n-Isofactorizations of 8-Regular Circulant Graphs

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Abstract

We investigate the problem of decomposing the edges of a connected circulant graph with n vertices and generating set S into isomorphic subgraphs each having n edges. For 8-regular circulants, we show this is always possible when $s + 2 \leq n/4$ for all edge lengths $s \in S$.

keywords: circulant graph; isomorphic factorization; n-isofactorization, forward edge

1 Introduction

A *circulant graph* $G = \text{CIRC}(n; S)$ is a Cayley graph whose underlying group is $\mathbb{Z}_n$. The edge set $E(G)$ has cardinality $n|S|/2$ and is defined by $\{x, y\} \in E(G) \Leftrightarrow x - y \in S$, where $S \subset \mathbb{Z} \setminus \{0\}$. The set S is called the *generating set* of G, and we require $s \in S \Leftrightarrow -s \in S$. This insures that G is an undirected graph. If we write $S = \{\pm s_1, \pm s_2, \ldots, \pm s_t\}$, then it is easy to see G is a connected graph if and only if $\mathbb{Z}_n$ is generated by S, equivalently, $\text{GCD}(n, s_1, s_2, \ldots, s_t) = 1$. We may assume without loss of generality that $s_i \in \{1, 2, \ldots \lfloor n/2 \rfloor\}$ for $i = 1, \ldots, t$, and write $S = S^+ \dot\cup S^-$ where $S^+ = \{s_i : \ i = 1, 2, \ldots, t\}$ and $S^- = \{-s_i : \ i = 1, 2, \ldots, t\}$. Consequently to simplify terminology, we denote $S = \{\pm s_1, \pm s_2, \ldots, \pm s_t\}$ as simply $S = \pm\{s_1, s_2, \ldots, s_t\}$. We say an edge $\{x, y\}$ is *generated by* s if $x - y = s$ or $x - y = -s$. A subgraph H is generated by s if every edge in H is generated by s. Clearly, if the additive order of s (denoted $\text{ORD}(s)$)

is 2, then s generates a 1-factor (or 1-regular spanning subgraph) and if $\text{ORD}(s) > 2$, s generates a 2-factor of G.

In general, a *Hamilton decomposition* is a partition of the edge set into k Hamilton cycles if the graph is $2k$-regular or k Hamilton cycles and a perfect matching if the graph is $(2k+1)$-regular. Alspach [3] conjectured that every connected $2k$-regular Cayley graph on a finite abelian group admits a Hamilton decomposition. For $k = 1$, the conjecture is trivially true, and Bermond et. al. [6] resolved the conjecture when $k = 2$.

Theorem 1.1 (Bermond, Favaron, Maheo [6]) *A connected 4-regular Cayley graph on a finite abelian group has a Hamilton decomposition.*

Dean considered $k = 3$, for circulant graphs, and proved the following result.

Theorem 1.2 (Dean [7, 8]) *Let* $G = \text{CIRC}(n; S)$ *be a connected 6-regular circulant graph. If either n is odd, or both n is even and some element $s \in S$ exists having order n, then G has a Hamilton decomposition.*

2 General n-isofactorizations

If $F \subseteq E(G)$, where $F \neq \emptyset$ and $E(G)$ denotes the set of edges of a graph G, then the subgraph of G induced by F, denoted $\langle F \rangle$, is defined to be the graph having as vertex set all vertices of G which are incident with at least one edge of F, and whose edge set is F. Given a graph G, a partition of the edge set $E(G)$ into subsets so that

$$E(G) = E_1 \cup E_2 \cup \cdots \cup E_t$$

where $|E_i| = |E_j|$ and $E_i \cap E_j = \emptyset$ for all $i \neq j$ is called an *isomorphic factorization* of G provided the t subgraphs induced on the edge sets $\langle E_i \rangle$, $i = 1, 2, \dots, t$ are pairwise isomorphic. If G has an isomorphic factorization into subgraphs with k edges (each being isomorphic to $\langle E_i \rangle$), we say G has a *k-isofactorization into the subgraph* $\langle E_i \rangle$.

In 1982, Alspach ([2],[3]) conjectured that every connected circulant graph of even valency has a k-isofactorization for all k dividing $|E(G)|$. The following two theorems give some recent results on k-isofactorizations.

Theorem 2.1 (Alspach, Dyer, Kreher [4]) *If $d|n$, then* $\text{CIRC}(n; S)$ *has an* $\frac{n|S|}{2d}$*-isofactorization unless it is the case that n is even, $n/2 \in S$ and n/d is odd.*

Theorem 2.2 (Alspach, Dyer, Kreher [4]) *For a connected circulant graph, $G = \mathrm{CIRC}(n;S)$ of order n, if k divides $|E(G)|$ and either k properly divides n or k divides $|S|$, then there exists a k-isofactorization of G.*

The case when $k = n$ is still vastly unresolved. Our goal is to find an n-isofactorization of $G = \mathrm{CIRC}(n;S)$ into $\frac{|S|}{2}$ subgraphs. We need only consider circulant graphs with even valency, for otherwise $|S|/2$ is not an integer. For odd n, G always has even valency (as $s \not\equiv -s \mod n$ for all $s \in S$), and for even n, we require $n/2 \notin S$, for otherwise the valency would be odd. As was just mentioned, it has been proved that the circulant graphs of valencies 2, 4, and partially for valency 6, all admit n-isofactorizations, in particular Hamilton decompositions. Additionally, the following is a direct consequence of Theorem 2.1, using $d = |S|/2$.

Corollary 2.3 *If $G = \mathrm{CIRC}(n;S)$ is connected with even valency, and $|S|/2$ divides n, then G has an n-isofactorization.*

Theorem 2.4 (Alspach, Dyer, Kreher [4]) *Let $G = \mathrm{CIRC}(n;S)$ be a connected circulant graph of order n with even valency. Partition S into 4-subsets, namely two elements together with their inverses, so that $S = \{\pm s_1, \pm s_2\} \cup \{\pm s_3, \pm s_4\} \cup \cdots \cup \{\pm s_{t-1}, \pm s_t\}$. If, for each pair, the $\mathrm{GCD}(n, s_i, s_{i+1}) = 1$, then G has a Hamilton decomposition.*

PROOF. The circulant graph $G_i = \mathrm{CIRC}(n; \{\pm s_i, \pm s_{i+1}\})$ is connected if and only if $\mathrm{GCD}(n, s_i, s_{i+1}) = 1$. Hence, by Theorem 1.1, there exists a Hamilton decomposition of G_i, for each i, $i = 1, 3, 5, ..., t-1$. Thus G has a Hamilton decomposition. ∎

Let $C = \mathrm{CIRC}(n; \{\pm 1\})$, the natural n-cycle. The edge joining vertex x and vertex y in the circulant graph G is said to have *length* equal to $\mathrm{DIST}_C(x, y)$, the length of the shortest path from x to y in C. This is easy to calculate as $\mathrm{DIST}_C(x, y) = |x - y|$ if we assume $x > y$ and $x, y \in \{0, 1, 2, \ldots, n-1\} \subset \mathbb{Z}$, the reduced residues modulo n. Note that if $\mathrm{ORD}(s) = t$, for some $s \in S$, then s generates n/t cycles of length t in G. This leads to a trivial n-isofactorization of G if every $s \in S$ has $\mathrm{ORD}(s) = n$, i.e. s is relatively prime to n (in particular if n is prime).

Theorem 2.5 *If $G = \mathrm{CIRC}(n;S)$ is a circulant graph, then there exists a Hamilton decomposition of G when $\mathrm{GCD}(n, s) = 1$ for all $s \in S$.*

Theorem 2.6 (Liu [11]) *Any circulant graph on 2p vertices, where p is a prime, is Hamilton decomposable.*

Theorem 2.7 (Liu [11]) *Any circulant graph of order at most 15 is Hamilton decomposable.*

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs with vertex set V_i and edge set E_i respectively. The *cartesian product* $G_1 \times G_2$ is the graph having vertex set $V_1 \times V_2$ and edge set,

$$\{(x, x')(y, y') : \; xy \in E(G_1) \text{ and } x' = y', \text{ or } x = y \text{ and } x'y' \in E(G_2)\}.$$

Theorem 2.8 (Aubert and Schneider [5]) *Let K_n be the complete graph on n vertices. The cartesian product $K_m \times K_n$ can be decomposed into $\frac{1}{2}(m+n-2)$ Hamilton cycles if $m+n$ is even and $\frac{1}{2}(m+n-3)$ Hamilton cycles and a perfect matching if $m+n$ is odd.*

Theorem 2.9 *Let $G = \mathrm{CIRC}(dr; S)$ with $S \subseteq H = \langle r \rangle$ the unique subgroup of index r in $\mathbb{Z}_{dr}$. Then G is isomorphic to r copies of $G' = \mathrm{CIRC}(d; S')$, where $S' = \{\frac{s}{r} : \forall\, s \in S\}$.*

PROOF. The the cosets of H in $\mathbb{Z}_{dr}$ are $H, H+1, \cdots, H+(r-1)$. Define $G_i = \langle H + i \rangle$ to be the subgraph of G induced on the elements in $H + i$. It is elementary to show that the mapping $\phi : V(G_i) \to V(G')$ given by $\phi : z \mapsto (z - i)/r$ is an isomorphism. ■

For example the the map $\phi : V(G) \to V(K_3 \times K_7)$ given by

$$\phi : v \mapsto (\, v \,(\mathrm{mod}\ 3)\,,\; v \,(\mathrm{mod}\ 7)\,)$$

shows that the circulant graph $G = \mathrm{CIRC}(21;\ \pm\{3, 6, 7, 9\})$ is isomorphic to $K_3 \times K_7$ This is because 7 generates 7 cycles of length 3 in G and as $\{3, 6, 9\} \subset \langle 3 \rangle$ we have $G' = \mathrm{CIRC}(21;\ \pm\{3, 6, 9\})$ is isomorphic to three copies of $\mathrm{CIRC}(7; \pm\{1, 2, 3\}) = K_7$ by Theorem 2.9 . As $3+7 = 10$ is even, there exists a decomposition of G into four Hamilton cycles by Theorem 2.8.

3 Valency 8: the n-isofactorization for small lengths

The results in this section give a partial solution to the n-isofactorization problem of $G = \mathrm{CIRC}(n; S)$ when $S^+ = \{s_1, s_2, s_3, s_4\}$ and $\mathrm{ORD}(s_i) > 2$

for all i. For the remainder of this paper, we shall assume that G is of this form and $s_1 < s_2 < s_3 < s_4$.

Consider a subset $U \subseteq V(G)$ of vertices of $G = \mathrm{CIRC}(n; S)$. The subset of edges

$$F_U = \{\{x,\ x+s\} :\ s \in S^+,\ x \in U\}$$

is called the set of *forward edges on U*. Note that $F_U = \bigcup_{x\in U} F_{\{x\}}$ and if $G = \mathrm{CIRC}(n; \pm\{s_1, s_2, s_3, s_4\})$, then $|F_{\{x\}}| = 4$ for all $x \in V(G)$.

Theorem 3.1 *Any connected circulant graph of order $n = 4x+4$ has an n-isofactorization.*

PROOF. As $|S|/2 = 4 \mid n$, G has an n-isofactorization by Corollary 2.3. ■

Theorem 3.2 *If $G = \mathrm{CIRC}(n; S)$ is a circulant graph of order $n = 4x+5$, such that $x \geq 5$ and $s \leq x$ for all $s \in S^+$, then there exists an n-isofactorization of G.*

PROOF. Partition $\mathbb{Z}_n = T \,\dot\cup\, V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2 \,\dot\cup\, V_3$, where

$$\begin{aligned}
T &= \{0, 1, x+2, 2x+3, 3x+4\},\\
V_0 &= \{2, 3, \ldots, x+1\},\\
V_1 &= \{x+3, x+4, \ldots, 2x+2\},\\
V_2 &= \{2x+4, 2x+5, \ldots, 3x+3\},\\
V_3 &= \{3x+5, 3x+6, \ldots, 4x+4\}.
\end{aligned}$$

For $i = 0, 1, 2, 3$, let $X_i = \langle F_{V_i} \rangle$, i.e. the subgraph of G induced by the set of all forward edges on V_i. Because each V_i consists of x consecutive vertices, it is clear that X_0, X_1, X_2, and X_3 are pairwise isomorphic, each having $4x$ edges. In fact, $\varphi : V_i \to V_j$ defined by $\varphi : x \mapsto x + (j-i)(x+1)$ is the isomorphism. We consider two cases.

Case 1: $1 \notin S$

Add the edges $\{0, s_1\}$ and $\{0, s_2\}$ to X_1 and add $\{0, s_3\}$ and $\{0, s_4\}$ to X_2. X_1 and X_2 now have two connected components, for the restriction $s \leq x$ ensures that no edge has one end in V_i and the other in V_{i+2} and no edge in F_{V_i} is incident with a vertex in V_{i+3} (subscript modulo 4). As $\{\{0, s_1\}, \{0, s_2\}\}$ and $\{1, 1+s_2\}$ are vertex disjoint, add $\{1, 1+s_2\}$ to X_1 and add $\{1, 1+s_4\}$ to X_2 so that X_1 and X_2 now have three connected components each. Add the remaining two forward edges, $\{1, 1+s_1\}$ and $\{1, 1+s_3\}$, to X_0. To preserve isomorphism, add $\{x+2, x+2+s_1\}$ and

$\{x+2,x+2+s_3\}$ to X_1, add $\{2x+3,2x+3+s_1\}$ and $\{2x+3,2x+3+s_3\}$ to X_2, and add $\{3x+4,3x+4+s_1\}$ and $\{3x+4,3x+4+s_3\}$ to X_3. There now remain the last two forward edges from each of the vertices in $T\setminus\{0,1\}$, a total of six edges. Add the two forward edges from $3x+4$ and a single edge from $2x+3$ to X_0. Lastly, add the two forward edges from $x+2$ and the remaining single edge from $2x+3$ to X_3. A simple check shows this exhausts all forward edges and the resulting isofactors are in the form of three components, one of size $4x+2$, a 2-path, and a single edge.

Case 2: $1 \in S$

Let $S^+ = \{1,s_2,s_3,s_4\}$ where $1 < s_2 < s_3 < s_4$. Add $\{0,s_3\}$, $\{0,s_4\}$, and $\{1,2\}$ to X_1 and add $\{0,s_2\}$, $\{1,1+s_2\}$, and $\{1,1+s_3\}$ to X_2. Add $\{x+2,x+2+s_2\}$, $\{x+2,x+2+s_3\}$, and $\{2x+3,2x+3+s_3\}$ to X_3 and add $\{3x+4,3x+4+s_2\}$, $\{3x+4,3x+4+s_3\}$, and $\{2x+3,2x+3+s_2\}$ to X_0. Next, add $\{x+2,x+3\}$ to X_0, $\{2x+3,2x+4\}$ to X_1, $\{3x+4,3x+5\}$ to X_2, and $\{0,1\}$ to X_3. Lastly, add $\{1,1+s_4\}$, $\{x+2,x+2+s_4\}$, $\{2x+3,2x+3+s_4\}$, and $\{3x+4,3x+4+s_4\}$ to X_0, X_1, X_2, and X_3, respectively to achieve the desired result. ∎

Theorem 3.3 *If $G = \mathrm{CIRC}(n;S)$ is a circulant graph of order $n = 4x+6$, such that $x \geq 5$ and $s \leq x$ for all $s \in S^+$, then there exists an n-isofactorization of G.*

PROOF. Partition $\mathbb{Z}_n = T \,\dot\cup\, V_0 \,\dot\cup\, V_1 \,\dot\cup\, V_2 \,\dot\cup\, V_3$, where

$$\begin{aligned} T &= \{0,1,x+2,2x+3,2x+4,3x+5\}, \\ V_0 &= \{2,3,\ldots,x+1\}, \\ V_1 &= \{x+3,x+4,\ldots,2x+2\}, \\ V_2 &= \{2x+5,2x+6,\ldots,3x+4\}, \\ V_3 &= \{3x+6,3x+7,\ldots,4x+5\}. \end{aligned}$$

As before, the induced subgraphs $X_i = \langle F_{V_i}\rangle$ for $0 \leq i \leq 3$ are pairwise isomorphic, each having $4x$ edges. It remains to distribute the 24 edges in F_T. Once again, we consider two cases.

Case 1: $1 \notin S$

We add three disjoint 2-paths to each X_i as follows. Add $\{0,s_1\}$, $\{1,1+s_1\}$, $\{0,s_3\}$, and $\{1,1+s_3\}$ to X_1 and add $\{0,s_2\},\{0,s_4\},\{1,1+s_2\}$ and $\{1,1+s_4\}$ to X_2. Likewise, add $\{2x+3,2x+3+s_1\},\{2x+3,2x+3+s_3\},\{2x+4,2x+4+s_1\}$ and $\{2x+4,2x+4+s_3\}$ to X_3 and add $\{2x+3,2x+3+s_2\},\{2x+3,2x+3+s_4\},\{2x+4,2x+4+s_2\}$ and $\{2x+4,2x+4+s_4\}$

to X_0. It remains to distribute the 8 edges in $F_{\{x+2,3x+5\}}$. Clearly, adding $e_1, e_2 \in F_{\{x+2\}}$ to X_2, and adding $F_{\{x+2\}} \setminus \{e_1, e_2\}$ to X_3, ensures X_2 and X_3 each have three disjoint 2-paths. Likewise, adding $e_1, e_2 \in F_{\{3x+5\}}$ to X_0 and $F_{\{3x+5\}} \setminus \{e_1, e_2\}$ to X_1, gives the desired result.

Case 2: $1 \in S$

We add three pairs of edges to each $X_i = \langle F_{V_i} \rangle$ as shown in Figure 1, which depicts (for ease of visualization) the edges in $F_{\{0\}} \cup F_{\{1\}} \setminus \{0, 1\}$ as vertex disjoint. ∎

Theorem 3.4 *Let $G = \mathrm{CIRC}(n; S)$ be a circulant graph of order $n = 4x + 7$ where $x \geq 5$ and $s \leq x$, for all $s \in S^+$, then there exists an n-isofactorization of G.*

PROOF. Partition $\mathbb{Z}_n = T \dot\cup V_0 \dot\cup V_1 \dot\cup V_2 \dot\cup V_3$, where

$$\begin{aligned} T &= \{0, 1, x+2, x+3, 2x+4, 2x+5, 3x+6\}, \\ V_0 &= \{2, 3, \ldots, x+1\}, \\ V_1 &= \{x+4, x+5, \ldots, 2x+3\}, \\ V_2 &= \{2x+6, 2x+7, \ldots, 3x+5\}, \\ V_3 &= \{3x+7, 3x+8, \ldots, 4x+6\}. \end{aligned}$$

As before, the induced subgraphs $X_i = \langle F_{V_i} \rangle$ where $0 \leq i \leq 3$ are pairwise isomorphic, each having $4x$ edges. It remains to distribute the 28 edges in F_T. We consider two cases.

Case 1: $1 \notin S$

Add $\{1, 1+s_3\}$ and $\{1, 1+s_4\}$ to X_0, add $\{x+3, x+3+s_3\}$ and $\{x+3, x+3+s_4\}$ to X_1, $\{2x+5, 2x+5+s_3\}$ and $\{2x+5, 2x+5+s_4\}$ to X_2, and $\{3x+6, 3x+6+s_3\}$ and $\{3x+6, 3x+6+s_4\}$ to X_3. To the subgraph X_1, add $\{1, 1+s_1\}$ and $\{1, 1+s_2\}$. Select $e_1, e_2 \in F_{\{0\}}$ that are not incident with $1+s_1$ or $1+s_2$ and add e_1 and e_2 to X_1. Add $F_{\{0\}} \setminus \{e_1, e_2\}$ to X_2 and add $\{2x+5, 2x+5+s_1\}$ and $\{2x+5, 2x+5+s_2\}$ to X_0. Select $e_1, e_2 \in F_{\{2x+4\}}$ that are not incident with $2x+5+s_1$ and $2x+5+s_2$, and add e_1 and e_2 to X_0 and add $F_{\{2x+4\}} \setminus \{e_1, e_2\}$ to X_3. The vertices $x+3+s_1$ and $x+3+s_2$ may be incident with at most two edges in $F_{\{x+2\}} = \{e_1, e_2, e_3, e_4\}$. As such, it is always possible to relabel the edges in $F_{\{x+2\}}$ so that $\{\{x+3, x+3+s_1\}\} \cup \{e_1, e_2\}$ and $\{\{x+3, x+3+s_2\}\} \cup \{e_3, e_4\}$ are vertex disjoint. Hence, add the former set to X_2 and the latter to X_3. Finally, add $\{3x+6, 3x+6+s_1\}$ to X_0 and $\{3x+6, 3x+6+s_2\}$ to X_1 to achieve the desired result.

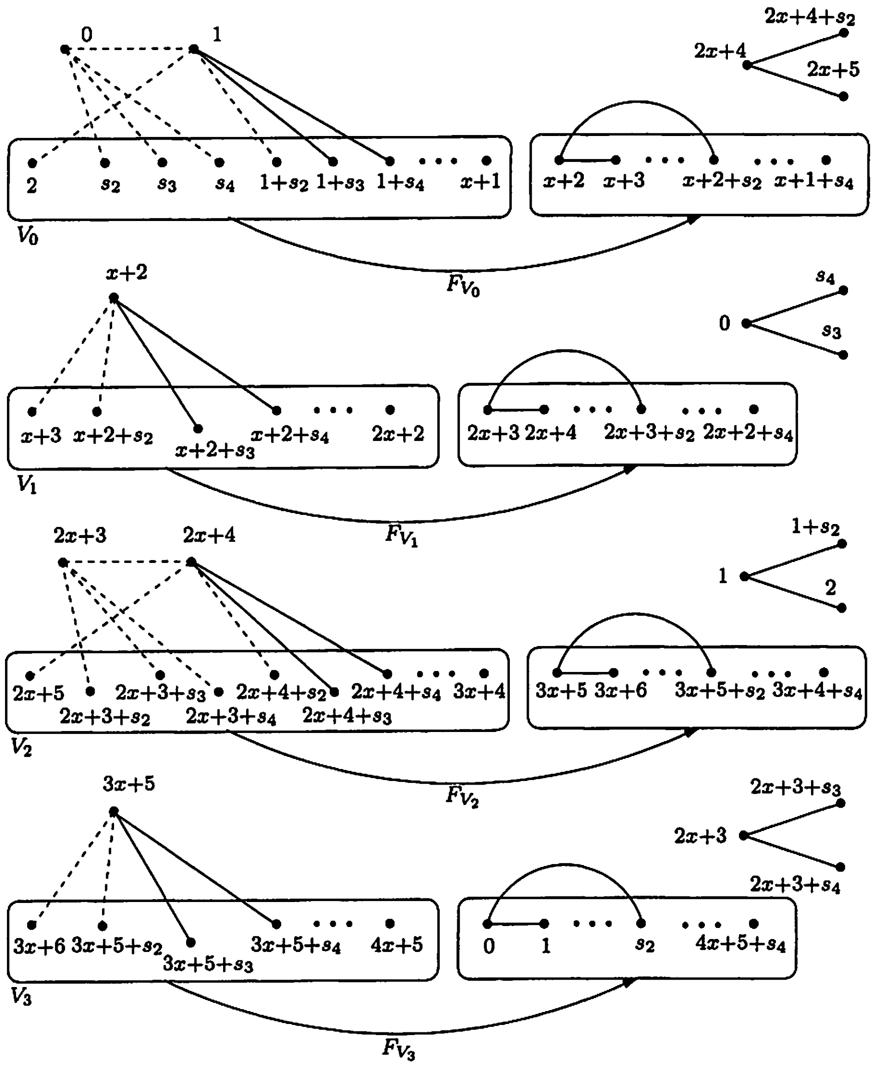

Figure 1: The n-isofactorization illustrating Case 2 of Theorem 3.3. The arcs labeled F_{V_i} denote the set of forward edges on the vertices in V_i, where $i = 0, 1, 2, 3$.

Case 2: $1 \in S$

Once again, let $S^+ = \{1, s_2, s_3, s_4\}$ and $1 < s_2 < s_3 < s_4$. Add $\{0, s_3\}$, $\{0, s_4\}$, and $\{1, 2\}$ to X_1, add $\{x+2, x+2+s_3\}$, $\{x+2, x+2+s_4\}$, and $\{1, 1+s_2\}$ to X_2, add $\{x+3, x+4\}$, $\{x+3, x+3+s_2\}$, and $\{2x+5, 2x+5+s_2\}$ to X_3, and add $\{2x+4, 2x+4+s_3\}$, $\{2x+4, 2x+4+s_4\}$, and $\{2x+5, 2x+6\}$ to X_0. Finally, add the remaining four edges in each of $F_{\{1,x+2\}}$, $F_{\{x+3,2x+4\}}$, $F_{\{2x+5,3x+6\}}$, and $F_{\{3x+6,0\}}$ to X_0, X_1, X_2, and X_3 respectively, to achieve the desired result. ∎

Theorem 3.5 *An 8-regular circulant graph $G = \mathrm{CIRC}(n; \pm\{s_1, s_2, s_3, s_4\})$ where $n \leq 23$, has an n-isofactorization.*

PROOF. We divide into three cases on the value of n.

Case 1: $n \equiv 0, 3 \pmod 4$

This is resolved in Theorem 3.1 when $n \equiv 0 \pmod 4$ or by Theorems 2.7 and 2.5 when $n \equiv 3 \pmod 4$.

Case 2: $n \equiv 1 \pmod 4$

Write $n = 4x + 5$. If $x \in \{1, 2, 3\}$, then $G = \mathrm{CIRC}(n; S)$ is Hamilton decomposable by Theorem 2.7 or Theorem 2.5. If $x = 4$, then $n = 21$, S^+ must be a 4-subset of $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. However, the only elements which are not co-prime with 21 are $3, 6, 7, 9$. Now if S^+ contains two or more elements relatively prime to 21, then we may remove one of the Hamilton cycles generated by them, and are left with a connected valency 6 circulant of odd order which has a Hamilton decomposition by Theorem 1.2. If S^+ contains exactly one element relatively prime with 21, then S^+ contains a 3-subset of $\{3, 6, 7, 9\}$. These cases are Hamilton decomposable by Theorem 2.4 except when $\{3, 6, 9\} \subset S^+$. The case $G = \mathrm{CIRC}(21; \pm\{3, 6, 7, 9\}) \cong K_3 \times K_7$ has a Hamilton decomposition by Theorem 2.8. The remaining cases on $G = \mathrm{CIRC}(21; \pm\{l, 3, 6, 9\})$, where $l \in \{1, 2, 4, 5, 8, 10\}$ are resolved in Table 2 of Section 4.

Case 3: $n \equiv 2 \pmod 4$

If $n = 4x + 6$ and $x = 1, 2, 3, 4$, then $n \in \{10, 14, 18, 22\}$. For $n \in \{10, 14, 22\}$, we are guaranteed Hamilton decompositions by Theorem 2.6 or by Theorem 2.7. When $n = 18$, and S^+ contains two or three elements co-prime with 18, there exists a Hamilton decomposition by Theorem 2.4. The graph $G = \mathrm{CIRC}(18; \pm\{2, 4, 6, 8\})$ is isomorphic to two copies of $G^* = \mathrm{CIRC}(9; \pm\{1, 2, 3, 4\})$ (namely the subgraphs induced on the even and odd vertices of G respectively) by Theorem 2.9. As G^* has a

9-isofactorization by [10] or a Hamilton decomposition by Theorem 2.4, we can create four pairs of subgraphs of G, each pair isomorphic to 2 copies of a 9-isofactor of G^*, and the isofactorization of G into four subgraphs is complete. There exist Hamilton decompositions for the 18 cases where $s_1 \in \{1,5,7\}$, $s_2 = 3$, and $s_3, s_4 \in \{2,4,6,8\}$ by Theorem 2.4. Therefore, the remaining cases on 18 vertices are $s_1 \in \{1,3,5,7\}$ and $s_i \in \{2,4,6,8\}$ for $i = 2,3,4$. See Table 1 of Section 4 for Hamilton decompositions when $s_1 \in \{1,3\}$ and apply the automorphisms $\alpha : x \mapsto 5x$ and $\beta : x \mapsto 7x$ to CIRC$(18, \{1, s_2, s_3, s_4\})$ to resolve the remaining cases. ■

Combining Theorems 3.1 through 3.5, and noting that $s \leq \frac{n}{4} - 2 \Rightarrow s \leq x$, we have the main result:

Theorem 3.6 *If the maximum edge length in a connected 8-regular circulant graph G of order n is at most $\frac{n}{4} - 2$, then G has an n-isofactorization.*

4 Data

This section provides Hamilton decompositions for the remaining cases on 18 and 21 vertices which were not covered in Theorem 3.5. Employing a back-tracking algorithm in a random computer search, Hamilton decompositions were found for the case when $n = 18$ and $s_1 = 3$, and the cases for $n = 21$. Hamilton decompositions when $n = 18$ and $s_1 = 1$ were found using ad-hoc methods. An example is given for each of these cases in Table 1 and Table 2.

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Table 1: Hamilton cycle decomposition $\{H_1, H_2, H_3, H_4\}$ for Case 3 of Theorem 3.5.

CIRC(18; $\pm\{1,2,4,6\}$)
$H_1 = (0,1,15,3,9,13,17,5,11,7,6,10,4,16,12,8,2,14)$
$H_2 = (0,2,3,1,17,15,13,11,9,7,5,4,6,8,10,12,14,16)$
$H_3 = (0,4,8,14,15,9,5,1,13,7,3,17,11,10,16,2,6,12)$
$H_4 = (0,6,5,3,4,2,1,7,8,9,10,14,13,12,11,15,16,17)$
CIRC(18; $\pm\{1,2,4,8\}$)
$H_1 = (0,1,11,3,13,5,15,7,17,9,8,16,6,14,4,12,2,10)$
$H_2 = (0,4,8,12,16,2,3,17,13,9,5,1,15,11,7,6,10,14)$
$H_3 = (0,2,4,6,8,10,12,13,11,9,7,5,3,1,17,15,14,16)$
$H_4 = (0,8,7,3,4,5,6,2,1,9,10,11,12,14,13,15,16,17)$
CIRC(18; $\pm\{1,2,6,8\}$)
$H_1 = (0,1,13,3,11,5,17,7,15,9,8,14,6,16,4,10,2,12)$
$H_2 = (0,10,11,1,7,13,5,15,3,9,17,16,8,2,14,4,12,6)$
$H_3 = (0,2,4,6,8,10,12,13,11,9,7,5,3,1,17,15,14,16)$
$H_4 = (0,8,7,6,5,4,3,2,1,9,10,16,15,13,14,12,11,17)$
CIRC(18; $\pm\{1,4,6,8\}$)
$H_1 = (0,1,13,3,11,5,17,7,15,9,8,14,6,16,4,10,2,12)$
$H_2 = (0,4,8,12,16,2,3,17,13,9,5,1,15,11,7,6,10,14)$
$H_3 = (0,2,4,6,8,10,12,13,11,9,7,5,3,1,17,15,14,16)$
$H_4 = (0,8,7,6,5,4,3,2,1,9,10,16,15,13,14,12,11,17)$
CIRC(18; $\pm\{2,3,4,6\}$)
$H_1 = (0,3,6,10,12,8,14,16,4,7,1,15,9,13,11,5,17,2)$
$H_2 = (0,14,10,8,5,1,16,13,15,3,17,11,7,9,12,6,2,4)$
$H_3 = (0,12,14,17,15,11,9,3,5,2,8,6,4,1,13,7,10,16)$
$H_4 = (0,6,9,5,7,3,1,17,13,10,4,8,11,14,2,16,12,15)$
CIRC(18; $\pm\{2,3,4,8\}$)
$H_1 = (0,10,12,15,17,14,4,6,8,16,2,5,1,9,13,11,7,3)$
$H_2 = (0,8,10,2,6,3,17,9,12,14,11,1,16,13,5,15,7,4)$
$H_3 = (0,16,12,2,17,1,4,8,11,15,13,3,5,9,7,10,6,14)$
$H_4 = (0,2,4,12,8,5,7,17,13,10,14,16,6,9,11,3,1,15)$
CIRC(18; $\pm\{2,3,6,8\}$)
$H_1 = (0,12,4,10,8,16,13,7,17,1,11,14,2,5,15,3,9,6)$
$H_2 = (0,8,11,9,7,1,13,15,12,6,14,4,16,10,2,17,5,3)$
$H_3 = (0,10,13,5,8,6,3,11,17,14,12,2,4,7,15,9,1,16)$
$H_4 = (0,15,17,9,12,10,7,5,11,13,3,1,4,6,16,14,8,2)$
CIRC(18; $\pm\{3,4,6,8\}$)
$H_1 = (0,3,7,10,14,17,13,16,1,11,15,5,9,6,2,8,4,12)$
$H_2 = (0,15,12,16,10,6,3,9,13,1,4,7,11,8,5,17,2,14)$
$H_3 = (0,6,12,8,14,4,16,2,5,1,7,15,9,17,11,3,13,10)$
$H_4 = (0,4,10,2,12,9,1,15,3,17,7,13,5,11,14,6,16,8)$

Table 2: Hamilton cycle decomposition $\{H_1, H_2, H_3, H_4\}$ for Case 2 of Theorem 3.5.

$\mathrm{CIRC}(21; \pm\{1,3,6,9\})$
$H_1 = (0,15,16,19,13,1,10,4,7,6,12,18,17,14,2,20,11,5,8,9,3)$
$H_2 = (0,6,5,2,8,14,15,3,4,1,7,13,16,10,19,20,17,11,12,9,18)$
$H_3 = (0,9,10,7,8,20,5,17,16,1,19,4,13,14,11,2,3,18,6,15,12)$
$H_4 = (0,20,14,5,4,16,7,19,18,15,9,6,3,12,13,10,11,8,17,2,1)$
$\mathrm{CIRC}(21; \pm\{2,3,6,9\})$
$H_1 = (0,12,14,20,11,17,2,4,13,16,18,9,15,6,8,5,7,19,10,1,3)$
$H_2 = (0,6,3,9,12,18,20,2,14,5,11,8,17,15,13,1,4,10,7,16,19)$
$H_3 = (0,9,7,4,16,1,19,13,10,8,2,11,14,17,20,5,3,12,6,18,15)$
$H_4 = (0,18,3,15,12,10,16,14,8,20,1,7,13,11,9,6,4,19,17,5,2)$
$\mathrm{CIRC}(21; \pm\{3,4,6,9\})$
$H_1 = (0,17,13,16,7,19,4,8,2,14,10,6,9,15,12,3,20,11,5,1,18)$
$H_2 = (0,9,18,15,6,12,8,5,2,11,17,14,20,16,4,1,13,19,10,7,3)$
$H_3 = (0,4,13,10,16,19,1,7,11,14,8,20,2,17,5,9,12,18,6,3,15)$
$H_4 = (0,12,16,1,10,4,7,13,9,3,18,14,5,20,17,8,11,15,19,2,6)$
$\mathrm{CIRC}(21; \pm\{3,5,6,9\})$
$H_1 = (0,15,12,7,1,4,20,11,14,19,10,16,13,8,5,17,2,18,9,6,3)$
$H_2 = (0,12,6,15,18,3,9,14,17,11,16,19,13,1,10,4,7,2,8,20,5)$
$H_3 = (0,6,18,12,9,4,19,7,13,10,15,3,8,11,2,5,14,20,17,1,16)$
$H_4 = (0,18,13,4,16,7,10,5,11,6,1,19,3,12,17,8,14,2,20,15,9$
$\mathrm{CIRC}(21; \pm\{3,6,8,9\})$
$H_1 = (0,18,15,9,12,20,2,14,17,4,19,6,3,11,5,8,16,7,10,1,13)$
$H_2 = (0,6,18,3,12,15,2,5,13,7,4,10,16,1,19,11,14,20,8,17,9$
$H_3 = (0,15,3,16,19,13,4,1,7,20,11,8,14,5,17,2,10,18,9,6,12)$
$H_4 = (0,8,2,11,17,20,5,18,12,4,16,13,10,19,7,15,6,14,1,9,3$
$\mathrm{CIRC}(21; \pm\{3,6,9,10\})$
$H_1 = (0,3,12,1,7,17,6,15,18,9,19,16,13,10,4,14,20,2,5,8,11)$
$H_2 = (0,6,12,9,20,17,14,8,18,3,15,4,19,1,13,2,11,5,16,7,10$
$H_3 = (0,12,15,5,20,8,19,13,3,14,2,17,11,1,10,16,4,7,18,6,9)$
$H_4 = (0,18,12,2,8,17,5,14,11,20,10,19,7,13,4,1,16,6,3,9,15)$

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