Locating-Domination in Complementary Prisms

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Abstract

Let $G = (V, E)$ be a graph and $\overline{G}$ be the complement of G. The complementary prism of G, denoted $G\overline{G}$, is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. A set $D \subseteq V(G)$ is a locating-dominating set of G if for every $u \in V(G)\backslash D$, its neighborhood $N(u) \cap D$ is nonempty and distinct from $N(v) \cap D$ for all $v \in V(G)\backslash D$ where $v \neq u$. The locating-domination number of G is the minimum cardinality of a locating-dominating set of G. In this paper, we study locating-domination of complementary prisms. We determine the locating-domination number of $G\overline{G}$ for specific graphs G and characterize the complementary prisms with small locating-domination numbers. We also present upper and lower bounds on the locating-domination numbers of complementary prisms, and we show that all values between these bounds are achievable.

Keywords: graph product, cartesian product, complementary prism, locating-domination number, complementary product.

1 Introduction

Complementary products, introduced in [2], are generalizations of cartesian products. Some of the most interesting and well-studied problems in domination involve determining domination invariants of cartesian products [4, 5]. When studying invariants of graph products, the standard question is what can be determined about the invariant of the graph product if the

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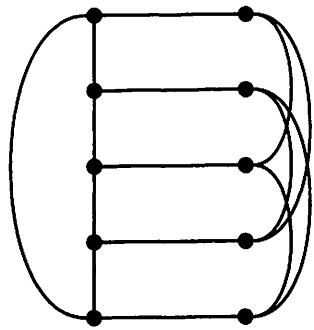

Figure 1: The Petersen Graph $C_5\overline{C}_5$

value of the invariant for the individual factors is known. Accordingly, in this paper, we investigate locating-domination in complementary products.

We restrict our attention to a subset of the complementary products, called complementary prisms, and refer the reader to [2] for the more general case. For a graph G, the *complementary prism*, denoted $G\overline{G}$, is formed from the disjoint union of G and its complement $\overline{G}$ by adding a perfect matching between corresponding vertices of G and $\overline{G}$. For each $v \in V(G)$, let $\overline{v}$ denote the vertex v in the copy of $\overline{G}$. Formally, the graph $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$. We note that complementary prisms are a generalization of the Petersen Graph. In particular, the Petersen graph is the complementary prism $C_5\overline{C}_5$. See Figure 1.

For any graph $G = (V, E)$ and a vertex $v \in V(G)$, the *open neighborhood* of v is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, its *open neighborhood* $N(S) = \cup_{v \in S} N(v)$, and its *closed neighborhood* $N[S] = N(S) \cup S$. A set S is a *dominating set* if $N[S] = V$ and is a *total dominating set* if $N(S) = V$. The minimum cardinality of any dominating set (respectively, total dominating set) of G is the *domination number* $\gamma(G)$ (respectively, *total domination number* $\gamma_t(G)$). For more details on domination, see the book [1].

The need to uniquely identify one vertex from another is the motivation for studying locating-domination in graphs. This concept was first introduced by Slater [6]. A set $L \subseteq V(G)$ is a *locating-dominating set* (abbreviated LDS) of G, if for every $u \in V(G)\backslash L$, its neighborhood $N(u) \cap L$ is nonempty and distinct from $N(v) \cap L$ for all $v \in V(G)\backslash L$ where $v \neq u$.

The *locating-domination number* of G, denoted $\gamma_L(G)$, is the minimum cardinality of a locating-dominating set of G. An LDS of G with cardinality $\gamma_L(G)$ is called a *$\gamma_L(G)$-set*. See Figure 2 for an example of an LDS for the path P_6, where the darkened vertices represent the $\gamma_L(G)$-set, L. Notice that $N(v_2) \cap L = \{v_1\}, N(v_3) \cap L = \{v_4\}$ and $N(v_5) \cap L = \{v_4, v_6\}$, so each of the vertices, v_2, v_3, and v_5 have unique neighborhoods $V(G) \cap L$. If a set L locating-dominates a set X, then we denote this as $L \succ_L X$.

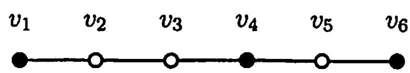

Figure 2: Locating-Dominating Set in P_6

Domination and total domination numbers of complementary prisms were studied in [2] and [3]. In these papers, bounds on the (total) domination numbers of $G\overline{G}$ were given in terms of the (total) domination numbers of G and $\overline{G}$. Here, we study locating-domination in complementary prisms.

As examples, we determine the value of $\gamma_L(G\overline{G})$ for some specific graphs G and present characterizations of the complementary prisms with small locating-domination numbers in Section 2. In Section 3, we present bounds on the locating-domination number of $G\overline{G}$, and we show that all possible values of $\gamma_L(G\overline{G})$ between the lower and upper bounds are attainable.

We will use the following terminology. The degree of a vertex v is $deg_G(v) = |N(v)|$. A vertex of degree 0 is an *isolated vertex*. A vertex of degree one is called a *leaf* or an *endvertex*, and its neighbor is called a *support vertex*. For any $S \subseteq V(G)$, the subgraph of G induced by S is denoted $\langle S \rangle$. For ease of discussion, we refer to the complementary prism $G\overline{G}$ as a copy of G and a copy of $\overline{G}$ with a perfect matching between corresponding vertices. For a set $P \subseteq V(G)$, let $\overline{P}$ be the corresponding set of vertices in $V(\overline{G})$.

2 Examples

To illustrate locating-domination in complementary prisms, we determine $\gamma_L(G\overline{G})$ where G is a complete graph and a complete bipartite graph. Further, we characterize the complementary prisms having small locating-domination numbers.

First, we determine the locating-domination number of $G\overline{G}$ when G is a complete graph. The *corona* of a graph G, denoted $G \circ K_1$, is formed from G by adding for each $v \in V(G)$, a new vertex v' and the pendant edge vv'.

Proposition 1 *If G is the complete graph K_n, then $\gamma_L(G\overline{G}) = n$.*

Proof. For $G = K_n$, the complementary prism $G\overline{G}$ is the corona $K_n \circ K_1$. Any $\gamma_L(G\overline{G})$-set must contain each leaf or its support vertex. Therefore $\gamma_L(G\overline{G}) \geq n$. The set of leaves forms an LDS, so $\gamma_L(G\overline{G}) \leq n$. Hence $\gamma_L(G\overline{G}) = n$. □

Next we determine the locating-domination number of $G\overline{G}$ when G is a complete bipartite graph.

Proposition 2 *Let G be the complete bipartite graph $K_{r,s}$, where $r+s = n$ and $1 \leq r \leq s$.*

$$\gamma_L(G\overline{G}) = \begin{cases} n, & if\ r = 1 \\ n-1, & if\ r = 2 \\ n-2, & otherwise. \end{cases}$$

Proof. Let $G = K_{r,s}$, $1 \leq r \leq s$, where R and S are the bipartite sets of G with cardinality r and s, respectively. Let $R = \{x_1, x_2, ..., x_r\}$ and $S = \{y_1, y_2, ..., y_s\}$. Let L be a $\gamma_L(G\overline{G})$-set.

First let $r = 1$, that is, $G = K_{1,s}$, $1 \leq s$. Clearly $V(G)$ is an LDS of $G\overline{G}$, so $\gamma_L(G\overline{G}) \leq n$. To see that $\gamma_L(G\overline{G}) \geq n$, note that $\overline{x}_1$ is a leaf in $G\overline{G}$. This implies that at least one of x_1 and $\overline{x}_1$ is in L. If $x_1 \in L$, then x_1 locating-dominates at most one of its neighbors. Thus, there are $n-1$ vertices in $N_{G\overline{G}}(x_1)$ that must either be in L or have another neighbor in L. Hence, $\gamma_L(G\overline{G}) \geq 1 + n - 1 = n$. If $x_1 \notin L$, then $\overline{x}_1 \in L$. This implies that at least one of y_i and $\overline{y}_i$ is in L to dominate y_i, $1 \leq i \leq n-1$. And again $\gamma_L(G\overline{G}) \geq n$. Thus, if $G = K_{1,s}$, $\gamma_L(G\overline{G}) = n$.

Now assume that $2 \leq r \leq s$. We first show that $|L \cap (S \cup \overline{S})| \geq s-1$. Assume that there are two vertices in S, say y_i and y_j, such that none of y_i, y_j, $\overline{y}_i$, and $\overline{y}_j$ are in L. Then $N_{G\overline{G}}(y_i) \cap L = N_G(y_i) \cap L = R \cap L = N_{G\overline{G}}(y_j) \cap L$. Thus, there exists at most one vertex, $y_i \in S$ such that y_i and $\overline{y}_i$ are in $V(G)\backslash L$. This implies that $|L \cap (S \cup \overline{S})| \geq s-1$ as desired.

We consider two cases.

Case 1. $2 = r \le s$. To show that $\gamma_L(G\overline{G}) \le n-1$, we note that $R \cup (\overline{S} \setminus \{\overline{y}_1, \overline{y}_2\}) \cup \{y_1\}$ is an LDS for $G\overline{G}$. To see this, notice that $N_{G\overline{G}}(\overline{x}_i) \cap L = \{x_i\}$, for $i \in \{1, 2\}$. Also $N_{G\overline{G}}(y_2) \cap L = \{x_1, x_2\}$, $N_{G\overline{G}}(\overline{y}_1) \cap L = \{y_1, \overline{y}_i \mid i \ge 3\}$, $N_{G\overline{G}}(\overline{y}_2) \cap L = \{\overline{y}_i \mid i \ge 3\}$. For $i \ge 3$, $N_{G\overline{G}}(y_i) \cap L = \{x_1, x_2, \overline{y}_i\}$. Thus, every vertex in $V(G\overline{G})\setminus L$ is locating-dominated by L. Hence $\gamma_L(G\overline{G}) \le |R| + |S| - 2 + 1 = r + s - 1 = n - 1$.

Next we show that $\gamma_L(G\overline{G}) \ge n - 1 = s + 1$. We have shown that $|L \cap (S \cup \overline{S})| \ge s - 1$. Assume to the contrary, that $\gamma_L(G\overline{G}) \le s$. Hence $|L \cap (R \cup \overline{R})| = 1$. Without loss of generality, either $L \cap (R \cup \overline{R}) = \{x_1\}$ or $L \cap (R \cup \overline{R}) = \{\overline{x}_1\}$. In the former, $\overline{x}_2$ is not dominated by L, a contradiction. In the later, at least one vertex from S is not dominated by L, a contradiction. And so, $\gamma_L(G\overline{G}) \ge s + 1 = n - 1$.

Case 2. $3 \le r \le s$. We show that $(\overline{R}\setminus\{\overline{x}_1, \overline{x}_2\}) \cup (\overline{S}\setminus\{\overline{y}_1, \overline{y}_2\}) \cup \{x_1, y_1\}$ is an LDS of $G\overline{G}$. To see this, notice that $N_{G\overline{G}}(x_2) \cap L = \{y_1\}$, $N_{G\overline{G}}(y_2) \cap L = \{x_1\}$, $N_{G\overline{G}}(\overline{x}_1) \cap L = \{x_1, \overline{x}_i \mid i \ge 3\}$, $N_{G\overline{G}}(\overline{x}_2) \cap L = \{\overline{x}_i \mid i \ge 3\}$, $N_{G\overline{G}}(\overline{y}_1) \cap L = \{y_1, \overline{y}_i \mid i \ge 3\}$, $N_{G\overline{G}}(\overline{y}_2) \cap L = \{\overline{y}_i \mid i \ge 3\}$. And for $i \ge 3$, $N_{G\overline{G}}(x_i) \cap L = \{y_1, \overline{x}_i\}$, and $N_{G\overline{G}}(y_i) \cap L = \{x_1, \overline{y}_i\}$. Thus, every vertex in $V(G\overline{G})\setminus L$ is locating-dominated by L. Hence, $\gamma(G\overline{G}) \le |R| - 2 + 2 + |S| - 2 = r + s - 2 = n - 2$.

We have shown $|L \cap (S \cup \overline{S})| \ge s - 1$. A similar argument for $R \cup \overline{R}$ will lead to $|L \cap (R \cup \overline{R})| \ge r - 1$. Thus, $\gamma_L(G\overline{G}) \ge s - 1 + r - 1 = r + s - 2 = n - 2$. □

Next we consider complementary prisms with small locating-domination numbers.

Proposition 3 *For a graph G of order n and its complementary prism $G\overline{G}$,*

(1) $\gamma_L(G\overline{G}) = 1$ *if and only if* $n = 1$.

(2) $\gamma_L(G\overline{G}) = 2$ *if and only if* $n = 2$.

(3) $\gamma_L(G\overline{G}) = 3$ *if and only if* $n \in \{3, 4\}$ *and* $G \notin \{K_4, \overline{K}_4, K_{1,3}, \overline{K}_{1,3}\}$.

Proof. (1) If $|V(G)| = 1$, then $G\overline{G} = K_2$. Thus, $\gamma_L(G\overline{G}) = 1$. Now assume that $\gamma_L(G\overline{G}) = 1$, and without loss of generality, L is a $\gamma_L(G\overline{G})$-set and $L \subseteq V(G)$. Since L must locating-dominate $\overline{G}$ in $G\overline{G}$, it follows that $|V(\overline{G})| = 1$ and $G = K_1$.

(2) If $|V(G)| = 2$, then $G \in \{K_2, \overline{K}_2\}$ so $G\overline{G} = P_4$ and $\gamma_L(G\overline{G}) = 2$.

Assume that $\gamma_L(G\overline{G}) = 2$, and let L be a $\gamma_L(G\overline{G})$-set. If $L \subseteq V(G)$, then since L must dominate $\overline{G}$, it follows that $|V(G)| = 2$ and so $G\overline{G} = P_4$. Now assume $L \cap V(G) = 1$ and $L \cap V(\overline{G}) = 1$. Without loss of generality, let $L = \{x, \overline{y}\}$.

If $\overline{y} = \overline{x}$, then $\{x\} \succ_L V(G)\backslash\{x\}$ and $\{\overline{x}\} \succ_L V(\overline{G})\backslash\{\overline{x}\}$. Let $w \in V(G)\backslash\{x\}$. Then w is adjacent to x and $\overline{w}$ is adjacent to $\overline{x}$, a contradiction. Thus, $V(G)\backslash\{x\} = \emptyset$, that is, $|V(G)| = 1$. Then $\gamma_L(G\overline{G}) = 1$, a contradiction.

Hence, $\overline{y} \neq \overline{x}$. Then $x \succ_L V(G)\backslash\{x, y\}$ and $\overline{y} \succ_L V(\overline{G})\backslash\{\overline{x}, \overline{y}\}$. Without loss of generality, we may assume that $xy \in E(G\overline{G})$ and $\overline{xy} \notin E(G\overline{G})$. Let $w \in V(G)\backslash\{x, y\}$. Then $N_{G\overline{G}}(w) \cap L = \{x\} = N_{G\overline{G}}(\overline{x}) \cap L$, contradicting that S is an LDS of $G\overline{G}$. Hence $V(G)\backslash\{x, y\} = \emptyset$, that is, $|V(G)| = 2$.

(3) Let $n \in \{3, 4\}$. By (2), $\gamma_L(G\overline{G}) \geq 3$. If $n = 3$, then $V(G)$ is an LDS of $G\overline{G}$, so $\gamma_L(G\overline{G}) \leq 3$ and hence $\gamma_L(G\overline{G}) = 3$. If $n = 4$, then again $V(G)$ is an LDS of $G\overline{G}$, so $\gamma_L(G\overline{G}) \leq 4$. If $G \in \{K_4, \overline{K}_4, K_{1,3}, \overline{K}_{1,3}\}$, then by Propositions 1 and 2, $\gamma_L(G\overline{G}) = 4$. So assume $G \notin \{K_4, \overline{K}_4, K_{1,3}, \overline{K}_{1,3}\}$.

Figure 3 illustrates an LDS of $G\overline{G}$ for all remaining non-isomorphic graphs G of order four. The darkened vertices represent the LDS. Since each has an LDS of cardinality three, $\gamma_L(G\overline{G}) \leq 3$. Hence for these graphs, $\gamma_L(G\overline{G}) = 3$.

Again by Propositions 1 and 2, for $G \in \{K_4, \overline{K}_4, K_{1,3}, \overline{K}_{1,3}\}$, $\gamma_L(G\overline{G}) = 4$. Assume that G is a graph of order n such that $\gamma_L(G\overline{G}) = 3$. We only need to show that $n \in \{3, 4\}$. Clearly $n \geq 3$ by part (2) of this proof. Let L be a $\gamma_L(G\overline{G})$-set. If $L \subseteq V(G)$ or $L \subseteq V(\overline{G})$, then it follows that $n = 3$. Hence assume that $L \cap V(G) \neq \emptyset$ and $L \cap V(\overline{G}) \neq \emptyset$. Without loss of generality, let $L = \{x, y, \overline{z}\}$.

Assume first that $\overline{z} \in \{\overline{x}, \overline{y}\}$, and without loss of generality, $\overline{z} = \overline{x}$. Then $\{\overline{x}\} \succ_L V(\overline{G})\backslash\{\overline{x}, \overline{y}\}$ in $G\overline{G}$, implying that there is at most one vertex in $V(\overline{G})\backslash\{\overline{x}, \overline{y}\}$ in $G\overline{G}$. Hence $n = 3$.

Assume that $\overline{z} \notin \{\overline{x}, \overline{y}\}$. Thus, $\{\overline{z}\} \succ_L V(\overline{G})\backslash\{\overline{x}, \overline{y}, \overline{z}\}$. This implies that there is at most one vertex in $V(\overline{G})\backslash\{\overline{x}, \overline{y}, \overline{z}\}$. Hence, $n \leq 4$. □

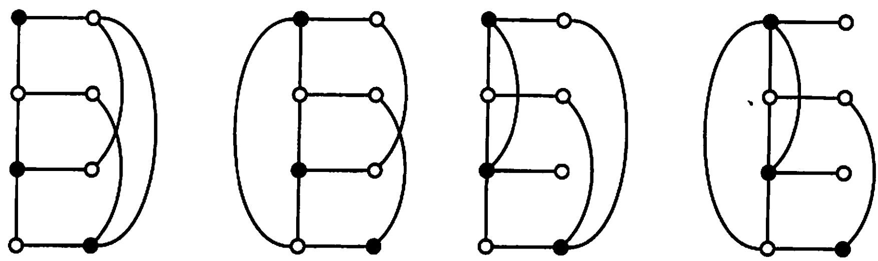

Figure 3: LDS of $G\overline{G}$ when $n = 4$ and $G \notin \{K_4,\ \overline{K}_4,\ K_{1,3},\ \overline{K}_{1,3}\}$

3 Bounds and Realizability

We give lower and upper bounds on $\gamma_L(G\overline{G})$ in terms of $\gamma_L(G)$ and $\gamma_L(\overline{G})$.

Proposition 4 *For any graph G,* $\max\{\gamma_L(G),\ \gamma_L(\overline{G})\} \le \gamma_L(G\overline{G}) \le \gamma_L(G)+\gamma_L(\overline{G})$.

Proof. By Proposition 1, if $G = K_n$, then $\max\{\gamma_L(G),\ \gamma_L(\overline{G})\} = n = \gamma_L(G\overline{G}) \le 2n-1 = \gamma_L(G)+\gamma_L(\overline{G})$. Thus, we may assume G is not complete. Let D be a $\gamma_L(G\overline{G})$-set, and let $D_1 = D \cap V(G)$ and $D_2 = D \cap V(\overline{G})$. Assume, without loss of generality, that $\gamma_L(G) \ge \gamma_L(\overline{G})$. If D_1 locating-dominates G, then we are finished. So assume there exists a set $T \subseteq V(G)$ such that T is not locating-dominated by D_1. Thus, T is located and/or dominated by D_2. Also, each vertex in D_2 is adjacent to at most one vertex in T. Thus, $|T| \le |D_2|$. But $D_1 \cup T$ is a locating-dominating set of G. So $\gamma_L(G) \le |D_1 \cup T| = |D_1| + |T| \le |D_1| + |D_2| = |D| = \gamma_L(G\overline{G})$.

For the upper bound, let S_1 be a $\gamma_L(G)$-set and S_2 be a $\gamma_L(\overline{G})$-set, and $S = S_1 \cup S_2$. Also, let $x \in V(G)\backslash S_1$ and $\overline{y} \in V(\overline{G})\backslash S_2$. Then,

$$N_{G\overline{G}}(x) = \begin{cases} N_G(x) \cap S_1 \cup \{\overline{x}\}, & if\ \overline{x} \in S_2 \\ N_G(x) \cap S_1, & otherwise \end{cases}, and$$

$$N_{G\overline{G}}(\overline{y}) = \begin{cases} N_{\overline{G}}(\overline{y}) \cap S_2 \cup \{y\}, & if\ y \in S_1 \\ N_{\overline{G}}(\overline{y}) \cap S_2, & otherwise. \end{cases}$$

Since S_1 and S_2 locating-dominate G and $\overline{G}$, respectively, it follows that S is an LDS of $G\overline{G}$, Hence $\gamma_L(G\overline{G}) \le |S| = |S_1| + |S_2| = \gamma_L(G) + \gamma_L(\overline{G})$. □

Let $a = \gamma(G) \geq \gamma(\overline{G})$. By Proposition 4, $\gamma_L(G\overline{G}) \leq 2a$. We have not been able to find a family of graphs that achieve this upper bound, but we note that for $G = P_5$, $\gamma_L(G) = 2 \geq \gamma_L(\overline{G})$ and $\gamma_L(G\overline{G}) = 4$ attaining the bound. Moreover, we conclude this paper by showing that there exists a graph G with $\gamma_L(G) = a$ and $\gamma_L(G\overline{G}) = b$ for all integers a and b such that $a \leq b \leq 2a - 1$.

Proposition 5 *For positive integers a and b such that $a \leq b \leq 2a - 1$, there exists a graph G such that $\gamma_L(\overline{G}) \leq \gamma_L(G) = a$ and $\gamma_L(G\overline{G}) = b$.*

Proof. Given a and b such that $a \leq b \leq 2a - 1$, we give a graph G with $\gamma_L(G) = a$ and $\gamma_L(G\overline{G}) = b$. Let $m = b-a+1$ and $G = (a-m)K_1 \cup mK_2$. Clearly, $\gamma_L(\overline{G}) \leq \gamma_L(G) = a$. We wish to show that $\gamma_L(G\overline{G}) = b = m+a-1$.

To aid in our discussion, we label the vertices of G as follows. Let $Z = \{z_i \,|\, 1 \leq i \leq a - m\}$ be the set of isolates. Let the m edges be labeled x_iy_i for $1 \leq i \leq m$, and let $X = \{x_i \,|\, 1 \leq i \leq m\}$ and $Y = \{y_i \,|\, 1 \leq i \leq m\}$. Note that for each i where $1 \leq i \leq m$, $N_{G\overline{G}}[x_i] = \{x_i,\ y_i,\ \overline{x}_i\}$, $N_{G\overline{G}}[y_i] = \{x_i,\ y_i,\ \overline{y}_i\}$, $N_{G\overline{G}}[\overline{x}_i] = \{x_i\} \cup \overline{X} \cup \overline{Z} \cup \overline{Y} \setminus \{\overline{y}_i\}$, and $N_{G\overline{G}}[\overline{y}_i] = \{y_i\} \cup \overline{Y} \cup \overline{Z} \cup \overline{X} \setminus \{\overline{x}_i\}$. Also, for $1 \leq i \leq a - m$, $N_{G\overline{G}}[z_i] = \{z_i, \overline{z}_i\}$ and $N_{G\overline{G}}[\overline{z}_i] = \overline{X} \cup \overline{Y} \cup \overline{Z} \cup \{z_i\}$.

Let $L = Z \cup X \cup \overline{Y} \setminus \{\overline{y}_1\}$. Then $N_{G\overline{G}}(y_1) \cap L = \{x_1\}$. For $2 \leq i \leq m$, $N_{G\overline{G}}(y_i) \cap L = \{x_i,\ \overline{y}_i\}$ and $N_{G\overline{G}}(\overline{x}_i) \cap L = \{x_i\} \cup \overline{Y} \setminus \{\overline{y}_1, \overline{y}_i\}$. And for $1 \leq i \leq a-m$, $N_{G\overline{G}}(\overline{z}_i) \cap L = \{z_i\} \cup \overline{Y} \setminus \{\overline{y}_1\}$. Finally, $N_{G\overline{G}}(\overline{y}_1) \cap L = \overline{Y} \setminus \{\overline{y}_1\}$ and $N_{G\overline{G}}(\overline{x}_1) \cap L = \{x_1\} \cup \overline{Y} \setminus \{\overline{y}_1\}$. Since every vertex in $V(G\overline{G}) \setminus L$ has a unique non-empty neighborhood in L, L is an LDS of $G\overline{G}$. Thus, $\gamma_L(G\overline{G}) \leq |L| = a - m + m + m - 1 = a + m - 1 = b$.

Let L be a $\gamma_L(G\overline{G})$-set. First note that $|L \cap (Z \cup \overline{Z})| \geq |Z| = a - m$ in order to dominate Z. Note also that to dominate x_i, $|L \cap \{x_i,\ y_i,\ \overline{x}_i\}| \geq 1$, and to dominate y_i, $|L \cap \{x_i,\ y_i,\ \overline{y}_i\}| \geq 1$.

To show that $\gamma_L(G\overline{G}) \geq b$, it suffices to show that there exists at most one i such that $|L \cap \{x_i,\ y_i,\ \overline{x}_i,\ \overline{y}_i\}| = 1$. For the purposes of contradiction, assume that $|L \cap \{x_i,\ y_i,\ \overline{x}_i,\ \overline{y}_i\}| = 1$ and $|L \cap \{x_j,\ y_j,\ \overline{x}_j,\ \overline{y}_j\}| = 1$ for some $i \neq j$. Since $|L \cap \{x_i,\ y_i,\ \overline{x}_i\}| \geq 1$ and $|L \cap \{x_i,\ y_i,\ \overline{y}_i\}| \geq 1$, it follows that $L \cap \{x_i,\ y_i,\ \overline{x}_i,\ \overline{y}_i\} \subset \{x_i, y_i\}$. Similarly, $L \cap \{x_j,\ y_j,\ \overline{x}_j,\ \overline{y}_j\}| \subset \{x_j, y_j\}$. Without loss of generality, assume that $L \cap \{x_i,\ y_i,\ \overline{x}_i,\ \overline{y}_i\} = \{x_i\}$. Then $N_{G\overline{G}}(\overline{y}_i) \cap L = V(\overline{G}) \cap L$. Moreover, $V(\overline{G}) \cap L \neq \emptyset$ because $\overline{y}_i$ is dominated by L. Since neither $\overline{x}_j$ nor $\overline{y}_j$ is in L and exactly one of x_j and y_j is in L, it follows that $N_{G\overline{G}}(\overline{y}_j) \cap L = L \cap V(\overline{G})$ or $N_{G\overline{G}}(\overline{x}_j) \cap L = L \cap V(\overline{G})$. Hence $N_{G\overline{G}}(\overline{y}_i) \cap L = N_{G\overline{G}}(\overline{y}_j) \cap L$ or $N_{G\overline{G}}(\overline{y}_i) \cap L = N_{G\overline{G}}(\overline{x}_j) \cap L$, contradicting that L is an LDS.

Hence $|L \cap \{x_i, y_i, \overline{x}_i, \overline{y}_i\}| \geq 2$ for $1 \leq i \leq m$, except for possibly one j where $|L \cap \{x_j, y_j, \overline{x}_j, \overline{y}_j\}| = 1$. Thus, $\gamma_L(G\overline{G}) = |L| \geq 2(m-1)+1+a-m = 2m-2+1+a-m = m+a-1 = b$. Hence $\gamma_L(G\overline{G}) = b$, completing the proof. □

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