}.$$

Proof. Since G is (a, d)-H-antimagic, the largest H-weight is not larger than $(v_G + e_G - v_H - e_H + 1) + (v_G + e_G - v_H - e_H + 2) + (v_G + e_G - v_H - e_H + 3) + \cdots + (v_G + e_G - 1) + (v_G + e_G)$ and the least H-weight is not smaller than $1 + 2 + \ldots + (v_H + e_H)$. Hence,

$$a + (t-1)d \leq \frac{1}{2}(v_H + e_H)[2v_G + 2e_G - v_H - e_H + 1] \qquad (1)$$

and

$$a \geq \frac{1}{2}(v_H + e_H)(v_H + e_H + 1). \qquad (2)$$

From (1) and (2), we have

$$\begin{aligned}(t-1)d &\leq \frac{1}{2}(v_H + e_H)[2v_G + 2e_G - v_H - e_H + 1] - \\ &\quad \frac{1}{2}(v_H + e_H)(v_H + e_H + 1) \\ &= (v_H + e_H)[v_G + e_G - v_H - e_H].\end{aligned}$$

Since $t \geq 2$, then

$$d \leq \frac{(v_H + e_H)[v_G + e_G - v_H - e_H]}{t-1}.$$

□

The next two theorems show that we are able to construct some new (a, d)-H-antimagic labelings from an "old" (a, d)-H-antimagic labeling.

Theorem 2. *If G has an (a, d)-H-antimagic total labeling, then G has an (a', d)-H-antimagic total labeling, where*

$$a' = (v_H + e_H)(v_G + e_G + 1) - a - (t-1)d.$$

Proof. Let ξ be an (a, d)-H-antimagic total labeling of G and $M = v_G + e_G + 1$. Define ξ' as the dual labeling in the following way:

$$\xi'(x) = M - \xi(x) \text{ for each } x \in V(G) \cup E(G).$$

For each subgraph H' isomorphic to H, the H-weight under the dual labeling ξ' is

$$\begin{aligned} w'(H') &= \sum_{v \in V(H')} \xi'(v) + \sum_{e \in E(H')} \xi'(e) \\ &= \sum_{v \in V(H')} (M - \xi(v)) + \sum_{e \in E(H')} (M - \xi(e)) \\ &= (v_{H'} M + e_{H'} M) - \left[\sum_{v \in V(H')} \xi(v) + \sum_{e \in E(H')} \xi(e) \right] \\ &= (v_H + e_H)M - w(H'). \end{aligned}$$

Let $M^* = (v_H + e_H)M - a$, then the set of H-weights under the dual labeling ξ' is

$$\{M^*, M^* - d, M^* - 2d, \ \dots \ , M^* - a - (t-2)d, M^* - (t-1)d\}.$$

Thus, G has an (a', d)-H-antimagic total labeling, where

$$a' = M^* - (t-1)d = (v_H + e_H)(v_G + e_G + 1) - a - (t-1)d.$$

□

Theorem 3. *If G has a super (a, d)-H-antimagic total labeling, then G has a super (a'', d)-H-antimagic total labeling, where*

$$a'' = [(v_G + 1)v_H + (2v_G + e_G + 1)e_H] - a - (t-1)d.$$

Proof. Let ξ be a super (a,d)-H-antimagic total labeling of G and $N = v_G + 1$ and $N' = 2v_G + e_G + 1$. Define ξ'' as the super dual labeling in the following way:

$$\xi''(x) = \begin{cases} N - \xi(x) \text{ for each } x \in V(G), \\ N' - \xi(x) \text{ for each } x \in E(G). \end{cases}$$

For each subgraph H' isomorphic to H, the H-weight under the dual labeling ξ'' is

$$\begin{aligned} w''(H') &= \sum_{v \epsilon V(H')} \xi''(v) + \sum_{e \epsilon E(H')} \xi''(e) \\ &= \sum_{v \epsilon V(H')} (N - \xi(v)) + \sum_{e \epsilon E(H')} (N' - \xi(e)) \\ &= (v_{H'} N + e_{H'} N') - \left[\sum_{v \epsilon V(H')} \xi(v) + \sum_{e \epsilon E(H')} \xi(e) \right] \\ &= (v_H N + e_H N') - w(H'). \end{aligned}$$

Let $N^* = (v_H N + e_H N') - a$, then the set of H-weights under the dual labeling ξ'' is

$$\{N^*, N^* - d, N^* - 2d, \ldots, N^* - (t-2)d, N^* - (t-1)d\}.$$

Thus, G has a super (a'', d)-H-antimagic total labeling, where

$$a'' = N^* - (t-1)d = [(v_G + 1)v_H + (2v_G + e_G + 1)e_H] - a - (t-1)d.$$

□

3 (a,d)-C_h-antimagic total labelings of fans

In this section, we provide a family of graphs obtaining (a,d)-C_h-antimagic total labelings.

A *fan* F_n for $n \geq 2$ is a graph obtained by joining all vertices of a path P_n to a further vertex, called the *center*. Thus, F_n contains $n+1$ vertices, namely one vertex which is the center, denoted by c and n vertices on the path, denoted by $v_1, v_2, v_3, \ldots, v_n$, and contains $2n-1$ edges, namely cx_i for $1 \leq i \leq n$, and $v_i v_{i+1}$ for $1 \leq i \leq n-1$. It is easy to observe that F_n admits a C_h-covering for $3 \leq h \leq n+1$. We denote by C_h^j, the subgraph of F_n isomorphic to C_h formed by $c, v_j, v_{j+1}, v_{j+2}, \ldots, v_{j+h-1}$. From Theorem 1, we have the following upper bound for d.

Lemma 1. *If F_n is (a,d)-C_h-antimagic, then*

$$d \le \frac{(6nh - 4h^2)}{n+1-h}.$$

In the next three theorems, we provide three (super) (a,d)-C_h-antimagic total labelings for fans whose differences are functions of the length of the cycles.

Theorem 4. *For $3 \le h \le n$, the fan F_n is super $(h^2+2h.n-h+3, 2h-5)$-C_h-antimagic.*

Proof. We label the vertices and edges of F_n as follows.

$$\begin{array}{lll} \xi_1(c) & = 1 & \\ \xi_1(v_i) & = n+2-i & \text{for } 1 \le i \le n, \\ \xi_1(v_iv_{i+1}) & = 3n+1-i & \text{for } 1 \le i \le n-1, \\ \xi_1(cv_i) & = n+1+i & \text{for } 1 \le i \le n, \end{array}$$

It is easy to see that the labeling ξ_1 is a bijective function from $E(F_n) \cup V(F_n)$ to $\{1,2,3, \ldots, 3n\}$ and $\xi_1(V(F_n)) = \{1,2, \ldots, n+1\}$.

For $1 \le j \le n-h+2$, the C_h-weight of a subgraph C_h^j of F_n under the labeling ξ_1 is

$$\begin{aligned} w(C_h^j) &= \xi_1(c) + \sum_{k=j}^{j+h-2} \xi_1(v_k) + \xi_1(cv_j) + \sum_{k=j}^{j+h-3} \xi_1(v_kv_{k+1}) + \xi_1(cv_{j+h-2}) \\ &= -h^2 + 4.h.n + 8h - 5n - 7 - (2h-5)j. \end{aligned}$$

Since $w(C_h^j) - w(C_h^{j+1}) = 2h-5$ and $w(C_h^{n-h+2}) = h^2 + 2h.n - h + 3$, ξ_1 is a super $(h^2+2h.n-h+3,\ 2h-5)$-C_h-antimagic total labeling of F_n. □

The three following corollaries are about the (super) dual of the label ξ_1 defined in the proof of Theorem 4. They can be proved by using Theorem 4 and Theorem 2 for Corollary 1, Theorem 4 and Theorem 3 for Corollary 2, and Corollary 2 and Theorem 2 for Corollary 3.

Corollary 1. *For $3 \le h \le n$, the fan F_n is $(h^2+2h.n-4h+5n+2, 2h-5)$-C_h-antimagic.*

Corollary 2. *For $3 \le h \le n$, the fan F_n is super $(h^2+h.n+5n+2, 2h-5)$-C_h-antimagic.*

Corollary 3. *For $3 \le h \le n$, the fan F_n is $(h^2+3h.n-5h+3, 2h-5)$-C_h-antimagic.*

Theorem 5. *For $3 \leq h \leq n$, the fan F_n is super $(h^2 + 2h.n + h - 2n + 1, 2h - 1)$-$C_h$-antimagic.*

Proof. We label the vertices and edges of F_n as follows.

$$\begin{array}{lll} \xi_2(c) & = 1 & \\ \xi_2(v_i) & = 1+i & for \quad 1 \leq i \leq n, \\ \xi_2(v_i v_{i+1}) & = 2n+1+i & for \quad 1 \leq i \leq n-1, \\ \xi_2(cv_i) & = n+1+i & for \quad 1 \leq i \leq n. \end{array}$$

It is easy to see that the labeling ξ_2 is a bijective function from $E(F_n) \cup V(F_n)$ to $\{1, 2, 3, ..., 3n\}$ and $\xi_2(V(F_n)) = \{1, 2, \ldots, n+1\}$.

For $1 \leq j \leq n - h + 2$, the C_h-weight of a subgraph C_h^j of F_n under the labeling ξ_2 is

$$\begin{aligned} w(C_h^j) &= \xi_2(c) + \sum_{k=j}^{j+h-2} \xi_2(v_k) + \xi_2(cv_j) + \sum_{k=j}^{j+h-3} \xi_2(v_k v_{k+1}) + \xi_2(cv_{j+h-2}) \\ &= h^2 + 2.h.n + h - 2n - h + (2h-1)j. \end{aligned}$$

Since $w(C_h^{j+1}) - w(C_h^j) = 2h - 1$ and $w(C_h^1) = h^2 + 2h.n + h - 2n + 1$, ξ_2 is a super $(h^2 + 2h.n + h - 2n + 1,\ 2h - 1)$-$C_h$-antimagic total labeling of F_n. □

The three following corollaries are about the (super) dual of the label ξ_2 defined in the proof of Theorem 5. They can be proved by using Theorem 5 and Theorem 2 for Corollary 4, Theorem 5 and Theorem 3 for Corollary 5, and Corollary 5 and Theorem 2 for Corollary 6.

Corollary 4. *For $3 \leq h \leq n$, the fan F_n is $(h^2 + 2h.n - 2h + 3n, 2h - 1)$-$C_h$-antimagic.*

Corollary 5. *For $3 \leq h \leq n$, the fan F_n is super $(h^2 + h.n - 4h + 3n + 2, 2h - 1)$-$C_h$-antimagic.*

Corollary 6. *For $3 \leq h \leq n$, the fan F_n is $(h^2 + 3.h.n + 3h - 2n - 1, 2h - 1)$-$C_h$-antimagic.*

Theorem 6. *For $3 \leq h \leq n$, the fan F_n is $(3h^2 - 3h + 3, 6h - 3)$-C_h-antimagic.*

Proof. We label the vertices and edges of F_n as follows.

$$\begin{array}{lll} \xi_3(c) & = 1 & \\ \xi_3(v_i) & = 3i - 1 & for \quad 1 \leq i \leq n, \\ \xi_3(v_i v_{i+1}) & = 3i + 1 & for \quad 1 \leq i \leq n-1, \\ \xi_3(cv_i) & = 3i & for \quad 1 \leq i \leq n. \end{array}$$

It is easy to see that the labeling ξ_3 is a bijective function from $E(F_n) \cup V(F_n)$ to $\{1, 2, 3, \ldots, 3n\}$.

For $1 \leq j \leq n - h + 2$, the C_h-weight of a subgraph C_h^j of F_n under the labeling ξ_3 is

$$\begin{aligned} w(C_h^j) &= \xi_3(c) + \sum_{k=j}^{j+h-2} \xi_3(v_k) + \xi_3(cv_j) + \sum_{k=j}^{j+h-3} \xi_3(v_k v_{k+1}) + \xi_3(cv_{j+h-2}) \\ &= 3h^2 - 9h + 6 + (6h - 3)j. \end{aligned}$$

Since $w(C_h^{j+1}) - w(C_h^j) = 6h - 3$ and $w(C_h^1) = 3h^2 - 3h + 3$, ξ_3 is a $(3h^2 - 3h + 3,\ 6h - 3)$-C_h-antimagic total labeling of F_n. □

By using Theorem 6 and Theorem 2, we have the following corollary.

Corollary 7. *For $3 \leq h \leq n$, the fan F_n is $(3h^2 + 2h - 6h + 3n, 6h - 3)$-$C_h$-antimagic.*

In Theorem 4, 5, and 6 we provide some (a, d)-C_h-antimagic total labelings for fans where d is a function of h. In the following theorem we construct an (a, d)-C_h-antimagic total labeling whose a difference is constant.

Theorem 7. *For $3 \leq h \leq n$, the fan F_n is super $(3h.n + 3h - 4n, 3)$-C_h-antimagic.*

Proof. We label the vertices and edges of F_n as follows .

$$\begin{array}{lll} \xi_4(c) & = n+1 & \\ \xi_4(v_i) & = i & for \quad 1 \leq i \leq n, \\ \xi_4(v_i v_{i+1}) & = 3n + 1 - i & for \quad 1 \leq i \leq n-1, \\ \xi_4(cv_i) & = n + 1 + i & for \quad 1 \leq i \leq n. \end{array}$$

It is easy to see that the labeling ξ_4 is a bijective function from $E(F_n) \cup V(F_n)$ to $\{1, 2, 3, \ldots, 3n\}$ and $\xi_4(V(F_n)) = \{1, 2, \ldots, n+1\}$.

For $1 \leq j \leq n - h + 2$, the C_h-weight of a subgraph C_h^j of F_n under the labeling ξ_4 is

$$\begin{aligned} w(C_h^j) &= \xi_4(c) + \sum_{k=j}^{j+h-2} \xi_4(v_k) + \xi_4(cv_j) + \sum_{k=j}^{j+h-3} \xi_4(v_k v_{k+1}) + \xi_4(cv_{j+h-2}) \\ &= 3h.n + 3h - 4n - 3 + 3j \end{aligned}$$

Since $w(C_h^{j+1}) - w(C_h^j) = 3$ and $w(C_h^1) = 3h.n + 3h - 4n$, ξ_4 is a super $(3h.n + 3h - 4n,\ 3)$-C_h-antimagic total labeling of F_n. □

The three following corollaries are about the (super) dual of the label ξ_4 defined in the proof of Theorem 7. They can be proved by using Theorem 7 and Theorem 2 for Corollary 8, Theorem 7 and Theorem 3 for Corollary 9, and Corollary 9 and Theorem 2 for Corollary 10.

Corollary 8. *For $3 \le h \le n$, the fan F_n is $(2h^2 + h.n - 6h + 9n + 5, 3)$-$C_h$-antimagic.*

Corollary 9. *For $3 \le h \le n$, the fan F_n is super $(2h^2 - 2h + 9n + 5, 3)$-C_h-antimagic.*

Corollary 10. *For $3 \le h \le n$, the fan F_n is $(4h.n - h - 4n - 5, 3)$-C_h-antimagic.*

We conclude this section with an open problem.

Open Problem 1. Find (a,d)-C_h-antimagic total labeling for F_n with $d \notin \{3, 2h-5, 2h-1, 6h-3 \mid 3 \le h \le n+1\}$.

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