On distance Magic Labeling of Graphs

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Abstract

Distance magic labeling of a graph of order n is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer constant k such that for any vertex x, $\Sigma_{y \in N(x)} f(y) = k$, where $N(x)$ is the set of vertices adjacent to x. In this paper, we prove new results about the distance magicness of graphs that have minimum degree one or two. Moreover, we construct distance magic labeling for an infinite family of non-regular graphs.

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1 Introduction

The notion of magic labeling was introduced by Sedláček in [4]. Miller *et al.* [3] introduced the concept of a distance magic labeling, called 1-vertex magic vertex labeling. Let $G(V,E)$ be a graph of order n. A *1-vertex magic vertex labeling* is a bijection $f : V \to \{1, 2, \ldots, n\}$ with the property that there is a positive integer constant k such that for any vertex x, $\Sigma_{y \in N(x)} f(y) = k$, where $N(x)$ is the set of vertices adjacent to x. For simplicity, we call 1-vertex magic vertex labeling a distance magic labeling in the rest of the paper. Miller *et al.* proved that $H_{p,q}$, $p > 1$ and $q > 1$, has distance magic labeling if and only if either p is even or both p and q are odd, where $H_{p,q}$ denotes a complete symmetric multipartite graph with q parts, each containing p vertices. They also proved that if $G = H \times \overline{K_{2k}}$, where H is an r-regular graph, then G has a distance magic labeling.

Finding an r-regular distance magic labeling turns out equivalent to finding equalized incomplete tournament $\mathrm{EIT}(n,r)$. See [1, 2] for more information. A fair incomplete tournament of n teams with k rounds is a tournament in which every team plays exactly k other teams and the total strength of the opponents that each team misses during the tournament is the same for all teams. The result on $H_{p,q}$ has been generalized by Frončck [1] and Frončck, Kovář and Kovářová [2] in terms of EIT as follows. Let G be an r-regular graph with order n.

Theorem 1. *[2] For n even an r-regular distance magic graph exists if and only if $2 \leq r \leq n-2, r \equiv 0 \pmod 2$ and either $n \equiv 0 \pmod 4$ or $n \equiv r + 2 \equiv 2 \pmod 4$.*

Theorem 2. *[1] Let $n, q > 1$ be odd numbers, $s \geq 1$; $r = 2^s q$ and $n = tq$ for some odd t, $t \geq 2^s + 1$. Then an r-regular distance magic graph of order n exists.*

The following lemma gives a necessary condition for the existence of distance magic labeling.

Lemma 1. *[3] A necessary condition for the existence of a distance magic labeling f of a graph $G = (V,E)$ is $kv = \Sigma_{x \in V} d(x) f(x)$, where v is the order of G, $d(x)$ is the degree of vertex x and k is the magic constant.*

Lemma 2. *[3] If G contains two vertices u and v such that $|N(u) \bigcap N(v)| = d(v) - 1 = d(u) - 1$ then G has no distance magic labeling.*

Using Lemma 1, we can conclude that several families of graphs, such as fans and friendship graphs, cannot have distance magiclabeling. Figure

1 shows an example of a non distance magic graph, since there are two vertices of degree two that have one common neighbour.

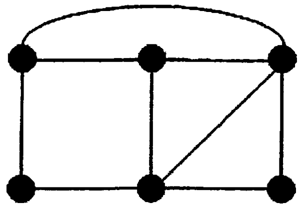

Figure 1: Example of a non distance magic graph.

Lemma 3. *[3] Let G be a graph on n vertices, with maximum degree Δ and minimum degree δ. If $\Delta(\Delta+1) > \delta(2n-\delta+1)$ then G does not have a distance magic labeling.*

Theorem 3. *[3] There exists a distance magic labeling of a tree if and only if the tree is either P_1 or P_3.*

Theorem 4. *[3] Every r-regular graph with odd r does not have a distance magic labeling.*

Using Theorem 4, we can conclude that all cubic graphs, such as prisms and the generalized Petersen graph, cannot have a distance magic labeling.

2 Properties of distance magic graph

Miller *et al.* [3] proved that all trees, except P_1 and P_3, cannot have distance magic labeling. However, here we give a stronger result regarding graphs containing a vertex of degree one.

Theorem 5. *If G is distance magic and $\delta(G) = 1$ then either G is isomorphic to P_3, or G contains exactly one component isomorphic to P_3 and all other components are isomorphic to $K_{2,2}$.*

Proof. In this proof we identify vertices with their labels, that is, $f(i) = i$.

Obviously, every vertex of degree one must be adjacent to the vertex n, hence there can be only one component containing vertices of degree one and all such vertices are adjacent to n. If x, y are other vertices adjacent to n and xy is an edge in G, then $w(x) \geq n + y > n$, a contradiction.

Hence, the only component containing vertices of degree one is a star. If G contains no other components, it is obvious that the star is P_3. Otherwise, we denote the components of G as $B_0, B_1, \ldots, B_p$, where the star is B_0.

Let b_1 be the smallest vertex which does not belong to B_0. We will show that $n - b_1$ must belong to B_1 as well, and b_1 and $n - b_1$ have a common neighbour. Let z be a neighbour of $n - b_1$ and y be another neighbour of z. Clearly, y cannot be greater than b_1, since then $w(z) \geq n - b_1 + y > n$. But y cannot be less than b_1 either, because all vertices smaller than b_1 are in B_0. Therefore, z has exactly two neighbours, b_1 and $n - b_1$, and the same is true for every neighbour of $n - b_1$. Let the neighbours be $x_1, x_2, \ldots, x_{n_1}$. Then $x_1 + x_2 + \cdots + x_{n_1} = n$, and indeed b_1 cannot have neighbours other than $x_1, x_2, \ldots, x_{n_1}$, because then $w(b_1)$ would be greater than n. Hence, B_1 is a complete bipartite graph K_{2,n_1}. Thus we have shown that the only component containing vertices of degree one is B_0. Therefore, $n_1 \geq 2$.

Now let b_2 be the smallest vertex not in $B_0 \cup B_1$. We can repeat the same argument as above to show that the component B_2 containing b_2 is isomorphic to K_{2,n_2} with $n_2 \geq 2$.

Repeating the argument finitely many times, we prove that $B_0 \cong K_{1,n_0}$ and $B_i \cong K_{2,n_i}$ for $i = 1, 2, \ldots, r$. If $n_i = 2$, for $i = 0, 1, \ldots, r$, we are done.

If not, we will utilize the same argument again to show that every n_i is equal to 2. Denote by $X_0, X_1, X_2, \ldots, X_s$ all partite sets of order one or two and by $Y_1, Y_2, \ldots, Y_t$ all partite sets of order greater than two. Let $y_1 \in Y_1$ be the smallest element of $Y = Y_1 \cup Y_2 \cup \cdots \cup Y_t$. Then $n - y_1$ cannot belong to $X = X_0 \cup X_1 \cup \cup \cdots \cup X_s$, because every X_i consists of v_i and $n - v_i$. If $(n - y_1) \in Y_1$, then $Y_1 = \{y_1, n - y_1\}$, which is impossible. So we must suppose that $(n - y_1) \in Y_i$ for some $i > 1$. Y_i must contain at least one vertex smaller than y_1, otherwise the neighbours of $n - y_1$ have weight greater than n, which is impossible. But y_1 was the smallest element of Y and we have a contradiction. Therefore, there is no partite set of order greater than two and the proof is complete. □

The following theorem gives a new result concerning graphs with adjacent vertices of degree 2.

Theorem 6. *If G is distance magic, $d(x) = d(y) = d(z) = 2$ and y is adjacent to x and z then either G is isomorphic to $K_{2,2}$, or G contains a component isomorphic to $K_{2,2}$.*

Proof. Let x_1 be adjacent to x. Let x_1 be adjacent to x and x_2 adjacent to y, then

$$w(x) = f(x_1) + f(y); \; w(z) = f(y) + f(x_2)$$

It is obvious that $x_1 = x_2$, say $u = x_1 = x_2$. Since $w(y) = f(x) + f(z)$ and y is adjacent to both x and z, then $\mathrm{d}(u) = 2$. Thus G has to be $K_{2,2}$ or has $K_{2,2}$ as a component.□

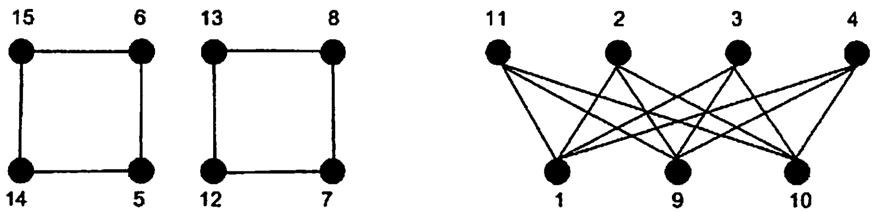

Figure 2: Distance magic labeling for disconnected graph with $K_{2,2}$ as a component.

The following result is a generalization of Theorem 2, part (a) by Miller, Rodger and Simanjuntak [3].

Theorem 7. *If G is a complete multipartite graph and G has a distance magic labeling then $G = K_{s_1,s_2,\ldots,s_r}$ with $1 \leq s_1 \leq s_2 \leq \ldots \leq s_r$ and $s_i \geq 2$, $i = 2, \ldots, r$.*

Proof. Let G be a distance magic multipartite graph of order n. Let k be the magic number of distance magic labeling f. Denote the partite sets $S_1, \ldots, S_r$. Suppose that there is more than one set that is a singleton, say $S_1 = \{u\}$ and $S_2 = \{v\}$. Thus $|N(u) \cap N(v)| = n - 2 = d(u) - 1 = d(v) - 1$ and by Lemma 2, G is not distance magic. Thus G can contain at most one singleton set. □.

3 Distance magic labeling of non-regular graphs

As mentioned in Section 1 (see Theorems 1 and 2), r-regular distance magic graphs on n vertices exist whenever r is even and either $n \equiv 0 \pmod 4$, or $n \equiv 2 \pmod 4$ and $r \equiv 0 \pmod 4$, or $n = tq$ is odd, $r = 2^s q$ where $s \geq 1$ and $t \geq 2^s + 1$.

The constructions were based on Kotzig arrays and lifted Kotzig arrays, which we define for reader's convenience below.

Definition A *Kotzig array* $KA(a,b)$ is an $a \times b$ matrix, where every row contains each of the integers $1, 2 \ldots, b$ exactly once and each column has the sum $a(b+1)/2$.

Definition A *lifted Kotzig array* $LKA(a,b;l)$ is an $a \times b$ matrix with entries $f_{ij} = e_{ij} + (i-1)b + l$, where e_{ij} is the entry in the i-th row and j-th column in the corresponding Kotzig array $KA(a,b)$. The sum of each column in $LKA(a,b;l)$ is $\sigma(a,b;l) = \frac{1}{2}a(ab+2l+1)$.

Neither $LKA(a,b;l)$ nor $KA(a,b)$ exist for every choice of integers a, b. Wallis proved the following.

Theorem 8. [5] *A Kotzig array $KA(a,b)$ exists if and only if either $a = b = 1$ or $a > 1$ and $a(b-1)$ is even.*

The main idea of the constructions was the following. Pick any p-regular graph H on b vertices $x_1, x_2, \ldots, x_b$ and construct $LKA(a,b;0)$. Then construct the graph $G = H[\overline{K}_a]$, which arises from H by replacing each vertex x_i by a set X_i of a independent vertices, and each edge $x_i x_j$ by the edges of the complete bipartite graph $K_{a,a}$ with bipartition X_i, X_j. Then label each set X_j of vertices of G by the entries $f_{kj}, 1 \leq k \leq a$ of $LKA(a,b;0)$. Now, from the existence theorems, $r = ap$ and $n = ab$.

We will now use the same idea to construct non-regular distance magic graphs. However, we will use two different lifted Kotzig arrays to "blow up" the vertices of the graph H into independent sets of different sizes, a and c.

Consider $LKA(a,b;0)$, and a p-regular graph H which has n' vertices with $n' = b + d$, where $b = |V_1(H)|$ and $d = |V_2(H)|$, for $V_1, V_2 \subset V(H)$, such that $V_1 = \{x_1, x_2, \ldots, x_b\}$ and $V_2\{y_1, y_2, \ldots, y_d\}$ form a partition of $V(H)$. Denote by $G = H[b \times a, d \times c]$ a graph arising from H by expanding each vertex $x_i \in V_1(H)$ into a set X_i of a independent vertices $\{x_{i1}, x_{i2}, \ldots, x_{ia}\}$ and, similarly, expanding each $y_j \in V_2(H)$ into a set of c independent vertices $\{y_{j1}, \ldots, y_{jc}\}$. Further, every edge $x_i x_j$ between two vertices of $V_1(H)$ will be replaced by a^2 edges of $K(a,a)$ while every edge $y_i y_j$ between two vertices of $V_2(H)$ will be replaced by c^2 edges of $K_{c,c}$. Also, any edge $x_i y_j$ between a vertex in $V_1(H)$ and a vertex in $V_2(H)$ will be replaced by ca edges of $K_{a,c}$. Denote $V_1(G) = X_1 \cup X_2 \cup \cdots \cup X_b$ and $V_2(G) = Y_1 \cup Y_2 \cup \cdots \cup Y_d$.

Through this process, we have taken the regular graph H and created a non-regular graph $G = H[b \times a, d \times c]$. Now we need to choose such value of d that the graph G would allow a distance magic labeling. As noted before, in $LKA(a, b; 0)$ the sum of each column will be $a(ab+1)/2$ and each integer from the set $1, ..., ab$ will appear in the array exactly once. One can observe that a distance magic labeling will be achieved when we label the vertices in $V_2(G)$ by integers $ab+1, ab+2, \dots, ab+cd$ and the sum of labels of all vertices in each set Y_j will be also equal to $a(ab+1)/2$. This is actually equivalent to the existence of $LKA(c, d; ab)$.

Lemma 4. *Let a, b, c, d be positive integers such that $a > c$, both $LKA(a, b; 0)$ and $LKA(c, d; ab)$ exist. Then $\sigma(a, b; 0) = \sigma(c, d; ab)$ if and only if $d = (a^2b - 2abc + a - c)/c^2$.*

Proof: Since the sum of each column in $LKA(a, b; 0)$ is $\frac{a(ab+1)}{2}$ while in $LKA(c, d; ab)$ the sum is $\frac{c(cd+2ab+1)}{2}$, we have

$$\sigma(a, b; 0) = \frac{a(ab+1)}{2} = \frac{c(cd+2ab+1)}{2} = \sigma(c, d; ab)$$

which yields

$$a(ab+1) = c^2d + 2abc + c$$

and

$$a^2b - 2abc + a - c = c^2d.$$

giving the desired result, $d = (a^2b - 2abc + a - c)/c^2$.

On the other hand, since the sum of each column in $LKA(c, d; ab)$ is $\frac{c(cd+2ab+1)}{2}$, we have

$$\sigma(c, d; ab) = \frac{c^2d + 2abc + c}{2} = \frac{a^2b - 2abc + a - c + 2abc + c}{2} = \frac{a^2b + a}{2}$$

which yields

$$\sigma(c, d; ab) = \frac{a(ab+1)}{2} = \sigma(a, b; 0).$$

□

For the special case $c = 2$ the following lemma holds.

Lemma 5. *Let a, b, d be positive integers such that both $LKA(a, b; 0)$ and $LKA(2, d; ab)$ exist and $\sigma(a, b; 0) = \sigma(2, d; ab)$. Then either $a \equiv 2 \pmod 4$ or a is odd and $a \equiv b \pmod 4$ and $a \geq 5$.*

Proof: We observe that if $a \equiv 0 \pmod 4$, then $\sigma(a,b;0) = a(ab+1)/2$ is obviously an even number. But this is impossible, since $\sigma(a,b;0) = \sigma(2,d;ab) = 2ab+2d+1$, which is odd. Therefore, $a \not\equiv 0 \pmod 4$. If $a \equiv 1 \pmod 4$ and $b \equiv 3 \pmod 4$ (or $a \equiv 3 \pmod 4$ and $b \equiv 1 \pmod 4$), then $ab \equiv 3 \pmod 4$ and $ab+1 \equiv 0 \pmod 4$. This again gives $\sigma(a,b;0) = a(ab+1)/2$ even, which is impossible. It is easy to check that in all remaining cases, i.e., when $a \equiv 2 \pmod 4$ or a is odd and $a \equiv b \pmod 4$, the parity conditions are satisfied.

Finally, we prove that $a \geq 5$. If $a = 1$, then by Theorem 8 also $b = 1$ and $\sigma(a,b;0) = 1$. On the other hand, $\sigma(2,d;ab) \geq 2$, which is impossible.

If $a = 2$, then $\sigma(a,b;0) = 2b+1$. On the other hand, $\sigma(2,d;ab) = 2ab + 2d+1 = 4b+2d+1 > 2b+1$, which is also a contradiction.

If $a = 3$, then $\sigma(a,b;0) = 3(3b+1)/2$ and $\sigma(2,d;ab) = 2ab+2d+1 = 6b+2d+1$. Because we must have $\sigma(a,b;0) = \sigma(2,d;ab)$, it follows that

$$3(3b+1)/2 = 6b+2d+1$$

and

$$9b+3 = 12b+4d+2$$

which yields

$$0 = 3b+4d-1.$$

Because $b > 0$, it follows that d must be negative, but d is also positive integer.

Finally, we already know that $a \not\equiv 0 \pmod 4$ and hence $a \geq 5$. □

We now present examples of two small connected bi-regular distance magic graphs.

Example 1. First we set $a = b = 5$. Then $d = (a^2b - 4ab + a - 2)/4 = 7$. We construct H_1 as follows. We take the complete bipartite graph $K_{5,7}$ with bipartition $X = \{x_1, \ldots, x_5\}, Y = \{y_1, \ldots, y_7\}$ and add the cycle $C_7 = y_1, y_2, \ldots, y_7$. This is clearly a 7-regular graph. Now we construct $G_1 = H_1[5 \times 5, 7 \times 2]$. In G_1, every vertex x_{ij} is adjacent to vertices of all sets $Y_1, Y_2, \ldots, Y_7$ and is therefore of degree 14. Every vertex y_{ij} is adjacent to all vertices in each $X_1, X_2, \ldots, X_5$ and to vertices $y_{i-1\,1}, y_{i-1\,2}, y_{i+1\,1}, y_{i+1\,2}$ and is of degree 29.

The sum of labels in each of the sets $X_1, \ldots, X_5, Y_1, \ldots, Y_7$ is equal to $\sigma(5,5;0) = 5(5 \cdot 5 + 1)/2 = 65 = 2 \cdot 5 \cdot 5 + 2 \cdot 7 + 1 = \sigma(2,7;25)$ and hence the magic constant is $m(G_1) = 7 \cdot 65 = 455$.

Example 2. Similarly, we can construct another graph with $a = 6, b = 9$. Then $d = (a^2b - 4ab + a - 2)/4 = (6^2 \cdot 9 - 4 \cdot 6 \cdot 9 + 6 - 2)/4 = 28$. We construct H_2 again as the complete bipartite graph $K_{9,28}$ with bipartition $X = \{x_1, \ldots, x_9\}, Y = \{y_1, \ldots, y_{28}\}$ and add any 19 1-factors belonging to an arbitrary 1-factorization of the complete graph K_{28} induced on the vertex set $Y = \{y_1, y_2, \ldots, y_{28}\}$. This is a 28-regular graph and we can build $G_2 = H_2[9 \times 6, 28 \times 2]$. In G, every vertex x_{ij} is adjacent to vertices of all sets $Y_1, Y_2, \ldots, Y_{28}$ and is therefore of degree 56. Every vertex y_{ij} is adjacent to all vertices in each $X_1, X_2, \ldots, X_9$ and to vertices of 9 sets Y_j and is of degree 92.

The sum of labels in each of the sets X_i, Y_j is equal to $\sigma(6,9;0) = \sigma(2,28;54) = 165$ and the magic constant is $m(G_2) = 28 \cdot 165 = 4620$.

The existence of non-regular distance magic graphs is guaranteed by the following theorem.

Theorem 9. *Let H be a p-regular graph on $b + d$ vertices and $G = H[b \times a, d \times c]$ be a graph with a, b, c, d satisfying conditions*

(1) $a > c$,

(2) both $LKA(a,b;0)$ and $LKA(c,d;ab)$ exist, and

(3) $d = (a^2b - 2abc + a - c)/c^2$.

Then G is a distance magic graph.

Proof: Denote the vertices of H by $x_1, x_2, \ldots, x_b$ and $y_1, y_2, \ldots, y_d$. To obtain $G = H[b \times a, d \times c]$, we expand each x_i into the set X_i of a independent vertices $x_{i1}, x_{i2}, \ldots, x_{ia}$ and each y_j into the set Y_j of c independent vertices $y_{j1}, y_{j2}, \ldots, y_{jc}$. We construct the arrays $LKA(a,b;0) = [e_{rs}], r = 1,2,\ldots,a; s = 1,2,\ldots,b$ and $LKA(c,d;ab) = [f_{rs}], r = 1,2,\ldots,c; s = 1,2,\ldots,d$ and label each vertex x_{ik} by the entry e_{ki} and each vertex y_{jm} by the entry f_{mj}. By Lemma 4 the sum of labels in each column equal to $\sigma = a(ab+1)/2$.

Now every vertex of G is adjacent to all vertices of precisely p sets of the collection $X_1, X_2, \ldots, X_b, Y_1, Y_2, \ldots, Y_d$. Because the sum of labels in each of these sets is equal to σ, it is obvious that the sum of labels of neighbours of every vertex is $p\sigma$, which completes the proof. □

4 Conclusion

We conclude this paper by the two following conjectures. The first conjecture was put forward by Miller *et al.* [3] and is still open.

Conjecture 1. *[3] Let $1 \leq a_1 \leq \cdots \leq a_p$ where $p \geq 4$. Let $s_i = \Sigma_{j=1}^{i} a_j$. There exists a labeling of the complete multipartite graph $K_{a_1,\ldots,a_p}$ if and only if the following conditions hold*
(a) $a_2 \geq 2$,
(b) $n(n+1) = 0 \pmod{2p}$, where $n = s_p = |V(H_{a_1,\ldots,a_p})|$, and
(c) $\Sigma_{j=1}^{s_i}(n+1-j) \geq \frac{in(n+1)}{2p}$ for $1 \leq i \leq p$.

We believe that the following conjecture is also true.

Conjecture 2. *If G is a distance magic graph different from $K_{1,2,2,\ldots,2}$, then the vertex set V can be partitioned into sets $V_1, V_2, \ldots, V_p$ such that each V_i has $|V_i| > 1$ and consists of mutually independent vertices.*

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