# Representations for complete graphs minus a disjoint union of paths

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**Abstract**

A graph $G$ has a representation modulo $n$ if there exists an injective map $f : V(G) \rightarrow \{0, 1, ..., n-1\}$ such that vertices $u$ and $v$ are adjacent if and only if $f(u) - f(v)$ is relatively prime to $n$. The representation number $rep(G)$ is the smallest $n$ such that $G$ has a representation modulo $n$. In 2000 Evans, Isaak, and Narayan determined the representation number of a complete graph minus a path. In this paper we refine their methods and apply them to the family of complete graphs minus a disjoint union of paths.

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## 1 Introduction

Let $G$ be a finite graph with vertices $\{v_1, \ldots, v_r\}$. A *representation* of $G$ modulo $n$ is an assignment of distinct labels to the vertices such that the label $a_i$ assigned to vertex $v_i$ is in $\{0, 1, \ldots, n-1\}$ and such that $a_i - a_j$ and $n$ are relatively prime if and only if $v_i$ and $v_j$ are adjacent. We call $\{a_1, \ldots, a_r\}$ a *representation* of $G$ modulo $n$. Erdős and Evans [3] showed that every finite graph can be represented modulo some positive integer. This result was used to give a simpler proof of a result of Lindner, Mendelsohn, Mendelsohn, and Wolk [9] that any finite graph can be realized as an orthogonal Latin square graph. The representation number of a graph $G$, $rep(G)$, is the smallest $n$ such that $G$ has a representation modulo $n$.

Modular representations have appeared in several recent publications. As part of an existence proof, Erdős and Evans [3] established a general upper bound for the representation number of a graph. Narayan [10] later refined this bound by proving that a graph of order $r > 1$ can be represented modulo a positive integer less than or equal to the product of the first $r-1$ primes greater than or equal to $r-1$, and this bound was shown to be the best possible.

The determination of $rep(G)$ for an arbitrary graph $G$ is a very difficult problem indeed. Evans, Isaak, and Narayan [6] showed that the determination of representation numbers for many disjoint unions of complete graphs is dependent upon the existence of sets of mutually orthogonal Latin squares. Recently Evans [4] used linked matrices and distance covering matrices to obtain new results involving representation numbers for the disjoint union of complete graphs.

Only a few papers have been written on graph representations. The initial theory was developed in [5]. In [13] the question of how many prime factors, counting multiplicity, $n$ must have for a given a graph $G$ to be representable modulo $n$ is partially answered in terms of a type of edge labeling of the complement of $G$. The closely related concept of the dimension of a graph is extensively studied in [8] and [12]. A survey of the tools used to work on graph representations, as well as several results, can be found in [7].

Evans, Isaak, and Narayan determined the representation number of a complete graph minus a path [6]. In this paper we refine their methods and apply them to the family of complete graphs minus a disjoint union of paths. Note that in this family of graphs the complement $\overline{G}$ is disjoint union of paths along with union a set of isolated vertices.

# 2 Background

The problem that has received the most interest is the following suggested in [3].

**Problem 1** *Given a graph $G$, determine rep($G$). More generally, given a class of graphs, determine rep($G$) for each graph $G$ in this class.*

As an example, consider the graph $G$ in Figure 1. The reader can easily verify that the labels $\{0,1,2,5,7\}$, assigned to the vertices of $G$, form a representation of $G$ modulo 105. Thus rep($G$) $\leq 105$. Equality will follow from Theorem 4 and Lemma 6.

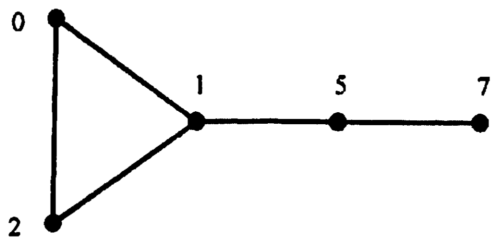


Figure 1. A representation modulo 105.

We restate a short list of known results that characterize graphs representable modulo certain types of integers.

**Theorem 2** *A graph is representable modulo a prime if and only if it is a complete graph.*

**Theorem 3** *A graph is representable modulo a power of a prime if and only if it is a complete multipartite graph.*

**Theorem 4** *If $G$ is representable modulo $n$, and $p$ is a prime divisor of $n$ then $p \geq \chi(G)$.*

We have the following corollary where $\omega(G)$ is the size of the largest complete subgraph in $G$.

**Corollary 5** *If $G$ is representable modulo $n$, and $p$ is a prime divisor of $n$ then $p \geq \omega(G)$.*

We restate Corollary 2.12 from Evans, Isaak, and Narayan [6] in the following lemma Here $p_{i+1}$ is the smallest prime greater than the prime $p_i$.

**Lemma 6** *If $G$ contains a $K_m + K_1$ and $p_i$ is the smallest prime satisfying $p_i \geq \chi(G)$ then $rep(G) \geq p_i p_{i+1} \cdots p_i + m - 1$.*

# 3 Complete graphs minus disjoint paths

In this section we investigate a complete graph minus a disjoint union of paths $K_m - P_{n_1} - P_{n_2} - \cdots - P_{n_r}$. As noted in [6] the lower bound for $K_m - P_l$ follows from Corollary 5 and Lemma 6. Note that if $n_1 + n_1 + \cdots n_r = n$, $\omega(K_m - P_{n_1} - P_{n_2} - \cdots P_{n_r}) \geq \omega(K_m - P_n)$. Hence our approach for lower bounds will essentially be the same in the case for $K_m - P_n$. In

[6] the upper bounds for $rep(K_m - P_n)$ were obtained by constructing an explicit representation using a table of coordinates. By modifying this table we are able to obtain upper bounds for certain graphs in the family $K_m - P_{n_1} - P_{n_2} - \cdots - P_{n_r}$. We demonstrate this refined approach in the following example.

**Example 7** *Using the methods of [6] (inverting the rows) we present a coordinate representation for $G = K_6 - P_5$. We note that since $G$ contains a complete subgraph of size $6 - 5 + \lceil \frac{5}{2} \rceil = 4$, $p_i = 5$ and $p_{i+1} = 7$.*

| $a_i/p_j$ | 5 | 7 |
|---|---|---|
| $a_1$ | 0 | 0 |
| $a_2$ | 1 | 1 |
| $a_3$ | 1 | 2 |
| $a_4$ | 2 | 2 |
| $a_5$ | 2 | 3 |
| $a_6$ | 3 | 3 |

*We modify the last two labels to obtain a representation for the graph $G = K_6 - P_3 - P_2$.*

| $a_i/p_j$ | 5 | 7 |
|---|---|---|
| $a_1$ | 0 | 0 |
| $a_2$ | 1 | 1 |
| $a_3$ | 1 | 2 |
| $a_4$ | 2 | 2 |
| $a_5$ | 3 | 3 |
| $a_6$ | 3 | 4 |

We start by examining the case of a complete graph minus two disjoint paths.

**Theorem 8** *Let $m \geq n \geq 3$ and $m - n + \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$ where $n = n_1 + n_2$. If $k$ is not prime then $rep(K_m - P_{n_1} - P_{n_2}) = p_i p_{i+1}$ where $p_i$ is the smallest prime greater than or equal to $k$.*

**Proof.** When $n \geq 3$, $K_m - P_{n_1} - P_{n_2}$ contains $K_2 + K_1$ and a complete graph of size $m - n + \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil$. So by Lemma 6, we have $rep(K_m - P_{n_1} - P_{n_2}) \geq p_i p_{i+1}$.

Next we show that $K_m - P_{n_1} - P_{n_2}$ is representable modulo $p_i p_{i+1}$. We will start with a coordinate representation $f$ for $K_m - P_n$ and then modify it to form a coordinate representation $g$ for $K_m - P_{n_1} - P_{n_2}$.

We give a coordinate representation $f$ with respect to $p_i$ and $p_{i+1}$ to the vertices $K_m - P_n$ as follows. Let $v_1, ..., v_n$ be the vertices of the removed path. Then we assign coordinates $f(v_i) = (i-1, i-1)$ for $i = 1, ..., m-n$ and $f(v_i) = \left(m-n-1+\lfloor \frac{i}{2} \rfloor, m-n-1+\lceil \frac{i}{2} \rceil\right)$ for $i = 1, ..., n$.

We form a new labeling $g$ as follows. The labels for $g(v_i) = f(v_i)$ for all $i = 1, .., m-n+n_1$. Then $g(v_i) = f(v_{i+1})$ for all $i = m-n+n_1+1, .., m-n+n_1+n_2-1$. Finally we define $g(v_m)$. If $g(v_{m-1}) = (t,t)$ then $g(v_m) = (t, t+1)$ and if $g(v_{m-1}) = (t, t+1)$ then $g(v_m) = (t+1, t+1)$. Then $g$ is a coordinate representation for $K_m - P_{n_1} - P_{n_2}$ modulo $p_ip_{i+1}$. Hence $rep(K_m - P_{n_1} - P_{n_2}) = rep(K_m - P_n) = p_ip_{i+1}$. ■

# 4 Conclusion

Examination of this refinement reveals we can modify the representation coordinate table for $K_m - P_n$ whenever there is 'room' to include the label for $v_m$. That is the label for $v_m$ must have its first coordinate less than $p_i$ and its second coordinate less than $p_{i+1}$. Another possibility is an entirely different set of coordinates where the first coordinates are all less than $p_i$ and the second coordinates are all less than $p_{i+1}$. It would be an interesting problem to determine necessary and sufficient conditions for the coordinates to fit within the table.

It turns out that this method can be generalized to determine representation numbers for all $K_m - P_{n_1} - P_{n_2} - \cdots - P_{n_r}$ where $n_r \geq 3$. We suggest a possible approach for solving the problem. In some cases the representation numbers are the same for various partitions of the removed path.

We illustrate this in the following example.

**Example 9** $rep(K_{20}-P_9) = rep(K_{20}-P_5-P_4) = rep(K_{20}-P_3-P_3-P_3) = rep(K_{20}-P_3-P_2-P_2-P_2) = 17 \cdot 19 = 323$.

Since each of the graphs contains a complete graph on 15 vertices. It follows by Corollary 5 and Lemma 6 that each has a representation number of at least $17 \cdot 19 = 323$. The coordinate representation found below shows $rep(K_{20} - P_9) \leq 323$ and hence $rep(K_{20} - P_9) = 323$.

| $a_i/p_j$ | 17 | 19 |
|---|---|---|
| $a_1$ | 0 | 0 |
| $a_2$ | 1 | 1 |
| $a_3$ | 2 | 2 |
| $a_4$ | 3 | 3 |
| $a_5$ | 4 | 4 |
| $a_6$ | 5 | 5 |
| $a_7$ | 6 | 6 |
| $a_8$ | 7 | 7 |
| $a_9$ | 8 | 8 |
| $a_{10}$ | 9 | 9 |
| $a_{11}$ | 10 | 10 |
| $a_{12}$ | 10 | 11 |
| $a_{13}$ | 11 | 11 |
| $a_{14}$ | 11 | 12 |
| $a_{15}$ | 12 | 12 |
| $a_{16}$ | 12 | 13 |
| $a_{17}$ | 13 | 13 |
| $a_{18}$ | 13 | 14 |
| $a_{19}$ | 14 | 14 |
| $a_{20}$ | 14 | 15 |

Modifying the last eight rows gives a representation for $K_{20} - P_5 - P_4$.

| | | |
|---|---|---|
| $a_{13}$ | 11 | 12 |
| $a_{14}$ | 12 | 12 |
| $a_{15}$ | 12 | 13 |
| $a_{16}$ | 13 | 13 |
| $a_{17}$ | 13 | 14 |
| $a_{18}$ | 14 | 14 |
| $a_{19}$ | 14 | 15 |
| $a_{20}$ | 15 | 15 |

Replacing the last eight rows gives a representation for $K_{20}-P_3-P_3-P_3$.

| | | |
|---|---|---|
| $a_{13}$ | 11 | 12 |
| $a_{14}$ | 12 | 12 |
| $a_{15}$ | 13 | 13 |
| $a_{16}$ | 13 | 14 |
| $a_{17}$ | 14 | 14 |
| $a_{18}$ | 15 | 15 |
| $a_{19}$ | 15 | 16 |
| $a_{20}$ | 16 | 16 |

Replacing the last eight rows gives a representation for $K_{20} - P_3 - P_2 - P_2 - P_2$.

| | | |
|---|---|---|
| $a_{13}$ | 11 | 12 |
| $a_{14}$ | 12 | 12 |
| $a_{15}$ | 13 | 13 |
| $a_{16}$ | 13 | 14 |
| $a_{17}$ | 14 | 15 |
| $a_{18}$ | 15 | 15 |
| $a_{19}$ | 16 | 16 |
| $a_{20}$ | 16 | 17 |

Hence we have $rep(K_{20} - P_9) = rep(K_{20} - P_5 - P_4) = rep(K_{20} - P_3 - P_3 - P_3) = rep(K_{20} - P_3 - P_2 - P_2 - P_2) = 17 \cdot 19 = 323$.

It would be interesting to see if this technique can be used for the general family of complete graphs minus a disjoint union of paths. We conclude with the following question.

**Question:** If $n = n_1 + n_2 + \cdots + n_r$, when is it true that $rep(K_m - P_n) = rep(K_m - P_{n_1} - P_{n_2} - \cdots - P_{n_r})$?

# References


[1] P. Alles, The dimension of sums of graphs, Discrete Math. **54** (1985) 229–233.

[2] B. Bollobás and A. Thomason, Graphs containing all small graphs, European J. Combin. **2** (1981) 13–15.

[3] P. Erdős and A. B. Evans, Representations of graphs and orthogonal Latin square graphs, J. Graph Theory **13** (1989) 593–595.

[4] A. B. Evans, Representations of disjoint unions of complete graphs, Discrete Math. **307** (2007) 1191–1198.

[5] A. B. Evans, G. H. Fricke, C. C. Maneri, T. A. McKee, and M. Perkel, Representations of graphs modulo $n$, J. Graph Theory **18** (1994) 801–815.

[6] A. B. Evans, G. Isaak, and D. A. Narayan, Representations of graphs modulo $n$, Discrete Math. **223** (2000) 109–123.

[7] A. B. Evans, D. A. Narayan, and J. Urick, Representations of graphs modulo $n$: Some problems, Bulletin of the Institute of Combinatorics and its Applications, in press.

[8] L. Lovász, J. Nešetřil, and A. Pultr, On a product dimension of graphs, J. Combinat. Theory B **29** (1980) 47–67.

[9] C. C. Lindner, E. Mendelsohn, N. S. Mendelsohn, and B. Wolk, Orthogonal latin square graphs, J. Graph Theory **3** (1979) 325–338.

[10] D. A. Narayan, An upper bound for representation number of graphs with fixed order, Integers **3** (2003) A12 1-4.

[11] D. A. Narayan and J. Urick, Representations of split graphs, their complements, stars, and hypercubes, Integers **7** (2007) A4, 1-13.

[12] J. Nešetřil and A. Pultr, A Dushnik–Miller type dimension of graphs and its complexity, in "Fundamentals of computation theory, Lecture Notes in Computer Science", vol. **56**, 482–493, Springer–Verlag, Berlin, 1977.

[13] F. C. Silva, Integral representations of graphs, Portugal. Math. **53** (1996) 137–142.