The metric dimensions of a complete n-partite graph and its cartesian product with a path

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Abstract. A set of vertices W *resolves* a graph G, if every vertex is uniquely determined by its coordinate of distances to the vertices in W. The minimum cardinality of a resolving set of G is called the *metric dimension* of G. In this paper, for any $n \geq 2$ and $k_1, k_2, \ldots, k_n \in \mathbb{Z}^+$, we determine the metric dimension of a *complete n-partite graph* $K_{k_1, k_2, \ldots, k_n}$. We also determine the metric dimension of a graph which is obtained from the *cartesian product* of $K_{k_1, k_2, \ldots, k_n}$ and a path.

Keywords: basis, cartesian product, complete n-partite graph, metric dimension, resolving set

1 Introduction

Throughout this paper, all graphs are finite, simple, and connected. Let G be a graph. We denote by $V(G)$ the vertex set of G, and $E(G)$ the edge set of G. The distance between two vertices $v, w \in V(G)$, denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest $u-v$ path in G. Let $W = \{w_1, \ldots, w_k\}$ be an ordered subset of $V(G)$. For any $v \in V(G)$, a *representation* of v with respect to W is defined as the k-tuple $r(v|W) = (d(v, w_1), \ldots, d(v, w_k))$. The set W is called a *resolving set* of G if every two distinct vertices x, $y \in V(G)$ satisfies $r(x|W) \neq r(y|W)$. A *metric basis* of G is a resolving set of G with the minimum cardinality, and the *metric dimension* of G refers to its cardinality and denoted by $\beta(G)$.

Resolving sets in general graphs were first studied by Harary and Melter [8], and independently by Slater [17, 18]. Slater referred to the metric dimension of a graph as its location number. This work was motivated by the

study of this invariant in the application to the placement of a minimum number of sonar/loran detecting devices in a network, so that the position of every vertex in the network can be uniquely described in terms of its distances to the devices in the set. Khuller *et al.* [11], then, studied the metric dimension motivated by the navigation of robots in a graph space. A resolving set for a graph corresponds to the presence of distinctively labelled "landmark" nodes in the graph. It is assumed that a robot navigating a graph can detect the distance to each of the landmarks, and hence uniquely determine its location in the graph.

In general, finding the metric dimension of a graph is a difficult problem. There is no efficient algorithm which can be used to determine the metric dimension of any graphs. Garey and Johnson [7], also Khuller *et al.* [11], showed that finding the metric dimension of any graph is an NP-problem. However, some results for some classes of graphs have been obtained. Chartrand *et al.* [4] and Khuler *et al.* [11] showed that a path is the only graph G with $\beta(G) = 1$. Furthermore, Chartrand *et al.* [4], characterized that K_n is the only graph G with $\beta(G) = n-1$. They also proved that $\beta(G) = n-2$ if and only if G is $K_{r,s}$ for $r, s \geq 1$, $K_r + \overline{K_s}$ for $r \geq 1, s \geq 2$, or $K_r + (K_1 \cup K_s)$ for $r, s \geq 1$. Therefore, for another G, the metric dimension $\beta(G)$ is at least 2 and at most $n-3$.

Characterizing all graphs with the metric dimension k, for some $k \in \{2, \ldots, n-3\}$, is still open. Since there is no polinomial time algorithm which can be used to determine the metric dimension of any graph, many researchers consider this problem for some particular classes of graphs. Up to now, we have known the metric dimension of cycles [5], trees [4, 8, 11], stars [4, 8, 11], fans [2], wheels [1, 2, 16], bipartites [4], unicylic graphs [14], grids [13], honeycomb networks [12], hexagonal networks [12], circulant networks [15], Cayley graphs [6], graphs with pendants [9], and amalgamation of cycles [10].

Many researcher also consider the metric dimension problem for a graph which obtained from a graph operation. Some results on sum product graph have been proved in [1–3, 16]. Caceres *et al.* [3], Khuler *et al.* [11], and Melter [13] showed the metric dimension of some graphs which obtained from the cartesian product between two or more graphs. Some graphs which constructed from the corona product between two graphs, have been studied by Iswadi *et al.* [9].

In the next section, we determine the metric dimension of a *complete n-partite graph* $K_{k_1, k_2, \ldots, k_n}$ for any $n \geq 2$ and $k_1, k_2, \ldots, k_n \in \mathbb{Z}^+$. Finally, in the last section, we study the metric dimension of a graph that is obtained from the *cartesian product* of $K_{k_1, k_2, \ldots, k_n}$ and a path. We use $[a, b]$ instead of $\{t \in \mathbb{N} | a \leq t \leq b\}$.

2 A complete n-partite graph

In this section, we consider a complete n-partite graph $K_{k_1,k_2,\ldots,k_n}$ for $n \geq 2$. Let $V_1, V_2, \ldots, V_n$ be the n independent sets of $K_{k_1,k_2,\ldots,k_n}$, with $|V_i| = k_i$ for every $i \in [1,n]$. Let $V_j = \left\{v_1^j, v_2^j, \ldots, v_{k_j}^j\right\}$ for every $j \in [1,n]$.

Lemma 1. *For any $n \geq 2$, let $G \cong K_{k_1,k_2,\ldots,k_n}$ with $1 \leq k_1 \leq k_2 \leq \ldots \leq k_n$. Let there exist m singleton partites. If*

$$A = \left\{ \begin{array}{ll} \left\{v_1^m, v_1^{m+1}, \ldots, v_1^n\right\} & for\ m > 0, \\ \left\{v_1^1, v_1^2, \ldots, v_1^n\right\} & for\ m = 0, \end{array} \right.$$

then $W = V(G) \backslash A$ is a resolving set.

Proof. For any two distinct vertices $x, y \in A$, we have $x \in V_p$ and $y \in V_q$, for some $p, q \in [1,n]$ with $p \neq q$ and $|V_q| \geq 2$. Let $z \in V_q \cap W$. Since $d(x, z) = 1$ and $d(y, z) = 2$, $r(x|W) \neq r(y|W)$. □

Lemma 2. *For any $n \geq 2$, let W be a resolving set of a complete n-partite graph G. Let $W' = V(G) \backslash W$. Then, W' contains at most one vertex from each partite whose cardinality at least 2, and contains at most one vertex from all singleton partites.*

Proof. Suppose there exist two distinct vertices $x, y \in V_i \cap W'$ for some $i \in [1,n]$. Since x and y are adjacent to every vertex in $V(G) \backslash V_i$, and are not adjacent to every vertex in V_i, it is clear that $r(x|W) = r(y|W)$, a contradiction.

Suppose there exist two vertices $s, t \in W'$ which satisfy $s \in V_p$, $t \in V_q$ for some distinct $p, q \in [1,n]$ and $|V_p| = 1 = |V_q|$. Since $sz, tz \in E(G)$ for every $z \in V(G) \backslash \{s, t\}$, clearly $r(s|W) = r(t|W)$, a contradiction. □

By using Lemma 1 and Lemma 2, we have the following theorem.

Theorem 1. *Let $n \geq 2$. Let $G \cong K_{k_1,k_2,\ldots,k_n}$ with $1 \leq k_1 \leq k_2 \leq \ldots \leq k_n$. If there are m singleton partites of G, then,*

$$\beta(G) = \left\{ \begin{array}{ll} |V(G)| - n + m - 1 & for\ m > 0 \\ |V(G)| - n & for\ m = 0. \end{array} \right.$$

□

Corollary 1. *For $n \geq 2$, let H_{2n} be a cocktail party graph. Then*

$$\beta(H_{2n}) = n.$$

Proof. Since H_{2n} is isomorphic with $K_{k_1,k_2,\ldots,k_n}$ where $k_i = 2$ for every $i \in [1,n]$, by Theorem 1, we obtain $\beta(H_{2n}) = n$. □

Remark 1. Theorem 1 strenghten the previous results, as follow.

- $\beta(K_{s,t}) = |V(K_{s,t})| - 2$, for $s + t \geq 3$ [4].
- $\beta(K_r) = r - 1$, for $r \geq 2$ [11].

3 The cartesian product of a complete n-partite graph and a path

The *Cartesian product* of graphs G and H, denoted by $G \times H$, is the graph with the vertex set $V(G) \times V(H) = \{(a,p) \,|\, a \in V(G), p \in V(H)\}$ where (a,p) is adjacent to (b,q) whenever either $a = b$ and $p, q \in E(H)$ or $p = q$ and $a, b \in E(G)$. We define a *column* of $G \times H$ corresponding to $a \in V(G)$ by a set of vertices $Col(a) = \{(a,p) \,|\, p \in V(H)\}$ and a *row* of $G \times H$ corresponding to $p \in V(H)$ by a set of vertices $Row(p) = \{(a,p) \,|\, a \in V(G)\}$. Observe that each row induces a copy of G and each column induces a copy of H.
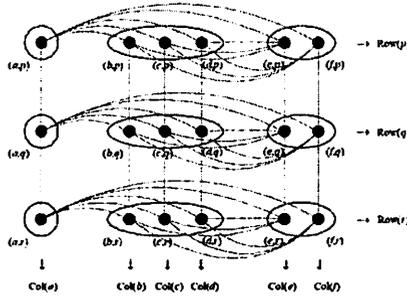

Fig. 1. $K_{1,3,2} \times P_3$

Let $r \geq 2$ and P_r be a path on r vertices. It is well known that the metric dimension of $G \times P_r$ for any graph G is satisfies the theorem below.

Theorem 2. *(Caceres et al [3])*
Let G be a graph and $r \geq 2$, then $\beta(G) \leq \beta(G \times P_r) \leq \beta(G) + 1$.

In this section, we determine the metric dimension of the cartesian product of $G \cong K_{k_1,k_2,\ldots,k_n}$ and P_r.

Let $V(P_r) = \{p_1, p_2, \ldots, p_r\}$. For $i \in [1,n]$ and $k_0 = 0$, let $t_i = \sum_{j=0}^{i-1} k_j$ and $V_i = \bigcup_{j=1}^{k_i} \{u_{t_i+j}\}$. First, we show a necessary condition for any resolving set of $G \times P_r$ as follows.

Lemma 3. *Let $n, r \geq 2$. Let $G \cong K_{k_1,k_2,\ldots,k_n}$ and W be a resolving set of $G \times P_r$. Let $u \in V_i$ and $v \in V_j$ for any $i, j \in [1,n]$. If either $i \neq j$ and $|V_i| = |V_j| = 1$ or $i = j$ and $u \neq v$ and $|V_i| \geq 2$, then $W \cap \{Col(u) \cup Col(v)\} \neq \varnothing$.*

Proof. Suppose that $W \cap \{Col(u) \cup Col(v)\} = \varnothing$. We have $d((u,p_1),w) = d((v,p_1),w)$ for every $w \in W$, a contradiction. □

Now, let S be any set of vertices of $G \times P_r$ which satisfies $S \cap \{Col(u) \cup Col(v)\} \neq \varnothing$ for any $u \in V_i$ and $v \in V_j$ for some $i,j \in [1,n]$ where either $i \neq j$ and $|V_i| = |V_j| = 1$ or $i = j$ and $u \neq v$ and $|V_i| \geq 2$. In Lemma 4 below, we show that S resolves every two different vertices in $Row(p_i)$ for some $i \in [1,r]$. We also give several conditions for S in Lemma 5 such that S will resolve two different vertices in the different row.

Lemma 4. *Let $n,r \geq 2$, $i \in [1,r]$, and $a,b \in V(G)$ with $a \neq b$. Then $r((a,p_i)|S) \neq r((b,p_i)|S)$.*

Proof. If $S \cap \{Col(u) \cup Col(v)\} \neq \varnothing$, then there exist $j \in [1,r]$ such that $(a,p_j) \in S$ or $(b,p_j) \in S$. We obtain:

$$d((a,p_j),(b,p_i)) = d((a,p_j),(a,p_i)) + d((a,p_i),(b,p_i)) \text{ if } (a,p_j) \in S; \text{ or}$$

$$d((b,p_j),(a,p_i)) = d((b,p_j),(b,p_i)) + d((b,p_i),(a,p_i)) \text{ if } (b,p_j) \in S.$$

Therefore, $r((a,p_i)|W) \neq r((b,p_i)|W)$.

If $S \cap \{Col(u) \cup Col(v)\} = \varnothing$, then a or b is not in a singleton partite. Without loss of generality, we may assume that b is in the same partite with $c \in V(G)$ where $(c,p_k) \in S$ for some $k \in [1,r]$. We obtain $d((c,p_k),(b,p_j)) = d((c,p_k),(a,p_i)) + d((a,p_i),(b,p_j))$. Therefore, $((a,p_i)|W) \neq r((b,p_i)|W)$. □

Lemma 5. *Let $x,y \in [1,n]$, $a \in V_x$, $b \in V_y$, and $1 \leq i < j \leq r$. Then $r((a,p_i)|S) \neq r((b,p_j)|S)$, if:*

(i) $(a,p_h) \in S$ with $1 \leq h \leq i$; or
(ii) $(b,p_k) \in S$ with $j \leq k \leq r$; or
(iii) $x = y$ and there exists $c \in V_x \setminus \{a,b\}$ such that $(c,p_k) \in S$ with $j \leq k \leq r$; or
(iv) $x \neq y$ and there exist $c \in V_x \setminus \{a\}$ such that $(c,p_k) \in S$ with $j \leq k \leq r$; or
(v) $x \neq y$ and there exist $c \in V_y \setminus \{b\}$ such that $(c,p_h) \in S$ with $1 \leq h \leq i$; or
(vi) there exist $c \in V_z$ with $x \neq z \neq y$ such that $(c,p_h) \in S$ with $1 \leq h \leq i$; or
(vii) there exist $c \in V_z$ with $x \neq z \neq y$ such that $(c,p_k) \in S$ with $j \leq k \leq r$.

Proof. From (i)-(vii), we have, respectively:

(i) $d((a,p_h),(b,p_j)) = d((a,p_h),(a,p_i)) + d((a,p_i),(b,p_j))$.
(ii) $d((b,p_k),(a,p_i)) = d((b,p_k),(b,p_j)) + d((b,p_j),(a,p_i))$.

(iii) $d\left(\left(c, p_k\right),\left(a, p_i\right)\right) = d\left(\left(c, p_k\right),\left(b, p_j\right)\right) + j - i.$

(iv) $d\left(\left(c, p_k\right),\left(a, p_i\right)\right) = d\left(\left(c, p_k\right),\left(b, p_j\right)\right) + j - i + 1.$

(v) $d\left(\left(c, p_h\right),\left(b, p_j\right)\right) = d\left(\left(c, p_h\right),\left(a, p_i\right)\right) + j - i + 1.$

(vi) $d\left(\left(c, p_h\right),\left(b, p_j\right)\right) = d\left(\left(c, p_h\right),\left(a, p_i\right)\right) + j - i.$

(vii) $d\left(\left(c, p_k\right),\left(a, p_i\right)\right) = d\left(\left(c, p_k\right),\left(b, p_j\right)\right) + j - i.$

All the above cases imply

$$r\left(\left(a, p_i\right) | S\right) \neq r\left(\left(b, p_j\right) | S\right).$$

$\square$

Theorem 3. *Let* $n, r \geq 2$ *and* $G \cong K_{k_1, k_2, \ldots, k_n}$. *Then*

$$\beta\left(G \times P_r\right) = \begin{cases} \beta\left(G\right) + 1 & \text{for } G \text{ is } K_{s,1} \text{ or } K_{s,2} \text{ or } K_{s,2,1} \text{ with } s \in [1,4], \\ \beta\left(G\right) & \text{otherwise.} \end{cases}$$

Proof. We distinguish two cases of G.

Case (1). G is $K_{s,1}$ or $K_{s,2}$ or $K_{s,2,1}$ with $s \in [1,4]$

For $G \cong K_{1,1}$ or $G \cong K_{1,2}$, $G \times P_r$ is isomorphic to a grid. Melter [13] showed that the metric dimension of a grid is two. For $G \cong K_{2,2}$, G is isomorphic to a cycle on 4 vertices. Caceres *et al.* [3] showed that the metric dimension of a cycle on even vertices and a path on greater than one vertex is three.

Now, assume $G \not\cong K_{i,j}$ with $i, j \in [1,2]$. Therefore, there are 8 remaining possibilities of G to be considered. By Theorem 2, we only need to show that $\beta\left(G \times P_r\right) \geq \beta\left(G\right) + 1$.

Suppose that $\beta\left(G \times P_r\right) = \beta\left(G\right)$ and W is a basis of $G \times P_r$. We show there always exist two distinct vertices having the same representation with respect to W. For some $t \in [1, r]$, we define

$$t \pm 1 = \begin{cases} t + 1 & \text{, if } t = 1 \\ t - 1 & \text{, otherwise.} \end{cases}$$

We consider 8 possibilities of G.

1. $G \cong K_{1,3}$

   By Theorem 1 and Lemma 3, let $W = \{(u_a, p_f), (u_b, p_g)\}$ for some $a, b \in [2,4]$ with $a \neq b$ and $f, g \in [1, r]$. Let $x = \min\{f, g\}$, $y = \max\{f, g\}$, and $c \in [2,4] \setminus \{a, b\}$. Then we obtain the representation of two vertices below.

   (a) $r\left(\left(u_c, p_x\right) | W\right) = r\left(\left(u_1, p_{x \pm 1}\right) | W\right)$, if $x = y$.

   (b) $r\left(\left(u_a, p_{x+1}\right) | W\right) = r\left(\left(u_1, p_x\right) | W\right)$, if $x \neq y$.

2. $G \cong K_{1,4}$

By Theorem 1 and Lemma 3, let $W = \{(u_a, p_f), (u_b, p_g), (u_c, p_h)\}$ for some $a, b, c \in [2, 5]$ with $a \neq b$, $b \neq c$, $a \neq c$ and $f, g, h \in [1, r]$. Let $x = \min\{f, g, h\}$, $z = \max\{f, g, h\}$, and $d \in [2, 5] \setminus \{a, b, c\}$. Then we obtain the representation of two vertices below.

(a) $r((u_d, p_x)|W) = r((u_1, p_{x\pm1})|W)$, if $x = z$.

(b) $r((u_a, p_{x+1})|W) = r((u_1, p_x)|W)$, if $x < z$.

3. $G \cong K_{2,3}$

By Theorem 1 and Lemma 3, let $W = \{(u_p, p_k), (u_a, p_f), (u_b, p_g)\}$ for some $p \in [1, 2]$; $a, b \in [3, 5]$ with $a \neq b$; and $k, f, g \in [1, r]$. Let $x = \min\{f, g\}$, $y = \max\{f, g\}$, $c \in [3, 5] \setminus \{a, b\}$, and $q \in [1, 2] \setminus \{p\}$. Then we obtain the representation of two vertices below.

(a) $r((u_c, p_{k+1})|W) = r((u_q, p_k)|W)$, if $x, y > k$.

(b) $r((u_a, p_{k+1})|W) = r((u_q, p_k)|W)$, if $y > k$ and $x \leq k$.

(c) $r((u_c, p_k)|W) = r((u_q, p_{k\pm1})|W)$, if $x = y = k$.

(d) $r((u_a, p_k)|W) = r((u_p, p_{k-1})|W)$, if $y = k$ and $x < k$.

(e) $r((u_c, p_{k-1})|W) = r((u_q, p_k)|W)$, if $x, y < k$.

4. $G \cong K_{2,4}$

By Theorem 1 and Lemma 3, let $W = \{(u_p, p_k), (u_a, p_f), (u_b, p_g), (u_c, p_h)\}$ for some $p \in [1, 2]$; $a, b, c \in [3, 6]$ with $a \neq b$, $b \neq c$, $a \neq c$; and $k, f, g, h \in [1, r]$. Let $x = \min\{f, g, h\}$, $z = \max\{f, g, h\}$, $y \in \{f, g, h\} \setminus \{x, z\}$, $q \in [1, 2] \setminus \{p\}$, and $d \in [3, 6] \setminus \{a, b, c\}$. Then we obtain the representation of two vertices below.

(a) $r((u_d, p_{k+1})|W) = r((u_q, p_k)|W)$, if $x, y, z > k$.

(b) $r((u_a, p_{k+1})|W) = r((u_q, p_k)|W)$, if $y, z > k$ and $x \leq k$.

(c) $r((u_c, p_k)|W) = r((u_p, p_{k+1})|W)$, if $z > k$ and $x, y \leq k$.

(d) $r((u_d, p_k)|W) = r((u_p, p_{k\pm1})|W)$, if $x = y = z = k$.

(e) $r((u_a, p_k)|W) = r((u_p, p_{k-1})|W)$, if $y = z = k$ and $x < k$.

(f) $r((u_c, p_{k-1})|W) = r((u_q, p_k)|W)$, if $z = k$ and $x, y < k$.

(g) $r((u_d, p_{k-1})|W) = r((u_q, p_k)|W)$, if $x, y, z < k$.

5. $G \cong K_{1,1,2}$

By Theorem 1 and Lemma 3, let $W = \{(u_a, p_f), (u_b, p_g)\}$ for some $a, b \in [2, 3]$ with $a \neq b$ and $f, g \in [1, r]$. Let $x = \min\{f, g\}$ and $y = \max\{f, g\}$. Then we obtain the representation of two vertices below.

(a) $r((u_4, p_x)|W) = r((u_2, p_{x\pm1})|W)$, if $x = y$.

(b) $r((u_a, p_{x+1})|W) = r((u_b, p_x)|W)$, if $x \neq y$.

6. $G \cong K_{1,2,2}$

By Theorem 1 and Lemma 3, let $W = \{(u_a, p_f), (u_b, p_g)\}$ for some $a, b \in \{2, 4\}$ with $a \neq b$ and $f, g \in [1, r]$. Let $x = \min\{f, g\}$ and $y = \max\{f, g\}$. Then we obtain the representation of two vertices below.

(a) $r\left((u_5, p_x) | W\right) = r\left((u_2, p_{x\pm 1}) | W\right)$, if $x = y$.

(b) $r\left((u_a, p_{x+1}) | W\right) = r\left((u_b, p_x) | W\right)$, if $x \neq y$.

7. $G \cong K_{1,2,3}$

By Theorem 1 and Lemma 3, let $W = \{(u_p, p_k), (u_a, p_f), (u_b, p_g)\}$ for some $p \in [2, 3]$; $a, b \in [4, 6]$ with $a \neq b$; and $k, f, g \in [1, r]$. Let $x = \min \{f, g\}$, $y = \max \{f, g\}$, $c \in [4, 6] \setminus \{a, b\}$, and $q \in [2, 3] \setminus \{p\}$. Then we obtain the representation of two vertices as same as subcase (3).

8. $G \cong K_{1,2,4}$

By Theorem 1 and Lemma 3, let $W = \{(u_p, p_k), (u_a, p_f), (u_b, p_g), (u_c, p_h)\}$ for some $p \in [2, 3]$; $a, b, c \in [4, 7]$ with $a \neq b$, $b \neq c$, $a \neq c$; and $k, f, g, h \in [1, r]$. Let $x = \min \{f, g, h\}$, $z = \max \{f, g, h\}$, $y \in \{f, g, h\} \setminus \{x, z\}$, $q \in [2, 3] \setminus \{p\}$, and $d \in [4, 7] \setminus \{a, b, c\}$. Then we obtain the representation of two vertices as same as subcase (4).

All the above subcases lead to a contradiction. Therefore, $\beta\left(G \times P_r\right) \geq \beta\left(G\right) + 1$.

Case (2). G is neither $K_{s,1}$ nor $K_{s,2}$ nor $K_{s,2,1}$ with $s \in [1, 4]$

By Theorem 2, we only need to show that $\beta\left(G \times P_r\right) \leq \beta\left(G\right)$. First, we define a vertex sets for each 5 possibilities of G as follows.

1. For $G \cong K_{1,s}$, with $s \in [5, \infty)$, we define

$$W = \{(u_a, p_1) | 2 \leq a \leq s, a \text{ is even}\} \cup \{(u_b, p_r) | 2 \leq b \leq s, b \text{ is odd}\}.$$

2. For $G \cong K_{2,s}$, with $s \in [5, \infty)$, we define

$$W = \{(u_1, p_1)\} \cup \{(u_a, p_1) | 3 \leq a \leq s+1, a \text{ is odd}\} \cup$$
$$\{(u_b, p_r) | 3 \leq b \leq s+1, b \text{ is even}\}.$$

3. For $G \cong K_{1,1,s}$, with $s \in [3, \infty)$, we define

$$W = \{(u_1, p_1)\} \cup \{(u_i, p_r) | 3 \leq i \leq s+1\}.$$

4. For $G \cong K_{1,2,s}$, with $s \in [5, \infty)$, we define

$$W = \{(u_2, p_1)\} \cup \{(u_a, p_1) | 4 \leq a \leq s+2, a \text{ is even}\} \cup$$
$$\{(u_b, p_r) | 4 \leq b \leq s+2, b \text{ is odd}\}.$$

5. If G is neither $K_{s,1}$ nor $K_{s,2}$ nor $K_{t,1,1}$ nor $K_{s,2,1}$ with $s \in [5, \infty)$ and $t \in [3, \infty)$, we define $W = \{(u, p_1) | u \in S\}$, where S is a basis of G.

For all the above subcases, we have $|W| = \beta\left(G\right)$. Next, we show that W is a resolving set of $G \times P_r$. Let $u, v \in V\left(G \times P_r\right) \setminus W$ with $u \neq v$. By Lemma 4, we have $r\left(u | W\right) \neq r\left(v | W\right)$ for $u, v \in Row\left(p_i\right)$ for some $i \in [1, r]$. Now, we assume that $u \in Row\left(p_i\right)$ and $v \in Row\left(p_j\right)$ for some $i, j \in [1, r]$ and $i \neq j$. Without loss of generality, let $u = (x, p_i)$ and $v = (y, p_j)$ with $i < j$, $x \in V_k$, and $y \in V_l$ for $k, l \in [1, n]$. We consider 3 possibilities of the position of u as follows.

1. $(x, p_1) \in W$

   For all 5 possibilities of G, by Lemma 5 (i), we obtain $r(u|W) \neq r(v|W)$.

2. $(x, p_r) \in W$

   This condition is not satisfied for the last possibilities of G. For the first 4 possibilities of G, there exists $(z, p_r) \in W$ with $z \neq x$. Then we have 3 possibilities of v.

   (a) if $y = z$, then by Lemma 5 (ii), we obtain $r(u|W) \neq r(v|W)$.

   (b) if $y \in \{u_1, \ldots, u_{k_1}\}$ for G is 2-partite, or $y \in \{u_1, \ldots, u_{k_1+k_2}\}$ for G is 3-partite, by Lemma 5 (iv), we obtain $r(u|W) \neq r(v|W)$.

   (c) if $y \in \{u_{k_1+1}, \ldots, u_{k_1+k_2}\} \setminus \{z\}$ for G is 2-partite, or $y \in \{u_{k_1+k_2+1}, \ldots, u_{k_1+k_2+k_3}\} \setminus \{z\}$ for G is 3-partite, then by Lemma 5 (iii), we obtain $r(u|W) \neq r(v|W)$.

3. If $(x, p_j) \notin W$ for every $j \in [1, r]$

   For $G \cong K_{1,s}$ or $G \cong K_{2,s}$ for $s \in [5, \infty)$, we have 2 possibilities of u.

   (a) if $x = u_{k_1}$, then there exists $(z, p_1) \in W$ with $z \neq y$. By Lemma 5 (v), we obtain $r(u|W) \neq r(v|W)$.

   (b) if $x = u_{k_1+k_2}$, then there exists $(z, p_r) \in W$. We have 3 possibilities of v.

      i. if $y \in \{u_1, \ldots, u_{k_1}\}$, then by Lemma 5 (iv), we obtain $r(u|W) \neq r(v|W)$.

      ii. if $y = z$, then by Lemma 5 (ii), we obtain $r(u|W) \neq r(v|W)$.

      iii. if $y \in \{u_{k_1+1}, \ldots, u_{k_1+k_2}\} \setminus \{z\}$, then by Lemma 5 (iii), we obtain $r(u|W) \neq r(v|W)$.

   For $G \cong K_{1,1,s}$ for $s \in [3, \infty)$, we have 2 possibilities of u.

   (a) if $x = u_2$, then we consider 2 possibilities of v. If $y = u_2$, then there exists $(z, p_r) \in W$ with $z \in \{u_3, \ldots, u_{s+2}\}$, so that by Lemma 5 (vii), we obtain $r(u|W) \neq r(v|W)$. Otherwise, there exists $(z, p_r) \in W$ with $z = u_2$, so that by Lemma 5 (vi), we obtain $r(u|W) \neq r(v|W)$.

   (b) if $x = u_{s+2}$, then there exists $(z, p_r) \in W$. We have 3 possibilities of v.

      i. if $y \in \{u_1, u_2\}$, then by Lemma 5 (iv), we obtain $r(u|W) \neq r(v|W)$.

      ii. if $y = z$, then by Lemma 5 (ii), we obtain $r(u|W) \neq r(v|W)$.

      iii. if $y \in \{u_3, \ldots, u_{s+2}\} \setminus \{z\}$, then by Lemma 5 (iii), we obtain $r(u|W) \neq r(v|W)$.

   For $G \cong K_{1,2,s}$ for $s \in [3, \infty)$, we have 3 possibilities of u.

   (a) if $x = u_1$, then there exists $(z, p_1) \in W$ with $z \in V_k$. By Lemma 5 (vi), we obtain $r(u|W) \neq r(v|W)$.

(b) if $x = u_3$, then there exists $(z, p_1) \in W$ with $z \neq y$. We have 2 possibilities of v. If $z \in V_l$, then by Lemma 5 (v), we obtain $r(u|W) \neq r(v|W)$; otherwise, by Lemma 5 (vi), we obtain $r(u|W) \neq r(v|W)$.

(c) if $x = u_{s+3}$, then there exists $(z, p_r) \in W$. We have 3 possibilities of v.

    i. if $y \in \{u_1, u_2, u_3\}$, then by Lemma 5 (vii), we obtain $r(u|W) \neq r(v|W)$.

    ii. if $y = z$, then by Lemma 5 (ii), we obtain $r(u|W) \neq r(v|W)$.

    iii. if $y \in \{u_4, \ldots, u_{s+3}\} \setminus \{z\}$, then by Lemma 5 (iii), we obtain $r(u|W) \neq r(v|W)$.

If G is neither $K_{s,1}$ nor $K_{s,2}$ nor $K_{t,1,1}$ nor $K_{s,2,1}$ with $s \in [5, \infty)$ and $t \in [3, \infty)$, we consider 2 possibilities of n.

(a) if $n \geq 3$, then there exists $(z, p_1) \in W$ with $z \in V_h$ and $k \neq h \neq l$. By Lemma 5 (vi), we obtain $r(u|W) \neq r(v|W)$.

(b) if $n \leq 2$, then have 2 possibilities of u and v.

    i. if $k = l$, then there exists $(z, p_1) \in W$ with $z \in V_h$ and $h \neq k$. By Lemma 5 (vi), we obtain $r(u|W) \neq r(v|W)$.

    ii. if $k \neq l$, then there exists $(z, p_1) \in W$ with $z \in V_l$ and $z \neq y$. By Lemma 5 (v), we obtain $r(u|W) \neq r(v|W)$.

$\square$

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