On the Existence of Regular Supermagic Graphs

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Abstract

A graph is called supermagic if it admits a labeling of its edges by consecutive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. In this paper we prove that the necessary conditions for an r-regular supermagic graph of order n to exist are also sufficient. All proofs are constructive and they are based on finding supermagic labelings of circulant graphs.

Keywords: regular graph, supermagic graph, circulant graph

1 Introduction

We consider finite undirected graphs without loops and multiple edges. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. The value $f^*(v)$ is often called the *weight* of the vertex v. An injective mapping f from $E(G)$ into positive integers is called a *magic labeling* of G for an *index* k if all the vertices have the same weight, thus

$$f^*(v) = k \quad \text{for all } v \in V(G).$$

A magic labeling f of G is called *supermagic* if the set $\{f(e) : e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labeling of G. For regular supermagic graphs we can always assume that the smallest edge label is 1.

The concept of magic graphs was introduced by Sedláček [7]. Supermagic graphs were introduced by M. B. Stewart [9]. There is by now a considerable number of papers published on magic and supermagic graphs; we single out [10, 4, 6, 3, 1, 8] as being more particularly relevant to the present paper, and refer the reader to [2] for comprehensive references.

There are strict bounds on the number of edges in a magic graph given in [11]. For supermagic graphs no similar result can be found. In [1] it is proved that if d is the greatest common divisor of integers n and ε, and if $\frac{n}{d}$ and ε are both even, then there exists no supermagic graph of order n and size ε. It seems that the existence of a supermagic graph with a given order and size is hard to solve. In this paper we deal with an easier problem, we focus only on regular graphs. We characterize all pairs n, r for which an r-regular supermagic graph of order n exists.

Theorem 1.1 (Main theorem). *Let r, n be positive integers, $n \geq r+1$. There exists an r-regular supermagic graph of order n if and only if one of the following statements holds.*

(i) $r = 1$ *and* $n = 2$,

(ii) $3 \leq r \equiv 1 \pmod 2$ *and* $n \equiv 2 \pmod 4$,

(iii) $4 \leq r \equiv 0 \pmod 2$ *and* $n > 5$.

In Section 2 we summarize all known results that will be relevant to our constructions. Section 3 brings several assertions that are used to prove the main result in Section 4.

2 Known results

The following necessary conditions hold for a regular supermagic graph.

Proposition 2.1. [4] *If G is an r-regular supermagic graph of order n, then the following statements hold:*

(i) *if* $r \equiv 1 \pmod 2$, *then* $n \equiv 2 \pmod 4$,

(ii) *if* $r \equiv 2 \pmod 4$ *and* $n \equiv 0 \pmod 2$, *then G contains no component of an odd order,*

(iii) *if* $n > 2$, *then* $r > 2$.

A regular spanning subgraph of a graph G is called the *factor* of G. A factor of degree k is called the *k-factor*. In [4] it is proved

Proposition 2.2. [4] *If G is a graph decomposable into pairwise edge-disjoint supermagic factors, then G is supermagic.*

The union of $m \geq 1$ disjoint copies of graph G is denoted by mG.

Proposition 2.3. [4] *If G is a $2k$-regular supermagic graph decomposable into two edge-disjoint supermagic k-factors, then the graph mG is supermagic for every positive integer m.*

To find a supermagic labeling of a regular graph it is convenient to use highly symmetric graphs. Let $n, m \geq 1$ and $a_1, \dots, a_m \leq \lfloor \frac{n}{2} \rfloor$ be pairwise different positive integers. An undirected graph with the set of vertices $\{v_1, \dots, v_n\}$ and the set of edges $\{v_i v_{i+a_j} : 1 \leq i \leq n, 1 \leq j \leq m\}$, the indices being taken modulo n, is called a *circulant graph* and it is denoted by $C_n(a_1, \dots, a_m)$.

The Möbius ladder M_n, where $6 \leq n \equiv 0 \pmod 2$, is the 3-regular circulant graph $C_n(1, \frac{n}{2})$. Sedláček [6] proved the following result.

Proposition 2.4. [6] *Let $n \geq 6$ be an even integer. The Möbius ladder M_n is supermagic if and only if* $n \equiv 2 \pmod 4$.

Notice that the complete graph K_n is the $(n-1)$-regular circulant graph $C_n(1, 2, \dots, \lfloor \frac{n}{2} \rfloor)$. Supermagic complete graphs were characterized in [10].

Proposition 2.5. [10] *A complete graph K_n of order n is supermagic if and only if $n = 2$ or* $5 < n \not\equiv 0 \pmod 4$.

The regular complete n-partite graph of order nk is denoted by $K_{n[k]}$. Evidently, every regular complete multipartite graph is a circulant graph, e.g., $K_{n[2]} = C_{2n}(1, 2, \dots, n-1)$. The existence of regular supermagic complete multipartite graphs was examined in [4]. Besides other the following was shown.

Proposition 2.6. [4] *The regular complete multipartite graph $K_{n[2]}$ is supermagic for every $n \geq 3$.*

By decomposing a graph into even number of Hamilton cycles the following was proved.

Proposition 2.7. [8] *Let $G := C_n(a_1, \dots, a_{2k})$ be a $4k$-regular circulant graph. If $n \equiv 0 \pmod 2$, $a_i \equiv 1 \pmod 2$ and $\gcd(a_{2j-1}, a_{2j}, n) = 1$ for every $i \in \{1, \dots, 2k\}$, $j \in \{1, \dots, k\}$, then G is supermagic.*

In the next we use the following corollary.

Corollary 2.8. *If n, k, a are positive integers, where a is odd and $2a+4k-2 \leq n \equiv 0 \pmod 2$, then the $4k$-regular circulant graph $C_n(a, a+2, \dots, a+2(k-1))$ is supermagic.*

The following result is proved in [5].

Proposition 2.9. [5] *If G is a $4k$-regular circulant graph of odd order, then the Cartesian product of graphs G and K_2 is supermagic.*

Since for n even the graph $C_n(2, 4, \dots, 4k, \frac{n}{2})$ is isomorphic to the Cartesian product of graphs $C_{\frac{n}{2}}(1, 2, \dots, 2k)$ and K_2, we have immediately.

Corollary 2.10. *If n, k are positive integers, $8k + 2 \leq n \equiv 2 \pmod 4$, then the circulant graph $C_n(2, 4, \dots, 4k, \frac{n}{2})$ is supermagic.*

3 New results on circulant graphs

Lemma 3.1. *Let $C_n(a, a+b)$ be a 4-regular circulant graph and let v be its vertex. If $\gcd(n, b) = 1$, then there exists a labeling $f : E(C_n(a, a+b)) \to \{1, 2, \dots, 2n\}$ such that for every vertex $u \in V(C_n(a, a+b))$ it holds*

$$f^*(u) = \begin{cases} 3n+3 & \text{if } u = v, \\ 4n+3 & \text{if } u \neq v. \end{cases}$$

Proof. As b and n are coprime, there exists a positive integer c (obviously, $c \equiv ab^{-1} \pmod{n}$) such that the graph $C_n(a, a+b)$ is isomorphic to the graph $C_n(c, c+1)$. Without loss of generality we can assume to have the graph $C_n(c, c+1)$ with vertex set $\{v_1, v_2, \dots, v_n\}$ where $v = v_1$. Define a mapping $f : E(C_n(c, c+1)) \to \{1, 2, \dots, 2n\}$ by

$$f(v_i, v_{i+c+1}) = i \qquad \text{if } 1 \le i \le n,$$

$$f(v_i, v_{i+c}) = \begin{cases} 2n+2-i & \text{if } 2 \le i \le n, \\ n+1 & \text{if } i = 1. \end{cases}$$

It is easy to check that

$$f^*(v_i) = \begin{cases} 3n+3 & \text{if } i = 1, \\ 4n+3 & \text{if } 2 \le i \le n. \end{cases}$$

Thus, the labeling f has required properties. □

Theorem 3.2. *Let $G := C_n(a, a+b, c, c+d)$ be a circulant graph of degree 8. If $\gcd(b, n) = 1$ and $\gcd(d, n) = 1$, then G is a supermagic graph.*

Proof. The graph G is decomposable into two edge-disjoint 4-factors $G_1 := C_n(a, a+b)$ and $G_2 := C_n(c, c+d)$. According to the previous lemma, there exist labelings $f : E(G_1) \to \{1, 2, \dots, 2n\}$ and $g : E(G_2) \to \{1, 2, \dots, 2n\}$ so that

$$f^*(v_1) = g^*(v_1) = 3n+3,$$
$$f^*(v_2) = \cdots = f^*(v_n) = g^*(v_2) = \cdots = g^*(v_n) = 4n+3.$$

Consider a mapping $h : E(G) \to \{1, \dots, 4n\}$ defined by

$$h(e) = \begin{cases} f(e) & \text{if } e \in E(G_1), \\ 4n+1-g(e) & \text{if } e \in E(G_2). \end{cases}$$

Obviously, $h^*(v) = 16n+4$ for every $v \in V(G)$ and thus h is a supermagic labeling of G. □

By choosing $b = d = 1$ and $c = a+2$ we get

Corollary 3.3. *Let n, a be positive integers. The 8-regular circulant graph $C_n(a, a+1, a+2, a+3)$ is supermagic for all $n > 2a+6$.*

By induction immediately follows

Corollary 3.4. *Let n, a, k be positive integers. The $8k$-regular circulant graph $C_n(a, a+1, \ldots, a+4k-1)$ is supermagic for all $n > 2a + 8k - 2$.*

In the following theorems we prove that the 4-regular circulant graph $C_n(1,3)$, the 6-regular circulant graph $C_n(1,2,3)$, and the 10-regular circulant graph $C_n(1,2,3,4,6)$ are supermagic for all feasible values of n. All proofs will be done in a similar way. The technique is most transparent for $C_n(1,2,3)$, therefore we start with this case.

Theorem 3.5. *The circulant graph $C_n(1,2,3)$ has a supermagic labeling for all $n \geq 7$.*

Proof. For any integer $n \geq 3$, let $H_n = H_n(1,2,3)$ be a graph with the vertex set $\{w_1, w_2, \ldots, w_{n+3}\}$ and edge set $\cup_{i=1}^{n}\{w_i w_{i+1}, w_{i+1} w_{i+3}, w_i w_{i+3}\}$. Notice that the vertices of the graph H_n are all of even degree, see Figures 1 through 4. The vertices w_1, w_{n+2}, and w_{n+3} are of degree 2, for $n \geq 3$ the vertices w_2, w_3, and w_{n+1} are of degree 4, and for $n \geq 4$ are all remaining vertices of degree 6. Also it is easy to observe, that by identifying the pairs w_1 and w_{n+1}, w_2 and w_{n+2}, w_3 and w_{n+3} we obtain for $n \geq 7$ from H_n the circulant graph $C_n(1,2,3)$.

The construction of the required labeling is done in two steps. First we find a labeling λ_n, called the auxiliary labeling, of H_n for any $n \geq 3$. Then we show how to obtain from λ_n a supermagic labeling f of $C_n(1,2,3)$.

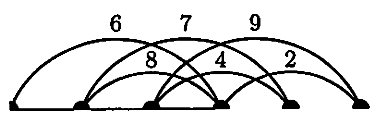

Figure 1: Auxiliary labeling of $H_3(1,2,3)$.

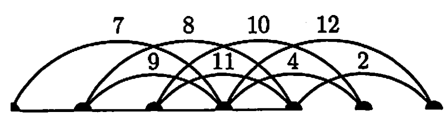

Figure 2: Auxiliary labeling of $H_4(1,2,3)$.

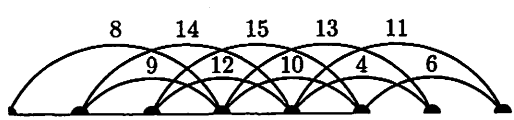

Figure 3: Auxiliary labeling of $H_5(1,2,3)$.

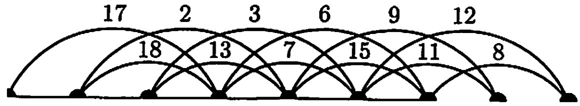

Figure 4: Auxiliary labeling of $H_6(1,2,3)$.

Since the graph H_n is a subgraph of a graph H_m, for $3 \leq n \leq m$, we can define the auxiliary labeling $\lambda_n : E(H_n) \to \{1, 2, \ldots, 3n\}$ recursively as follows. The labelings of the graphs H_3, H_4, H_5, and H_6 are given in the Figures 1 through 4. For larger n the auxiliary labeling is defined by

$$\lambda_{n+4}(e) = \begin{cases} \lambda_n(e) + 6 & \text{for } e \in E(H_n), \\ 1 & \text{for } e = w_{n+3}w_{n+4}, \\ 2 & \text{for } e = w_{n+5}w_{n+7}, \\ 3 & \text{for } e = w_{n+2}w_{n+3}, \\ 4 & \text{for } e = w_{n+4}w_{n+6}, \\ 5 & \text{for } e = w_{n+1}w_{n+2}, \\ 6 & \text{for } e = w_{n+4}w_{n+5}, \\ 3n+7 & \text{for } e = w_{n+1}w_{n+4}, \\ 3n+8 & \text{for } e = w_{n+2}w_{n+5}, \\ 3n+9 & \text{for } e = w_{n+2}w_{n+4}, \\ 3n+10 & \text{for } e = w_{n+3}w_{n+6}, \\ 3n+11 & \text{for } e = w_{n+3}w_{n+5}, \\ 3n+12 & \text{for } e = w_{n+4}w_{n+7}. \end{cases}$$

It is easy to verify that for every $n \in \{3, 4, 5, 6\}$ and all $i = 1, 2, \ldots, n+3$ it holds

$$\lambda_n^*(w_i) = \begin{cases} \frac{1}{2}\deg(w_i)(3n+1) - 1 & \text{for } i = 1, 2, 3, \\ \frac{1}{2}\deg(w_i)(3n+1) + 1 & \text{for } i = n+1, n+2, n+3, \\ \frac{1}{2}\deg(w_i)(3n+1) & \text{otherwise.} \end{cases} \qquad (1)$$

Similarly, for the first three vertices of H_{n+4} is

$$\begin{aligned} \lambda_{n+4}^*(w_1) &= \lambda^*(w_1) + 6\deg_{H_n}(w_1) = 3n + 12 = 3(n+4), \\ \lambda_{n+4}^*(w_2) &= \lambda^*(w_2) + 6\deg_{H_n}(w_2) = (3n+1) + 24 = 6(n+4) + 1, \\ \lambda_{n+4}^*(w_3) &= \lambda^*(w_3) + 6\deg_{H_n}(w_3) = (3n+1) + 24 = 6(n+4) + 1. \end{aligned}$$

For the vertices w_i, $i = 4, 5, \ldots, n$, we have

$$\lambda_{n+4}^*(w_i) = \lambda^*(w_i) + 6\deg_{H_n}(w_i) = (9n+3) + 36 = 9(n+4) + 3.$$

Now for the remaining vertices in $V(H_n)$ is

$$\begin{aligned}
\lambda^*_{n+4}(w_{n+1}) &= \lambda^*(w_{n+1}) + 6\deg_{H_n}(w_{n+1}) + 5 + (3n+7) \\
&= 9(n+4)+3, \\
\lambda^*_{n+4}(w_{n+2}) &= \lambda^*(w_{n+2}) + 6\deg_{H_n}(w_{n+2}) + 3 + 5 + (3n+8) \\
&+ (3n+9) = 9(n+4)+3, \\
\lambda^*_{n+4}(w_{n+3}) &= \lambda^*(w_{n+3}) + 6\deg_{H_n}(w_{n+3}) + 1 + 3 + (3n+10) \\
&+ (3n+11) = 9(n+4)+3.
\end{aligned}$$

Finally, for the four last vertices in $V(H_{n+4})$ is

$$\begin{aligned}
\lambda^*_{n+4}(w_{n+4}) &= 1+4+6+(3n+7)+(3n+9)+(3n+12) \\
&= 9(n+4)+3, \\
\lambda^*_{n+4}(w_{n+5}) &= 2+6+(3n+8)+(3n+11) = 6(n+4)+3, \\
\lambda^*_{n+4}(w_{n+6}) &= 4+(3n+10) = 3(n+4)+2, \\
\lambda^*_{n+4}(w_{n+7}) &= 2+(3n+12) = 3(n+4)+2.
\end{aligned}$$

Thus, the equations in (1) hold for every vertex of H_n and for all $n \geq 3$.

For $n \geq 7$ we obtain the 6-regular circulant graph $C_n(1,2,3)$ from the graph H_n by identifying vertices w_1 and w_{n+1}, w_2 and w_{n+2}, w_3 and w_{n+3}. More precisely, the mapping $\xi : V(H_n) \to V(C_n(1,2,3))$, given by $\xi(w_i) = v_i$, for all $i = 1, \dots, n$, and $\xi(w_{n+j}) = v_j$, for $j = 1,2,3$, is a homomorphism of graphs H_n and $C_n(1,2,3)$. Moreover, the homomorphism ξ induces a bijective mapping ξ_E from $E(H_n)$ to $E(C_n(1,2,3))$. Consider the labeling $f : E(C_n(1,2,3)) \to \{1, \dots, 3n\}$ defined by $f(e) = \lambda_n(\xi_E^{-1}(e))$. Evidently, it holds

$$f^*(v_i) = \begin{cases} \lambda^*_n(w_i) + \lambda^*_n(w_{n+i}) & \text{for } 1 \leq i \leq 3, \\ \lambda^*_n(w_i) & \text{for } 4 \leq i \leq n. \end{cases}$$

Therefore, $f^*(v) = 9n+3$ for every vertex $v \in V(C_n(1,2,3))$. Thus, f is a supermagic labeling of the circulant graph $C_n(1,2,3)$. □

According to Corollary 2.8 the circulant graph $C_n(1,3)$ is supermagic for every even integer $n \geq 8$. However we can extend this claim.

Theorem 3.6. *The circulant graph $C_n(1,3)$ has a supermagic labeling for all $n \geq 7$.*

Proof. For any integer $n \geq 4$, let $H_n = H_n(1,3)$ be a graph with the vertex set $\{w_1, w_2, \dots, w_{n+3}\}$ and edge set $\bigcup_{i=1}^{n}\{w_i w_{i+3}\} \cup \bigcup_{i=3}^{n}\{w_i w_{i+1}\} \cup \{w_1 w_2, w_{n+2} w_{n+3}\}$. Notice that the vertices of the graph H_n are all of even degree, see Figures 5 through 8. The vertices w_1, w_2, w_3, w_{n+1}, w_{n+2} and

w_{n+3} are of degree 2 and all remaining vertices are of degree 4. Also it is easy to observe, that by identifying the pairs w_1 and w_{n+1}, w_2 and w_{n+2}, w_3 and w_{n+3} we obtain for $n \geq 7$ from H_n the circulant graph $C_n(1,3)$.

Since the graph H_n is a subgraph of a graph H_m, for $4 \leq n \leq m-2$, we can define the auxiliary labeling $\lambda_n : E(H_n) \rightarrow \{1, 2, \ldots, 2n\}$ recursively for $n \neq 5$ as follows. The labelings of the graphs H_4, H_6, H_7, and H_9 are given in the Figures 5 through 8. For $n = 8$ and for $n \geq 10$ the auxiliary labeling is defined by

$$\lambda_{n+4}(e) = \begin{cases} \lambda_n(e) + 4 & \text{for } e \in E(H_n), \\ 1 & \text{for } e = w_{n+3}w_{n+4}, \\ 2 & \text{for } e = w_{n+6}w_{n+7}, \\ 3 & \text{for } e = w_{n+4}w_{n+5}, \\ 4 & \text{for } e = w_{n+1}w_{n+2}, \\ 2n+5 & \text{for } e = w_{n+2}w_{n+5}, \\ 2n+6 & \text{for } e = w_{n+1}w_{n+4}, \\ 2n+7 & \text{for } e = w_{n+3}w_{n+6}, \\ 2n+8 & \text{for } e = w_{n+4}w_{n+7}. \end{cases}$$

Analogously as in the proof of Theorem 3.5 it is easy to verify that for every vertex of H_n, $n \geq 6$, it holds

$$\lambda_n^*(w_i) = \begin{cases} \frac{1}{2}\deg(w_i)(2n+1) - 1 & \text{for } i = 1, n+3, \\ \frac{1}{2}\deg(w_i)(2n+1) + 1 & \text{for } i = 3, n+1, \\ \frac{1}{2}\deg(w_i)(2n+1) & \text{otherwise.} \end{cases}$$

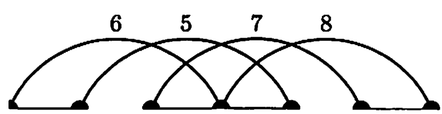

Figure 5: Auxiliary labeling of $H_4(1,3)$.

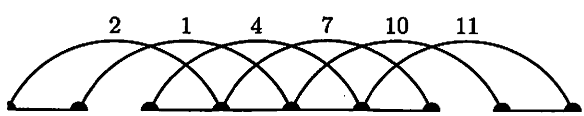

Figure 6: Auxiliary labeling of $H_6(1,3)$.

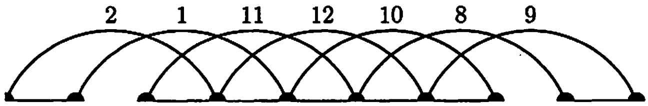

Figure 7: Auxiliary labeling of $H_7(1,3)$.

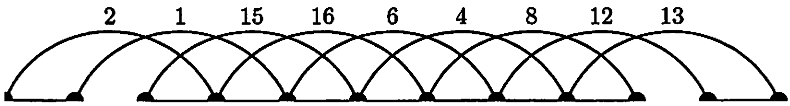

Figure 8: Auxiliary labeling of $H_9(1,3)$.

For $n \geq 7$ we obtain the 4-regular circulant graph $C_n(1,3)$ from the graph H_n by identifying vertices w_1 and w_{n+1}, w_2 and w_{n+2}, w_3 and w_{n+3}. More precisely, the mapping $\xi : V(H_n) \to V(C_n(1,3))$, given by $\xi(w_i) = v_i$, for all $i = 1, \ldots, n$, and $\xi(w_{n+j}) = v_j$, for $j = 1, 2, 3$, is a homomorphism of graphs H_n and $C_n(1,3)$. Moreover, the homomorphism ξ induces a bijective mapping ξ_E from $E(H_n)$ to $E(C_n(1,3))$. Now the edge labeling f, given by $f(e) = \lambda_n(\xi_E^{-1}(e))$ similarly as in Theorem 3.5, is a supermagic labeling of the circulant graph $C_n(1,3)$ with the index $4n+2$. □

Remark. Magic stars are well established among recreational mathematics. A magic star is a collection of consecutive integers connected with straight lines such that the sum along every line is constant. There is a one to one correspondence between magic stars and circulant graphs $C_n(1,t)$. A type B magic star corresponds to the circulant graph $C_n(1,3)$. Trenkler [12] showed that there exists a type B magic star (in literature also denoted by T_n) for every even $n > 7$. Theorem 3.6 extends the result to all integers $n \geq 7$, thus there exists a type B magic star T_n for every $n \geq 7$.

Theorem 3.7. *The circulant graph $C_n(1,2,3,4,6)$ has a supermagic labeling for all $n \geq 13$.*

Proof. For any integer $n \geq 4$, let $H_n = H_n(1,2,3,4,6)$ be a graph given by

$$\begin{aligned} V(H_n) &= \{w_1, w_2, \ldots, w_{n+6}\} \\ E(H_n) &= \cup_{i=1}^{n} \{w_{i+2}w_{i+3}, w_{i+2}w_{i+4}, w_{i+3}w_{i+6}, w_i w_{i+4}, w_i w_{i+6}\}. \end{aligned}$$

All vertices of the graph H_n are of even degree, see Figures 9 through 12. Also it is easy to observe, that for $n \geq 13$ we obtain from H_n the circulant graph $C_n(1,2,3,4,6)$ by identifying the pairs w_i and w_{n+i} for $1 \leq i \leq 6$.

Since the graph H_n is a subgraph of a graph H_m, for $4 \leq n \leq m$, we can define the auxiliary labeling $\lambda_n : E(H_n) \to \{1, 2, \dots, 5n\}$ recursively as follows. The labelings of the graphs H_4, H_5, H_6, and H_7 are given in the Figures 9 through 12. For $n \geq 8$ the auxiliary labeling is defined by

$$\lambda_{n+4}(e) = \begin{cases} \lambda_n(e) + 10 & \text{for } e \in E(H_n), \\ 1 & \text{for } e = w_{n+1}w_{n+5}, \\ 2 & \text{for } e = w_{n+4}w_{n+7}, \\ 3 & \text{for } e = w_{n+4}w_{n+8}, \\ 4 & \text{for } e = w_{n+2}w_{n+6}, \\ 5 & \text{for } e = w_{n+3}w_{n+5}, \\ 6 & \text{for } e = w_{n+6}w_{n+9}, \\ 7 & \text{for } e = w_{n+7}w_{n+10}, \\ 8 & \text{for } e = w_{n+3}w_{n+4}, \\ 9 & \text{for } e = w_{n+6}w_{n+8}, \\ 10 & \text{for } e = w_{n+5}w_{n+7}, \\ 5n+11 & \text{for } e = w_{n+6}w_{n+7}, \\ 5n+12 & \text{for } e = w_{n+5}w_{n+8}, \\ 5n+13 & \text{for } e = w_{n+3}w_{n+7}, \\ 5n+14 & \text{for } e = w_{n+4}w_{n+10}, \\ 5n+15 & \text{for } e = w_{n+5}w_{n+6}, \\ 5n+16 & \text{for } e = w_{n+3}w_{n+9}, \\ 5n+17 & \text{for } e = w_{n+4}w_{n+6}, \\ 5n+18 & \text{for } e = w_{n+2}w_{n+8}, \\ 5n+19 & \text{for } e = w_{n+4}w_{n+5}, \\ 5n+20 & \text{for } e = w_{n+1}w_{n+7}. \end{cases}$$

Analogously as in the proof of Theorem 3.5 it is easy to verify that for every vertex of H_n, $n \geq 4$, it holds

$$\lambda_n^*(w_i) = \begin{cases} \frac{1}{2}\deg(w_i)(5n+1) - 1 & \text{for } i = 2, n+5, \\ \frac{1}{2}\deg(w_i)(5n+1) + 1 & \text{for } i = 5, n+2, \\ \frac{1}{2}\deg(w_i)(5n+1) & \text{otherwise.} \end{cases}$$

For $n \geq 13$ we obtain the 10-regular circulant graph $C_n(1,2,3,4,6)$ from the graph H_n by identifying vertices w_i and w_{n+i} for $i = 1,2,3,4,5,6$. More precisely, the mapping $\xi : V(H_n) \to V(C_n(1,2,3,4,6))$, given by $\xi(w_i) = v_i$, for all $i = 1, \dots, n$, and $\xi(w_{n+j}) = v_j$, for $j = 1,2,3,4,5,6$, is a homomorphism of graphs H_n and $C_n(1,2,3,4,6)$. Moreover, the homomorphism

ξ induces a bijective mapping ξ_E from $E(H_n)$ to $E(C_n(1,2,3,4,6))$. Now the edge labeling f, given by $f(e) = \lambda_n(\xi_E^{-1}(e))$ similarly as in Theorem 3.5, is a supermagic labeling of the circulant graph $C_n(1,2,3,4,6)$ with the index $25n + 5$. □

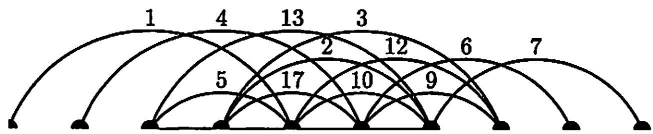

Figure 9: Auxiliary labeling of $H_4(1,2,3,4,6)$.

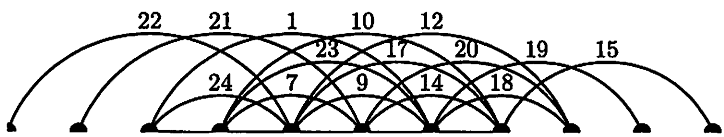

Figure 10: Auxiliary labeling of $H_5(1,2,3,4,6)$.

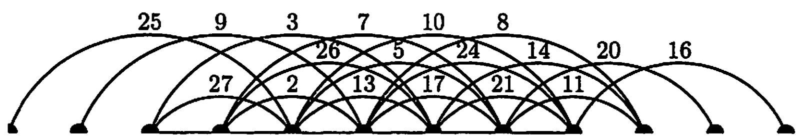

Figure 11: Auxiliary labeling of $H_6(1,2,3,4,6)$.

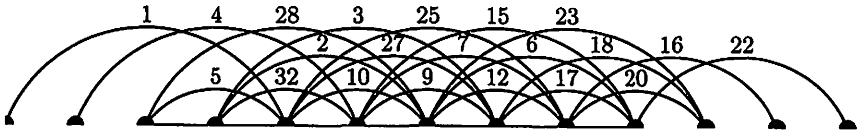

Figure 12: Auxiliary labeling of $H_7(1,2,3,4,6)$.

4 Proof of the main theorem

Summarizing all the previous results we are ready to prove Theorem 1.1 – the main theorem of this paper.

The necessity of the conditions (*i*), (*ii*) or (*iii*) follows from Proposition 2.1.

Sufficiency we examine case by case. Let k be a non-negative integer such that $0 \leq r - 8k \leq 7$. Consider the following cases.

A. Obviously the only graph satisfying (*i*) is K_2 and it is the only one 1-regular supermagic graph.

B. Suppose condition (*ii*) holds.

B.1. For $r = 8k + 1$, $k \geq 1$, and $n \equiv 2 \pmod 4$, let us consider the r-regular graph $G := C_n(1, 2, \dots, 4k, \frac{n}{2})$ of order n. According to Corollaries 2.10 and 2.8 the graphs $G_1 := C_n(2, 4, \dots, 4k, \frac{n}{2})$ and $G_2 := C_n(1, 3, \dots, 4k - 1)$ are supermagic for all $n > r$. As G can be decomposed into two edge-disjoint factors G_1 and G_2, the graph G is supermagic by Proposition 2.2.

B.2. For $r = 3$, let us consider the Möbius ladder $C_n(1, \frac{n}{2})$. By Proposition 2.4 it is supermagic.

For $r = 8k + 3$, $k \geq 1$, let us consider the r-regular circulant graph $G := C_n(1, 2, \dots, 4k + 1, \frac{n}{2})$ on $n > r$ vertices. By Proposition 2.4 there exists a 3-regular supermagic graph $G_1 := C_n(1, \frac{n}{2})$ for all feasible values of n. According to Corollary 3.4 the graph $G_2 := C_n(2, 3, \dots, 4k + 1)$

is supermagic. As G can be decomposed into two edge-disjoint factors G_1 and G_2, the existence of a supermagic labeling of G is guaranteed by Proposition 2.2.

B.3. For $r = 8k+5$ and $n = 8k+6$ there exists a unique graph, namely the complete graph K_n. It is supermagic by Proposition 2.5.

For $r = 5$ and $n > 6$, let us consider the circulant graph $C_n(2, 4, \frac{n}{2})$. It is supermagic by Corollary 2.10.

For $r = 8k+5$ and $k \geq 1$, let us consider the r-regular circulant graph $G := C_n(1, 2, \ldots, 4k+1, 4k+3, \frac{n}{2})$ of order n, $n > 8k+6$. The graph G is decomposable into two edge-disjoint factors $C_n(2, 4, \ldots, 4k, \frac{n}{2})$ and $C_n(1, 3, \ldots, 4k+1, 4k+3)$. Thus it is supermagic according to Corollary 2.10, Corollary 2.8 and Proposition 2.2.

B.4. For $r = 7$ and $n = 10$, let us consider the graph $C_{10}(1, 2, 4, 5)$. The graph $C_{10}(1, 5)$ is supermagic by Proposition 2.4. In [4] it is proved that the graph $2K_5$ (isomorphic to $C_{10}(2, 4)$) is supermagic, too. Then the graph $C_{10}(1, 2, 4, 5)$ is supermagic according to Proposition 2.2.

For $r = 7$ and $14 \leq n \equiv 2 \pmod 4$, let us consider the circulant graph $C_n(1, 2, 6, \frac{n}{2})$. The graph $C_n(1, \frac{n}{2})$ is supermagic by Proposition 2.4. The graph $C_n(2, 6)$ is isomorphic to $2C_{\frac{n}{2}}(1, 3)$ and so it is supermagic by Theorem 3.6 and Proposition 2.3. According to Proposition 2.2, the graph $C_n(1, 2, 6, \frac{n}{2})$ is supermagic.

For $r = 15$ and $18 \leq n \equiv 2 \pmod 4$, let us consider the circulant graph $G_1 := C_n(1, 2, 3, 4, 5, 6, 8, \frac{n}{2})$. The graphs $C_n(1, \frac{n}{2})$ and $C_n(3, 5)$ are supermagic by Proposition 2.4 and Corollary 2.8. The graph $C_n(2, 4, 6, 8)$, isomorphic to $2C_{\frac{n}{2}}(1, 2, 3, 4)$, is supermagic by Corollary 3.3 and Proposition 2.3. Therefore, according to Proposition 2.2 the graph G_1 is supermagic.

For $r = 8k+7$, $k > 1$, $n > 8(k+1)$, let us consider the circulant graph $G := C_n(1, 2, 3, 4, 5, 6, 8, 9, \ldots, 8(k+1), \frac{n}{2})$. The graph G is decomposable into G_1 (from the previous paragraph) and $G_2 := C_n(9, 10, \ldots, 8(k+1))$. Since G_2 is a supermagic graph by Corollary 3.4, the graph G is also supermagic by Proposition 2.2.

C. Suppose condition (*iii*) holds.

C.1. For $r = 8k$, let us consider the circulant graph $C_n(1, 2, \ldots, 4k)$ which is supermagic by Corollary 3.4.

C.2. For $r = 8k + 2$ and $n = 8k + 3$ we have the complete graph K_n which is supermagic due Proposition 2.5.

The unique graph for $r = 8k + 2$ and $n = 8k + 4$ is the complete $(4k + 2)$-partite graph $K_{4k+2[2]}$ which is supermagic due Proposition 2.6.

For $r = 10$ and $n \geq 13$, let us consider the circulant graph $C_n(1, 2, 3, 4, 6)$ which is supermagic by Theorem 3.7.

Finally, for $r = 8k + 2$, $k > 1$, $n \geq 8k + 5$, let us consider the circulant graph $G := C_n(1, 2, 3, 4, 6, 7, \ldots, 4k + 2)$ which is decomposable into two edge-disjoint factors $C_n(1, 2, 3, 4, 6)$ and $C_n(7, 8, \ldots, 4k + 2)$. These factors are supermagic by Theorem 3.7 and Corollary 3.4. Therefore, G is supermagic due Proposition 2.2.

C.3. For $r = 8k + 4$ and $n = 8k + 5 > 5$ we have the complete graph K_n which is supermagic due Proposition 2.5.

. The only graph for $r = 8k + 4$ and $n = 8k + 6$ is the complete $(4k + 3)$-partite graph $K_{4k+3[2]}$ which is supermagic due Proposition 2.6.

For $r = 4$ and $n \geq 7$, let us consider the circulant graph $C_n(1, 3)$ which is supermagic by Theorem 3.6.

For $r = 8k + 4$, $k \geq 1$, $n \geq 8k + 7$, let us consider the circulant graph $G := C_n(1, 3, 4, \ldots, 4k + 3)$ which is decomposable into two edge-disjoint factors $C_n(1, 3)$ and $C_n(4, 5, \ldots, 4k + 3)$. These factors are supermagic by Theorem 3.6 and Corollary 3.4. Therefore, G is supermagic due Proposition 2.2.

C.4. For $r = 6$ and $n \geq 7$, we consider the circulant graph $C_n(1, 2, 3)$ which is supermagic by Theorem 3.5.

Finally, for $r = 8k + 6 > 6$ and $n > r$ we can construct an r-regular circulant graph $C_n(1, 2, \ldots, 4k + 3)$ on n vertices by Corollary 3.4 and Proposition 2.2, from circulant graphs $C_n(1, 2, 3)$ and $C_n(4, 5, \ldots, 4k + 3)$.

Therefore, there is a supermagic r-regular circulant graph of order n for an arbitrary pair n, r satisfying (i), (ii) or (iii). This completes the proof.

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