On the path covering number of given subdigraphs of regular multipartite tournaments

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Abstract

A tournament is an orientation of a complete graph, and a multipartite or c-partite tournament is an orientation of a complete c-partite graph. If we speak of a path, then we mean a directed path

Let D be a regular c-partite tournament with r vertices in each partite set and let $X \subseteq V(D)$ be an arbitrary set with exactly 2 vertices from each partite set. For all $c \geq 4$ the authors determined in a recent article the minimal value $g(c)$ such that $D-X$ is Hamiltonian for every regular multipartite tournament with $r \geq g(c)$. In this paper we will supplement this result by postulating a given path covering number instead of the Hamilonicity of the digraph $D - X$. This means, for all $c \geq 4$ and $k \geq 1$ we will determine the minimal value $h(k, c)$ such that $D - X$ can be covered by at most k paths for every regular c-partite tournament with $r \geq h(k, c)$. Moreover, we will present the minimal path covering number of $D - X$, if D is a regular 3-partite tournament and X contains exactly s vertices ($s \geq 2$) of every partite set.

Keywords: Multipartite tournaments; Regular multipartite tournaments; Path covering number

1 Terminology and introduction

In this paper all digraphs are finite without loops and multiple arcs. The vertex set and the arc set of a digraph D are denoted by $V(D)$ and $E(D)$, respectively. If xy is an arc of a digraph D, then we write $x \to y$ and say x *dominates* y, and if X and Y are two disjoint vertex sets or subdigraphs of D such that every vertex of X dominates every vertex of Y, then we say that X *dominates* Y, denoted by $X \to Y$. Furthermore, $X \leadsto Y$ denotes the fact that there is no arc leading from Y to X. For the number of arcs from X to Y we write $d(X, Y)$.

If D is a digraph, then the *out-neighborhood* $N_D^+(x) = N^+(x)$ of a vertex x is the set of vertices dominated by x and the *in-neighborhood* $N_D^-(x) = N^-(x)$ is the set of vertices dominating x. Therefore, if there is the arc $xy \in E(D)$, then y is an *outer neighbor* of x and x is an *inner neighbor* of y. The numbers $d_D^+(x) = d^+(x) = |N^+(x)|$ and $d_D^-(x) = d^-(x) = |N^-(x)|$ are called the *outdegree* and the *indegree* of x, respectively. Furthermore, the numbers $\delta_D^+ = \delta^+ = \min\{d^+(x)|x \in V(D)\}$ and $\delta_D^- = \delta^- = \min\{d^-(x)|x \in V(D)\}$ are the *minimum outdegree* and the *minimum indegree*, respectively.

For a vertex set X of D, we define $D[X]$ as the subdigraph induced by X. If we replace in a digraph D every arc xy by yx, then we call the resulting digraph the *converse* of D, denoted by D^{-1}.

If we speak of a *cycle* or *path*, then we mean a directed cycle or dircted path, and a cycle of length n is called an *n-cycle*. The length of a cycle C is denoted by $L(C)$. A cycle in a digraph D is *Hamiltonian*, if $L(C) = |V(D)|$. A *cycle-factor* of a digraph D is a spanning subdigraph consisting of disjoint cycles. The *path covering number* $pc(D)$ of a digraph D is the minimum number of paths in D that are pairwise vertex disjoint and cover the vertices of D.

A digraph D is *strongly connected* or *strong*, if for each pair of vertices u and v, there is a path from u to v in D. A digraph D with at least $k+1$ vertices is *k-connected* if for any set A of at most $k-1$ vertices, the subdigraph $D - A$ obtained by deleting A is strong. The *connectivity*, denoted by $\kappa(D)$, is then defined to be the largest value of k such that D is k-connected. If $\kappa(D) = 1$ and x is a vertex of D such that $D - x$ is not strong, then we say that x is a *cut-vertex* of D.

There are several measures of how much a digraph differs from being regular. In [16], Yeo defines the *global irregularity* of a digraph D by

$$i_g(D) = \max_{x \in V(D)} \{d^+(x), d^-(x)\} - \min_{y \in V(D)} \{d^+(y), d^-(y)\}$$

and the *local irregularity* by $i_l(D) = \max\{|d^+(x) - d^-(x)|\,|x \in V(D)\}$. Clearly $i_l(D) \leq i_g(D)$. If $i_g(D) = 0$, then D is *regular* and if $i_g(D) \leq 1$, then D is called *almost regular*.

A *c-partite* or *multipartite tournament* is an orientation of a complete c-partite graph. A *tournament* is a c-partite tournament with exactly c vertices. If $V_1, V_2, \ldots, V_c$ are the partite sets of a c-partite tournament D and the vertex x of D belongs to the partite set V_i, then we define $V(x) = V_i$.

If D is a c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$, then $|V_c| = \alpha(D)$ is the independence number of D.

There is an extensive literature on cycles in multipartite tournaments, see e.g., Bang-Jensen and Gutin [1], Guo [2], Gutin [3], Volkmann [8, 9], Winzen [14] and Yeo [15]. A new approach on cycles was presented by the authors in [13]:

Problem 1.1 (Volkmann, Winzen [13]) *Let D be a regular c-partite tournament with $c \geq 4$ and exactly r vertices in each partite set. Furthermore, let $X \subseteq V(D)$ be an arbitrary set with exactly $s < r$ vertices of each partite set. For all $s < r$ and $c \geq 4$ find the minimal value $g(s,c)$ such that $D - X$ is Hamiltonian for every regular multipartite tournament with $r \geq g(s,c)$.*

In [11] and [13], Volkmann and Winzen solved this problem for the cases $s = 1$ and $s = 2$.

Theorem 1.2 (Volkmann, Winzen [11, 13]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D such that $|V_1| = |V_2| = \ldots = |V_c| = r$. If $g(s,c)$ is defined as in Problem 1.1, then it follows that*

$$\begin{aligned} &g(1,c) = 4, \quad \textit{if} \quad c \geq 4 \quad \textit{is odd}, \quad g(1,c) = 3, \quad \textit{if} \quad c \geq 4 \quad \textit{is even}, \\ &g(2,4) = g(2,5) = 9, \quad g(2,6) = 7, \quad g(2,7) = 8 \\ &\textit{and} \quad g(2,c) = 7 \quad \textit{if} \ c \geq 8. \end{aligned}$$

The idea is now to replace the condition that $D - X$ is Hamiltonian in Problem 1.1 by the weaker condition that $pc(D-X) \leq k$ for a given integer $1 \leq k \leq |V(D)|$.

Problem 1.3 *Let D be a regular c-partite tournament with $c \geq 4$ and exactly r vertices in each partite set. Furthermore, let $X \subseteq V(D)$ be an arbitrary set with exactly $s < r$ vertices of each partite set. For all integers $1 \leq k \leq |V(D - X)|$, $s < r$ and $c \geq 4$ find the minimal value $h(s,k,c)$ such that $pc(D - X) \leq k$ for every regular multipartite tournament with $r \geq h(s,k,c)$.*

Note that the condition $s < r$ in Problem 1.3 implies that $h(s,k,c) \geq s + 1$.

The following result of the authors [12] gives a quick answer of Problem 1.3 for the case that $s = 1$.

Theorem 1.4 (Volkmann, Winzen [12]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of exactly s vertices from each partite set for $1 \leq s \leq r-1$. If*

$$r \geq 3s + \left\lceil \frac{4s-5}{c-3} \right\rceil,$$

then D contains a path P such that $V(P) = V(D) - X$.

The case $s = 1$ directly implies that $h(1,1,c) \leq 3$ for all $c \geq 4$. According to the well known result of Rédei [6] that every tournament contains a Hamiltonian path we even have $h(1,1,c) \leq 2$ and thus $h(1,1,c) = h(1,k,c) = 2$ for all $1 \leq k \leq |V(D-X)|$.

Since Theorem 1.4 is not applicable for $c = 3$ we pose the following similar problem.

Problem 1.5 *Let D be a regular c-partite tournament with $c \geq 2$ and exactly r vertices in each partite set. Furthermore, let $X \subseteq V(D)$ be an arbitrary set with exactly $s < r$ vertices from each partite set. For all integers s and $c \geq 2$ find the minimal value $f(s,c)$ such that $pc(D-X) \leq f(s,c)$ for every regular c-partite tournament with $r > s$.*

Theorem 1.4 with the same considerations as above directly implies that $f(1,c) = 1$ for all $c \geq 4$.

In this article we will determine $h(2,k,c)$, $f(2,c)$ and $f(s,3)$ for all $c \geq 4$, $1 \leq k \leq |V(D-X)|$ and $s \geq 1$. Furthermore, we will prove that $f(s,c) \leq 2s-1$, if $c \geq 4$.

Section 2 presents the most important old results used throughout this paper. In Section 3 we will give a solution of Problem 1.3 for the case that $s = 2$ and $k = 1$, and Section 4 deals with Problem 1.3 for the case that $s = 2$ and $k \geq 2$ and with Problem 1.5 for the case that $c \geq 4$. Finally, in Section 5 we will determine $f(s,3)$ for all integers s.

2 Preliminary results

The following results play an important role in our investigations.

A characterization whether a digraph D has a cycle-factor or not was given by Ore [5].

Theorem 2.1 (Ore [5]) *A digraph D has a cycle-factor if and only if $|N_D^+(S)| \geq |S|$ for each subset $S \subseteq V(D)$.*

In 1999, Yeo [16] rewrote Theorem 2.1 in the following useful form.

Theorem 2.2 (Yeo [16]) *A digraph D has no cycle-factor if and only if $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 such that*

$$R_1 \rightsquigarrow Y,\ (R_1 \cup Y) \rightsquigarrow R_2,\ Y \text{ is an independent set} \qquad (1)$$

and $|Y| > |Z|$.

Gutin and Yeo [4] generalized this result to digraphs with a path covering number $pc(D) > k$.

Theorem 2.3 (Gutin, Yeo [4]) *For a digraph D we have $pc(D) > k$ if and only if $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 that satisfy (1) and $|Y| > |Z| + k$.*

The following theorem is a useful supplement to Lemma 4.3 in [16] and Theorem 3.2 in [4].

Theorem 2.4 (Stella, Volkmann, Winzen [7]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 0$. According to Theorem 2.3, $V(D)$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) such that $|Z| + k + 1 \leq |Y| \leq |V_c| - t$ with an integer $t \geq 0$. Let V_i be the partite set with the property that $Y \subseteq V_i$. Let $Q = V(D) - Z - V_i$, $Q_1 = Q \cap R_1$, $Q_2 = Q \cap R_2$, $Y_1 = R_1 \cap V_i$ and $Y_2 = R_2 \cap V_i$. Then*

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|}{2},$$

if $Q_1 = \emptyset$,

$$i_g(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_1|}{2},$$

if $Q_2 = \emptyset$, and

$$i_g(D) \geq i_l(D) \geq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t}{2},$$

if $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$.

The following corollary presents a direct consequence of the last theorem.

Corollary 2.5 (Volkmann, Winzen [10]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \dots, V_c$ such that $|V_1| \leq |V_2| \leq \dots \leq |V_c|$. If there exists a positive integer k such that*

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2},$$

then $pc(D) \leq k$.

An analysis of the proof of Theorem 2.4 yields the following result.

Corollary 2.6 (Stella, Volkmann, Winzen [7]) *Let D be a semicomplete multipartite digraph with the partite sets $V_1, V_2, \ldots, V_c$ such that $|V_1| \leq |V_2| \leq \ldots \leq |V_c|$. Assume that $pc(D) > k$ for an integer $k \geq 1$. Let $Y, Z, R_1, R_2, Q, Q_1, Q_2, V_i, Y_1$ and Y_2 be defined as in Theorem 2.4.*

If $Q_1 = \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+|Y_2|}{2}$, then the following holds.

i) $\min\{d^-(w)|w \in V_i\} = |Z| = |Y| - k - 1$.

ii) $|Y| = |V_i| - |Y_2|$, *which means that* $|Y_1| = 0$ *and* $|V_i \cap Z| = 0$.

iii) $Y \rightarrow Q_2 \rightarrow (Y_2 \cup Z)$.

iv) $d^-(q_2) = d^+(q_2) + k - |Y_2| + 1$ *for all* $q_2 \in Q_2$.

v) $\max\{d^+(w), d^-(w)|w \in V(D) - V_i\} = d^-(q)$ *for a vertex* $q \in Q_2$ *such that* $|V(q)| = |V_{c-1}|$

vi) $i_g(D) = \max\{d^-(q)|q \in Q_2\} - \min\{d^-(w)|w \in V_i\}$.

vii) $|V_i| = |V_c|$.

viii) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + |Y_2|$ *is even.*

Let $j = c-1$, if $i = c$ and $j = c$, if $i < c$. If $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$ and $i_g(D) = \frac{|V(D)|-|V_{c-1}|-2|V_c|+3k+3+t}{2}$, then we conclude that

a) $i_g(D) = i_l(D)$.

b) $\{|V_i|, |V_j|\} = \{|V_c|, |V_{c-1}|\}$.

c) $V_i \cap Z = \emptyset$, $|Z| = |Y| - 1 - k$, $|Y| = |V_c| - t$.

d) $|V_m \cap Q_1| = |V_l \cap Q_1|$ *and* $|V_m \cap Q| = |V_l \cap Q|$ *for all* $1 \leq l, m \leq c$ *such that* $V_m \cap Q \neq \emptyset$ *and* $V_l \cap Q \neq \emptyset$.

e) $V_j \subseteq Q$.

f) $\frac{d(Q_1,Q_2)}{|Q_1|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} - |Y_2| + |Y_1|$ *and*
$\frac{d(Q_1,Q_2)}{|Q_2|} = \frac{|V(D)|-|V_{c-1}|-2|V_c|+1+k+t}{2} + |Y_2| - |Y_1|$.

g) $d^+(q_1) = d^-(q_1) + i_g(D)$ *for all* $q_1 \in Q_1$ *and* $d^-(q_2) = d^+(q_2) + i_g(D)$ *for all* $q_2 \in Q_2$.

h) $Q_2 \rightarrow (Z \cup Y_2)$, $(Z \cup Y_1) \rightarrow Q_1$.

j) $|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 3 + t$ *is even.*

Theorem 2.7 (Volkmann, Winzen [10]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete c-partite digraph D such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r+p$ for an integer $p \geq 0$. If $c \geq \max\{2, 3+\frac{2i_g(D)-5+p}{r}\}$, then D contains a Hamiltonian path.*

In [12] the authors presented a result about the existence of a path with all but s vertices from each partite set in regular multipartite tournaments. The proof of this theorem (Theorem 3.9) provides more than the theorem states.

Theorem 2.8 (Volkmann, Winzen [12]) *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 5$ such that $|V_1| = |V_2| = \ldots = |V_c| = r$. Furthermore let X be an arbitrary subset of $V(D)$ with exactly $s \geq 2$ vertices from each partite set. If $r \geq 5s-3$, then $D-X$ contains a Hamiltonian path.*

The following remark concerning regular multipartite tournaments is well-known but important for this article.

Remark 2.9 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament. Then it follows that $r = |V_1| = |V_2| = \ldots = |V_c|$ and*

$$d^+(x), d^-(x) = \frac{(c-1)r}{2}$$

for all $x \in V(D)$. That means especially that r is even, if c is even.

3 The determination of $h(2,1,c)$

Let D be a regular c-partite tournament with $c \geq 4$ and exactly $r \geq 3$ vertices from each partite set. Furthermore let X be an arbitrary subset of $V(D)$ with exactly two vertices from each partite set. In this section we will find the minimal value for r that guarantees the existence of a Hamiltonian path in $D-X$. This means that we will determine $h(2,1,c)$ of Problem 1.3 for all $c \geq 4$.

Applying Theorem 1.4 and Theorem 2.8 with $s = 2$, it is obvious that $h(2,1,c) \leq 9$ for all $c \geq 4$ and $h(2,1,c) \leq 7$ for all $c \geq 5$. The following example demonstrates that $h(2,1,c) \geq 7$ for all $c \geq 4$.

Example 3.1 *Let $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{6,i}\}$ $1 \leq i \leq c$ be the partite sets of a regular c-partite tournament D such that $R_1 = (\{v_{1,i}, v_{2,i} \mid 1 \leq i \leq c-2\} \cup \{v_{3,c-1}\}) \rightarrow (Y = \{v_{1,c}, v_{2,c}, v_{3,c}, v_{4,c}\}) \rightarrow (R_2 = (\{v_{3,i}, v_{4,i} \mid 1 \leq i \leq c-2\} \cup \{v_{4,c-1}\})) \rightarrow (Z = \{v_{1,c-1}, v_{2,c-1}\}) \rightarrow R_1 \rightsquigarrow R_2$, $Y \rightarrow Z$, $(R_1 - \{v_{3,c-1}\}) \rightarrow v_{3,c-1}$ and $v_{4,c-1} \rightarrow (R_2 - \{v_{4,c-1}\}) \rightsquigarrow (X = \{v_{i,j} \mid 5 \leq i \leq$*

6, $1 \le j \le c\}) \leadsto (R_1 - \{v_{3,c-1}\})$. Furthermore, let $R_1 - \{v_{3,c-1}\}$ as well as $R_2 - \{v_{4,c-1}\}$ be regularly connected. Moreover let $\{v_{1,c}, v_{2,c}\} \to (\{v_{i,j} \mid 5 \le i \le 6,\ 1 \le j \le \lfloor \frac{c-2}{2} \rfloor\} \cup \{v_{i,j} \mid i = 5$ and $j = \lceil \frac{c-2}{2} \rceil$, if c is odd$\}) \to \{v_{3,c}, v_{4,c}\} \to (\{v_{i,j} \mid 5 \le i \le 6, \lceil \frac{c-2}{2} \rceil + 1 \le j \le c-2\} \cup \{v_{i,j} \mid i = 6$ and $j = \lceil \frac{c-2}{2} \rceil$, if c is odd$\}) \to \{v_{1,c}, v_{2,c}\}$, $\{v_{5,c-1}, v_{6,c-1}\} \to Y$, $\{v_{3,c-1}, v_{4,c-1}\} \to (X_1 = \{v_{i,j} \mid 5 \le i \le 6, 1 \le j \le \lfloor \frac{c-3}{2} \rfloor\} \cup \{v_{5,j} \mid j = \lceil \frac{c-3}{2} \rceil$, if c is even$\}) \to Z \to (X_2 = \{v_{i,j} \mid 5 \le i \le 6, \lceil \frac{c-3}{2} \rceil + 1 \le j \le c-3\} \cup \{v_{6,j} \mid j = \lceil \frac{c-3}{2} \rceil$, if c is even$\}) \to \{v_{3,c-1}, v_{4,c-1}\}$, $(Z \cup \{v_{4,c-1}\}) \to \{v_{5,c-2}, v_{6,c-2}, v_{5,c}, v_{6,c}\} \to \{v_{3,c-1}, v_{5,c-1}, v_{6,c-1}\}$, $v_{5,c} \to v_{5,c-2} \to v_{6,c} \to v_{6,c-2} \to v_{5,c}$ and $X_1 \to \{v_{5,c-2}, v_{6,c-2}, v_{6,c-1}\} \to X_2 \to \{v_{5,c}, v_{6,c}, v_{5,c-1}\} \to X_1$. If finally the vertices in $X_1 \cup X_2$ are regularly connected, then it is straightforward to see that D is a regular c partite tournament and $D - X$ consists of the sets Y, R_1, R_2 and Z satisfying (1) with $k = 1$. According to Theorem 2.3 it follows that $D - X$ contains no Hamiltonian path.

Using this example and some of the results of Section 2, we may determine $h(2, 1, c)$ for all $c \ge 4$.

Theorem 3.2 *Let D be a regular c-partite tournament with $c \ge 4$ and exactly $r \ge 3$ vertices in each partite set. If $h(s, k, c)$ is defined as in Problem 1.3, then it follows that*

$$h(2, 1, c) = 7 \quad \text{for all} \quad c \ge 4.$$

Proof. Let D be a regular c-partite tournament with the partite sets $V_1, V_2, \dots, V_c$ each of the cardinality r. To prove this theorem we distinguish different cases.

Case 1. Let $c \ge 5$. According to Theorem 1.4 and Theorem 2.8 with $s = 2$ we conclude that $h(2, 1, c) \le 7$. Example 3.1 implies that $h(2, 1, c) \ge 7$ such that we arrive at the desired result in this case.

Case 2. Let $c = 4$. Remark 2.9 yields that r has to be even. With Theorem 1.4 and Example 3.1 we deduce that $7 \le h(2, 1, 4) \le 9$. Hence, we have to investigate the case that D contains exactly 8 vertices from each partite set. Let X be an arbitrary subset of $V(D)$ with exactly 2 vertices from each partite set. Then the multipartite tournament $D' := D - X$ has the partite sets $V'_1, V'_2, \dots, V'_c$ with $r' := |V'_i| = 6$ for all $1 \le i \le c$. Moreover it follows that $i_g(D') \le 6$. If $i_g(D') \le 5$, then because of

$$3 + \frac{2i_g(D') - 5}{r'} = 3 + \frac{2i_g(D') - 5}{6} \le 3 + \frac{5}{6} \le c = 4$$

Theorem 2.7 yields the desired result that D' contains a Hamiltonian path. Hence, let $i_g(D') = 6$.

Suppose that D' does not contain any Hamiltonian path which means that $pc(D') > 1$. Then $V(D')$ can be partitioned into subsets Y, Z, R_1 and R_2 satisfying (1) with $k = 1$. Furthermore let Q_1, Q_2, Y_1, Y_2, V_i' and t be defined as in Theorem 2.4.

Subcase 2.1. Let $Q_1 = \emptyset$. In this case we observe that $|V_i'| + |Z| \leq 6 + 4 = 10$. Now for an arbitrary vertex $y \in Y \neq \emptyset$ we arrive at the contradiction

$$12 = d^+(y) \geq |Q_2| = |Q| \geq |V(D')| - |V_i'| - |Z| \geq 24 - 10 = 14.$$

Subcase 2.2. Let $Q_2 = \emptyset$. Observing the coverse D^{-1} of D Subcase 2.1. yields a contradiction.

Subcase 2.3. Assume that $Q_1 \neq \emptyset$ and $Q_2 \neq \emptyset$. Since

$$\frac{|V(D')| - |V_{c-1}'| - 2|V_c'| + 3 + 3}{2} = 6 = i_g(D')$$

Theorem 2.4 implies that $|Y| = |V_c'|$ and thus $|Y_1| = |Y_2| = 0$. Applying Corollary 2.6 *a*), *c*), *f*) and *h*) we deduce that $i_g(D') = i_l(D') = 6$, $|Y| = 6 = |Z| + 2$, $|Q_1| = |Q_2| = 7$ and $Q_2 \to Z \to Q_1$. The fact that $r' = 6$ yields that for every vertex $z \in Z$ there is a vertex $q \in Q = Q_1 \cup Q_2$ such that $q \in V(z)$, a contradiction to $Q_2 \to Z \to Q_1$.

Summarizing our results we see that D' contains a Hamiltonian path. Since r has to be even we conclude that $h(2,1,4) \leq 7$ and hence $h(2,1,4) = 7$. This completes the proof of this theorem. □

4 The determination of $h(2,k,c)$ and $f(2,c)$ and an estimation for $f(k,c)$ if $c \geq 4$

To make statements about multipartite tournaments D having a path covering number $pc(D) > 1$ we firstly need the following generalizations of the Theorems 1.4 and 2.7.

Theorem 4.1 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of the semicomplete c-partite digraph D such that $1 \leq r = |V_1| \leq |V_2| \leq \ldots \leq |V_c| \leq r + p$ for an integer $p \geq 0$. If $c \geq \max\left\{2, 3 + \frac{2i_g(D)+p-3k-2}{r}\right\}$, then it follows that $pc(D) \leq k$.*

Proof. According to Corollary 2.5, it is sufficient to show that

$$i_g(D) \leq \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2}.$$

Since $c \geq 3 + \frac{2i_g(D)-3k-2+p}{r}$, we conclude that $i_g(D) \leq \frac{(c-3)r+3k+2-p}{2}$, and together with $|V_1|, |V_2|, \ldots, |V_{c-2}| \geq r$, $|V_c| \leq r + p$ and $c \geq 2$ this implies

$$\begin{aligned} & \frac{|V(D)| - |V_{c-1}| - 2|V_c| + 3k + 2}{2} \\ = & \frac{|V_1| + |V_2| + \ldots + |V_{c-2}| - |V_c| + 3k + 2}{2} \\ \geq & \frac{(c-3)r - p + 3k + 2}{2} \geq i_g(D), \end{aligned}$$

the desired result. □

Theorem 4.2 *Let $V_1, V_2, \ldots, V_c$ be the partite sets of a regular c-partite tournament D with $c \geq 4$ and $|V_1| = |V_2| = \ldots = |V_c| = r \geq 2$. Furthermore, let X be an arbitrary subset of $V(D)$ consisting of exactly s vertices from each partite sets with $1 \leq s \leq r - 1$. If*

$$r \geq 3s + \left\lceil \frac{4s - 3k - 2}{c - 3} \right\rceil,$$

then $pc(D - X) \leq k$.

Proof. Let $D' = D - X$ with the partite sets $V_1', V_2', \ldots, V_c'$ such that $|V_1'| = |V_2'| = \ldots = |V_c'| = r - s$. Since D is regular, it follows that $i_g(D') \leq s(c-1)$. Using Theorem 4.1 with $p = 0$, we see that it is sufficient to show that

$$3 + \frac{2i_g(D') - 3k - 2}{r - s} \leq 3 + \frac{2s(c-1) - 3k - 2}{r - s} \leq c.$$

Since r is an integer the last inequality yields the following equivalent transformations

$$\begin{aligned} & 3 + \frac{2s(c-1) - 3k - 2}{r - s} \leq c \\ \Leftrightarrow \quad & (c-3)(r-s) \geq 2s(c-1) - 3k - 2 \\ \Leftrightarrow \quad & r \geq \left\lceil \frac{2s(c-1) - 3k - 2}{c - 3} \right\rceil + s = 3s + \left\lceil \frac{4s - 3k - 2}{c - 3} \right\rceil. \end{aligned}$$

According to the assumptions of this theorem the last inequality is valid. This completes the proof of the theorem. □

The following two examples demonstrate that $h(2, 2, 2p + 1) \geq 6$ for all $p \geq 2$.

Example 4.3 *Let* $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{5,i}\}$, $1 \le i \le 4m+1$, *be the partite sets of a regular* $(4m+1)$*-partite tournament* D_m $(m \in \mathbb{N})$ *such that* $R_1 = \{v_{i,k} \mid (1 \le i \le 2 \ \wedge\ 1 \le k \le 2m) \ \vee\ (i = 1 \ \wedge\ 2m+1 \le k \le 4m)\} \to Y = \{v_{1,4m+1}, v_{2,4m+1}, v_{3,4m+1}\} \to R_2 = \{v_{i,k} \mid (i = 3 \ \wedge\ 1 \le k \le 2m) \ \vee\ (2 \le i \le 3 \ \wedge\ 2m+1 \le k \le 4m)\}$ *and* $R_1 \leadsto R_2$. *If* $V_i' = V_i \cap R_1$, $V_i'' = V_i \cap R_2$ $(1 \le i \le 4m)$, $K_1' = \bigcup_{i=1}^{2m} V_i'$, $K_2' = \bigcup_{i=2m+1}^{4m} V_i'$, $K_1'' = \bigcup_{i=1}^{2m} V_i''$ *and* $K_2'' = \bigcup_{i=2m+1}^{4m} V_i''$, *then let the vertices* K_1' *be connected such that* $D[K_1']$ *is* $(2m-1)$*-regular and the vertices* K_2' *let be connected such that* $D[K_2']$ *is almost regular and* $d^+_{D[K_2']}(v_{1,k}) = d^-_{D[K_2']}(v_{1,k}) + 1 = m$ *for* $2m+1 \le k \le 3m$ *and* $d^-_{D[K_2']}(v_{1,k}) = d^+_{D[K_2']}(v_{1,k}) + 1 = m$ *for* $3m+1 \le k \le 4m$. *Furthermore let* $V_i' \to V_j'$, *if* $1 \le i \le 2m$ *and* $j \in \{2m+1+(i-1 \bmod 2m), 2m+1+(i \bmod 2m), \ldots, 2m+1+(i-2+m \bmod 2m)\}$ *and* $V_j' \to V_i'$ *otherwise for* $V_i' \subseteq K_1'$ *and* $V_j' \subseteq K_2'$. *Analogously let the vertices* K_2'' *connected such that* $D[K_2'']$ *is* $(2m-1)$*-regular and the vertices* K_1'' *let be connected such that* $D[K_1'']$ *is almost regular and* $d^+_{D[K_1'']}(v_{3,k}) = d^-_{D[K_1'']}(v_{3,k}) + 1 = m$ *for* $1 \le k \le m$ *and* $d^-_{D[K_1'']}(v_{3,k}) = d^+_{D[K_1'']}(v_{3,k}) + 1 = m$ *for* $m+1 \le k \le 2m$. *Furthermore let* $V_i'' \to V_j''$, *if* $2m+1 \le j \le 4m$ *and* $i \in \{1+(j-2m-1 \bmod 2m), 1+(j-2m \bmod 2m), \ldots, 1+(j-2-m \bmod 2m)\}$ *and* $V_j'' \to V_i''$ *otherwise for* $V_i'' \subseteq K_1''$ *and* $V_j'' \subseteq K_2''$. *If* $X = \{v_{i,j} \mid 4 \le i \le 5 \ \wedge\ 1 \le j \le 4m+1\}$ *and* $X_i = X \cap V_i$, *then let* $\{v_{1,4m+1}, v_{2,4m+1}\} \to (X_{2m+1} \cup X_{2m+2} \cup \ldots \cup X_{4m}) \to v_{3,4m+1} \to (X_1 \cup X_2 \cup \ldots \cup X_{2m}) \to \{v_{1,4m+1}, v_{2,4m+1}\}$. *The vertices of* X *let be regularly connected. Finally let* $R_2 \leadsto X \leadsto R_1$ *with exception of the following arcs:*

$$
\{v_{1,3m+1}, v_{1,3m+2}, \ldots, v_{1,4m}\}
\to \left\{ \begin{array}{ll} (X_{2m+1} \cup X_{2m+2} \cup \ldots \cup X_{2m+\frac{m}{2}}), & \text{if } m \text{ is even} \\ (X_{2m+1} \cup X_{2m+2} \cup \ldots \cup X_{2m-1+\lfloor \frac{m}{2} \rfloor} & \\ \cup \{v_{4,2m+\lfloor \frac{m}{2} \rfloor}\}), & \text{if } m \text{ is odd} \end{array} \right\}
\to \{v_{3,1}, v_{3,2}, \ldots, v_{3,m}\},
$$

$$
\{v_{1,2m+1}, v_{1,2m+2}, \ldots, v_{1,3m}\}
\to \left\{ \begin{array}{ll} (X_{3m+1} \cup X_{3m+2} \cup \ldots \cup X_{3m-1+\frac{m}{2}} & \\ \cup \{v_{4,3m+\frac{m}{2}}\}), & \text{if } m \text{ is even} \\ (X_{3m+1} \cup X_{3m+2} \cup \ldots \cup X_{3m+\lfloor \frac{m}{2} \rfloor}) & \text{if } m \ge 3 \text{ is odd} \end{array} \right\}
\to \{v_{3,m}, v_{3,m+1}, \ldots, v_{3,2m}\}
$$

and

$$v_{i,j} \to$$

$$\left\{\begin{array}{ll} v_{4,1+(j \bmod 2m)}, & \textit{if } m = 2 \\ (X_{1+(j \bmod 2m)} \cup \ldots \cup X_{1+(j-2+\frac{m}{2} \bmod 2m)} & \\ \cup \{v_{4,1+(j-1+\frac{m}{2} \bmod 2m)}\}), & \textit{if } m \geq 4 \textit{ is even} \\ X_{1+(j \bmod 2m)} \cup \ldots \cup X_{1+(j-1+\lfloor \frac{m}{2} \rfloor \bmod 2m)}, & \textit{if } m \geq 3 \textit{ is odd} \end{array}\right\}$$
$$\rightarrow v_{i+1,j+2m}$$

for all $1 \leq i \leq 2$ *and* $1 \leq j \leq 2m$.

The resulting digraph D_m *(see Fig. 1 for* D_1*) is a* $10m$*-regular* $(4m+1)$*-partite tournament with the property that* $pc(D - X) = 3$.

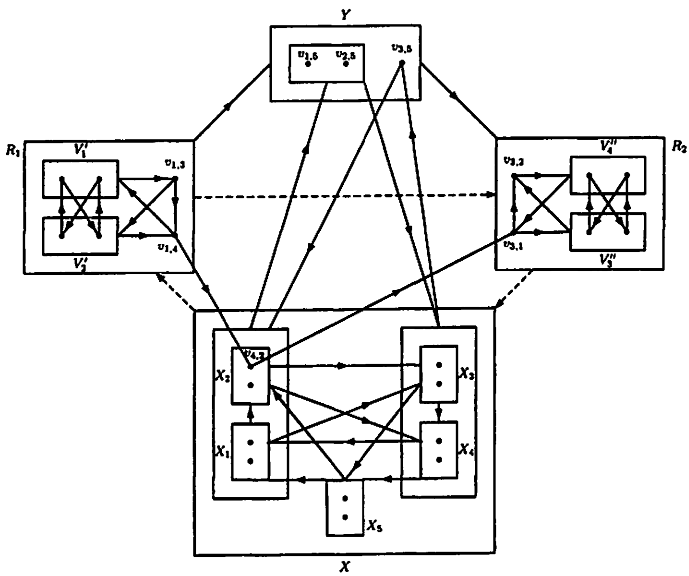

Figure 1: The regular 5-partite tournament D_1 with the property that $pc(D - X) = 3$.

Example 4.4 *Let* $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{5,i}\}$, $1 \leq i \leq 4m+3$, *be the partite sets of a regular* $(4m+3)$*-partite tournament* H_m *(*$m \in \mathbb{N}$*) such that* $R_1 = \{v_{i,k} \mid (1 \leq i \leq 2 \ \wedge \ 1 \leq k \leq 2m+1) \ \vee \ (i = 1 \ \wedge \ 2m+2 \leq k \leq 4m+2)\} \rightarrow Y = \{v_{1,4m+3}, v_{2,4m+3}, v_{3,4m+3}\} \rightarrow R_2 = \{v_{i,k} \mid (i = 3 \ \wedge \ 1 \leq k \leq 2m+1) \ \vee \ (2 \leq i \leq 3 \ \wedge \ 2m+2 \leq k \leq 4m+2)\}$ *and* $R_1 \rightsquigarrow R_2$. *If* $V_i' = V_i \cap R_1$, $V_i'' = V_i \cap R_2$ *(*$1 \leq i \leq 4m+2$*),* $K_1' = \bigcup_{i=1}^{2m+1} V_i'$, $K_2' = \bigcup_{i=2m+2}^{4m+2} V_i'$,

$K_1'' = \bigcup_{i=1}^{2m+1} V_i''$ *and* $K_2'' = \bigcup_{i=2m+2}^{4m+2} V_i''$, *then let the vertices* K_1' *be connected such that* $D[K_1']$ *is* $(2m)$*-regular and the vertices* K_2' *let be connected such that* $D[K_2']$ *is* m*-regular. Furthermore let* $V_i' \to V_j'$, *if* $1 \leq i \leq 2m+1$ *and* $j \in \{2m+2+(i-1 \bmod (2m+1)), 2m+2+(i-1 \bmod (2m+1)), \ldots, 2m+2+(i-1+m \bmod (2m+1))\}$, *and let* $V_j' \to V_i'$ *otherwise for* $V_j' \subseteq K_2'$ *and* $V_i' \subseteq K_1'$. *Analogously let the vertices* K_2'' *connected such that* $D[K_2'']$ *is* $(2m)$*-regular and the vertices* K_1'' *let be connected such that* $D[K_1'']$ *is* m*-regular. Furthermore let* $V_i'' \to V_j''$, *if* $2m+2 \leq j \leq 4m+2$ *and* $i \in \{1+(j-2m-2 \bmod (2m+1)), 1+(j-2m-1 \bmod (2m+1)), \ldots, 1+(j-2-m \bmod (2m+1))\}$ *and* $V_j'' \to V_i''$ *otherwise for* $V_i'' \subseteq K_1''$ *and* $V_j'' \subseteq K_2''$. *If* $X = \{v_{i,j} \mid 4 \leq i \leq 5 \wedge 1 \leq j \leq 4m+3\}$ *and* $X_i = X \cap V_i$, *then let* $\{v_{1,4m+1}, v_{2,4m+1}\} \to (X_{2m+2} \cup X_{2m+2} \cup \ldots \cup X_{4m+2}) \to v_{3,4m+1} \to (X_1 \cup X_2 \cup \ldots \cup X_{2m+1}) \to \{v_{1,4m+1}, v_{2,4m+1}\}$. *The vertices of* X *let be regularly connected. Finally let* $R_2 \rightsquigarrow X \rightsquigarrow R_1$ *with exception of the following arcs:*

$$\begin{aligned}&v_{1,j} \to \\ &(X_{2m+2+(j-2m-1 \bmod (2m+1))} \cup \ldots \cup X_{2m+2+(j+\frac{m}{2}-(2m+2) \bmod (2m+1))} \\ &\cup \{v_{4,2m+2+(j-2m-1+\frac{m}{2} \bmod (2m+1))}\}) \\ &\to v_{3,j-2m-1},\end{aligned}$$

if m *is even, and*

$$\begin{aligned}&v_{1,j} \to \\ &X_{2m+2+(j-2m-1 \bmod (2m+1))} \cup \ldots \cup X_{2m+2+(j-2m-1+\lfloor \frac{m}{2} \rfloor \bmod (2m+1))} \\ &\to v_{3,j-2m-1},\end{aligned}$$

if m *is odd, for all* $j \in \{2m+2, 2m+3, \ldots, 4m+2\}$, *and*

$$\begin{aligned}&v_{i,j} \to \\ &\left\{\begin{array}{ll} v_{4,1+(j \bmod (2m+1))}, & \text{if } m = 2 \\ (X_{1+(j \bmod (2m+1))} \cup \ldots \cup X_{1+(j-2+\frac{m}{2} \bmod (2m+1))} & \\ \cup \{v_{4,1+(j-1+\frac{m}{2} \bmod (2m+1))}\}), & \text{if } m \geq 4 \text{ is even} \\ X_{1+(j \bmod (2m+1))} \cup \ldots \cup X_{1+(j-1+\lfloor \frac{m}{2} \rfloor \bmod (2m+1))}, & \text{if } m \geq 3 \text{ is odd} \end{array}\right\} \\ &\to v_{i+1,j+2m+1}\end{aligned}$$

for all $i \in \{1,2\}$ *and* $j \in \{1, 2, \ldots, 2m+1\}$.

The resulting digraph H_m *is a* $(10m+5)$*-regular* $(4m+3)$*-partite tournament with the property that* $pc(D-X) = 3$.

Now we are able to determine the values of $h(2,k,c)$ and $f(2,c)$ for all integers $k \geq 2$ and $c \geq 4$.

Theorem 4.5 *Let D be a regular c-partite tournament with $c \geq 4$ and exactly $r \geq 3$ vertices in each partite set. If $h(s,k,c)$, s,k and X are defined as in Problem 1.3, then it follows that*

$$h(2,2,2m) = 3, \quad h(2,2,2m+1) = 6 \quad \text{and} \quad h(2,k,c) = 3$$

for all integers m,k,c with $m \geq 2$, $3 \leq k \leq |V(D-X)|$ and $c \geq 4$.

Proof. Let D be a regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ each of the cardinality r. Furthermore let $X \subseteq V(D)$ an arbitrary set with $|X \cap V_i| = 2$ for all $1 \leq i \leq c$ and $D' = D - X$. If k is defined as in Problem 1.3, then we distinguish different cases.

Case 1. Let $k = 2$. According to Theorem 4.2 with $k = s = 2$ we observe that $pc(D') \leq 2$, if $r \geq 6$. Hence we have $h(2,2,c) \leq 6$.

If $c = 2m+1$ for an integer $m \geq 2$, then the Examples 4.3 and 4.4 imply that $h(2,2,2m+1) \geq 6$ and thus $h(2,2,2m+1) = 6$.

If $c = 2m$ for an integer $m \geq 2$, then Remark 2.9 yields that r has to even. Hence, it remains to treat the case that $r = 4$. Suppose that $r = 4$ and $pc(D') > 2$. Then, according to Theorem 2.3, $V(D')$ can be partitioned into subsets Y, Z, R_1 and R_2 that satisfy (1) and $|Y| \geq |Z|+3 \geq 3$. Since Y is an independent set this would imply that $r = |V(y)| \geq 5$ for all $y \in Y$, a contradiction. Consequently, we have $pc(D') \leq 2$, and thus $h(2,2,2m) = 3$.

Case 2. Let $k \geq 3$. Assume that $pc(D') > k \geq 3$. Applying Theorem 2.3 this yields that $V(D')$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) and $|Y| \geq |Z| + k + 1 \geq 4$. Since Y is an independent set it follows that $r = |V(y)| \geq |Y| + 2 \geq 6$ for all $y \in Y$. According to Theorem 4.2 with $s = 2$ this is sufficient to show that $pc(D') \leq k$, because if $r \geq 6$, then the following inequality chain is fulfilled:

$$r \geq 6 + \left\lceil \frac{-3}{c-3} \right\rceil \geq 6 + \left\lceil \frac{8-3k-2}{c-3} \right\rceil.$$

Hence, we have a contradiction to our assumption that $pc(D') > k$. This implies that $h(2,k,c) = 3$ for all $3 \leq k \leq |V(D-X)|$ and $c \geq 4$.

This completes the proof of this theorem. □

Together with the Examples 3.1, 4.3 and 4.4 this theorem immediately implies the following corollary.

Corollary 4.6 *Let D be a regular c-partite tournament with $c \geq 4$ and exactly $r \geq 3$ vertices in each partite set. If $f(s,c)$, s and X are defined as in Problem 1.5, then it follows that*

$$f(2,2m) = 2 \quad \text{and} \quad f(2,2m+1) = 3$$

for all integers m, c with $m \geq 2$ and $c \geq 4$.

The following theorem gives an estimation for $f(s,c)$, if $c \geq 4$.

Theorem 4.7 *Let D be a regular c-partite tournament with $c \geq 4$ and exactly r vertices in each partite set. Furthermore, let $f(s,c)$, s and X with $1 \leq s < r$ are defined as in Problem 1.5. Then it follows that $f(s,c) \leq 2s-1$.*

Proof. Let D be a regular c-partite tournament with the partite sets $V_1, V_2, \ldots, V_c$ each of the cardinality r. Furthermore, for an integer $s < r$ let $X \subseteq V(D)$ an arbitrary set with $|X \cap V_i| = s$ for all $1 \leq i \leq c$ and $D' = D - X$. We assume that $pc(D') > k$ for an integer k with $2s - 1 \leq k \leq |V(D')|$. Applying Theorem 2.3 we see that $V(D')$ can be partitioned into subsets Y, Z, R_1, R_2 satisfying (1) and $|Y| \geq |Z| + k + 1 \geq 2s$. This yields that $r = |V(y)| \geq |Y| + s \geq 3s$ for all $y \in Y$. Since $k \geq 2s - 1 \geq \frac{4s-2}{3}$ for $s \geq 1$ and thus $3k + 2 \geq 4s$ we deduce that the following inequality chain is fulfilled:

$$r \geq 3s \geq 3s + \left\lceil \frac{4s - 3k - 2}{c-3} \right\rceil .$$

According to Theorem 4.2 we arrive at $pc(D') \leq k$, a contradiction to our assumption. This implies that $f(s,c) \leq 2s - 1$, the desired result. □

Since we have the condition $r > s$ in Problem 1.3 this yields the following result.

Corollary 4.8 *Let D be a regular c-partite tournament with $c \geq 4$ and exactly r vertices in each partite set. Furthermore, let $h(s,k,c)$, s, k and X with $1 \leq s < r$ are defined as in Problem 1.3. If $2s - 1 \leq k \leq |V(D - X)|$, then it follows that $h(s,k,c) = s + 1$.*

5 The determination of $f(s,3)$

To find the exact values of $f(s,3)$ we distinguish three cases depending on the rest that occures by dividing s by 3. The following three examples demonstrate that $f(3m-1,3) \geq 4m-2$, $f(3m-2,3) \geq 4m-3$ and $f(3m,3) \geq 4m$.

Example 5.1 *Let $m \geq 1$ be an integer and let $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{7m-3,i}\}$, $1 \leq i \leq 3$, be the partite sets of a regular 3-partite tournament G_m such that $(R_1 = \{v_{i,j} \mid 1 \leq i \leq 2m-1, 1 \leq j \leq 2\}) \rightarrow (Y = \{v_{i,3} \mid 1 \leq i \leq 4m-2\}) \rightarrow (R_2 = \{v_{i,j} \mid 2m \leq i \leq 4m-2, 1 \leq j \leq 2\})$ and $R_1 \rightsquigarrow R_2$. If $V_1' = V_1 \cap R_1$ and $V_2' = V_2 \cap R_1$, then let*

$$v_{i,1} \rightarrow$$

$$\{v_{1+(i-1 \bmod (2m-1)),2}, v_{1+(i \bmod (2m-1)),2}, \ldots, v_{1+(i+m-2 \bmod (2m-1)),2}\}$$

for all $1 \leq i \leq 2m-1$ *and* $v_{j,2} \to v_{i,1}$ *otherwise, if* $v_{j,2} \in V_2'$ *and* $v_{i,1} \in V_1'$. *Analogously, if* $V_1'' = V_1 \cap R_2$ *and* $V_2'' = V_2 \cap R_2$, *then let*

$$\{v_{2m+(j-1 \bmod (2m-1)),1}, v_{2m+(j \bmod (2m-1)),1}, \ldots, v_{2m+(j+m-2 \bmod (2m-1)),1}\} \to v_{j,2}$$

for all $2m \leq j \leq 4m-2$ *and* $v_{j,2} \to v_{i,1}$ *otherwise, if* $v_{j,2} \in V_2''$ *and* $v_{i,1} \in V_1''$. *Furthermore, let* $V_2'' \leadsto (X = \{v_{i,j} \mid 4m-1 \leq i \leq 7m-3, 1 \leq j \leq 3\}) \leadsto V_1'$, $V_2' \to v_{4m-1,3} \to V_1'' \leadsto (X \setminus \{v_{4m-1,3}\}) \leadsto V_2'$. *If* $X_i = X \cap V_i$, *then finally let* $X_1 \to \{v_{i,3} \mid 1 \leq i \leq 2m-1\} \to X_2 \to \{v_{i,3} \mid 2m \leq i \leq 4m-2\} \to X_1 \to X_2 \to X_3 \to X_1$. *The resulting digraph* G_m *(see Figure 2 for* $m = 1$*) is a* $(7m-3)$*-regular 3-partite tournament with the property that* $|X_i| = 3m-1$ $(1 \leq i \leq 3)$ *and* $pc(D - X) = 4m - 2$.

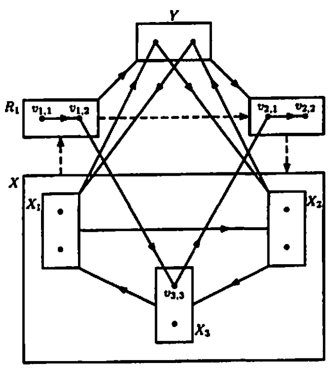

Figure 2: The regular 3-partite tournament G_1 with the property that $pc(D - X) = 2$.

Example 5.2 *Let* $m \geq 1$ *be an integer and let* $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{7m-5,i}\}$, $1 \leq i \leq 3$, *be the partite sets of a regular 3-partite tournament* A_m *such that* $(R_1 = \{v_{i,j} \mid (1 \leq i \leq 2m-2 \wedge 1 \leq j \leq 2) \vee (i = 2m-1 \wedge j = 1)\}) \to (Y = \{v_{i,3} \mid 1 \leq i \leq 4m-3\}) \to (R_2 = \{v_{i,j} \mid (2m \leq i \leq 4m-3 \wedge 1 \leq j \leq 2) \vee (i = 2m-1 \wedge j = 2)\})$ *and* $R_1 \leadsto R_2$. *If* $V_1' = V_1 \cap R_1$ *and* $V_2' = V_2 \cap R_1$, *then let*

$$v_{i,1} \to \{v_{1+(i-1 \bmod (2m-2)),2}, v_{1+(i \bmod (2m-2)),2}, \ldots, v_{1+(i+m-3 \bmod (2m-2)),2}\}$$

for all $1 \leq i \leq 2m-1$ *and* $v_{j,2} \to v_{i,1}$ *otherwise, if* $v_{j,2} \in V_2'$ *and* $v_{i,1} \in V_1'$.

Analogously, if $V_1'' = V_1 \cap R_2$ *and* $V_2'' = V_2 \cap R_2$, *then let*

$$\{v_{2m+(j-1 \bmod (2m-2)),1}, v_{2m+(j \bmod (2m-2)),1}, \ldots, v_{2m+(j+m-3 \bmod (2m-2)),1}\} \to v_{j,2}$$

for all $2m-1 \le j \le 4m-3$ *and* $v_{j,2} \to v_{i,1}$ *otherwise, if* $v_{j,2} \in V_2''$ *and* $v_{i,1} \in V_1''$. *Furthermore, let* $(V_2'' \cup \{v_{i,1} \mid 3m-1 \le i \le 4m-3\}) \leadsto (X = \{v_{i,j} \mid 4m-2 \le i \le 7m-5, 1 \le j \le 3\}) \leadsto (V_1' \cup \{v_{i,2} \mid m \le i \le 2m-2\})$, $\{v_{i,2} \mid 1 \le i \le m-1\} \to v_{4m-2,3} \to \{v_{i,1} \mid 2m \le i \le 3m-2\} \leadsto (X \setminus \{v_{4m-2,3}\}) \leadsto \{v_{i,2} \mid 1 \le i \le m-1\}$. *If* $X_i = X \cap V_i$ $(1 \le i \le 3)$, *then finally let* $X_1 \to \{v_{i,3} \mid 2m-1 \le i \le 4m-3\} \to X_2 \to \{v_{i,3} \mid 1 \le i \le 2m-2\} \to X_1 \to X_2 \to X_3 \to X_1$. *The resulting digraph* A_m *(see Figure 3 for* $m = 2$*) is a* $(7m-5)$*-regular 3-partite tournament with the property that* $|X_i| = 3m-2$ $(1 \le i \le 3)$ *and* $pc(D - X) = 4m - 3$.

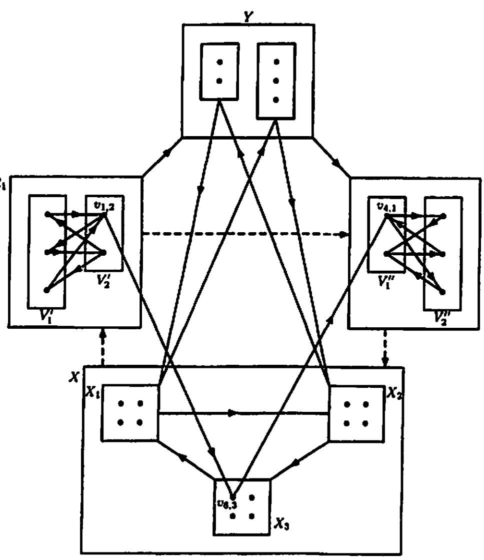

Figure 3: The regular 3-partite tournament A_2 with the property that $pc(D - X) = 5$.

Example 5.3 *Let* $m \ge 1$ *be an integer and let* $V_i = \{v_{1,i}, v_{2,i}, \ldots, v_{7m,i}\}$, $1 \le i \le 3$, *be the partite sets of a regular 3-partite tournament* B_m *such that* $(R_1 = \{v_{i,j} \mid 1 \le i \le 2m, 1 \le j \le 2\}) \to (Y = \{v_{i,3} \mid 1 \le i \le 4m\}) \to$

($R_2 = \{v_{i,j} \mid 2m+1 \le i \le 4m, 1 \le j \le 2\}$) and $R_1 \rightsquigarrow R_2$. If $V_1' = R_1 \cap V_1$ and $V_2' = R_1 \cap V_2$, then let

$$v_{i,1} \rightarrow \{v_{1+(i-1 \bmod 2m),2}, v_{1+(i \bmod 2m),2}, \ldots, v_{1+(i+m-2 \bmod 2m),2}\}$$

for all $1 \le i \le 2m$ and $v_{j,2} \rightarrow v_{i,1}$ otherwise, if $v_{j,2} \in V_2'$ and $v_{i,1} \in V_1'$.

Analogously, if $V_1'' = R_2 \cap V_1$ and $V_2'' = R_2 \cap V_2$, then let

$$v_{i,1} \rightarrow \{v_{2m+1+(i-1 \bmod 2m),2}, v_{2m+1+(i \bmod 2m),2}, \ldots, v_{2m+1+(i+m-2 \bmod 2m),2}\}$$

for all $2m+1 \le i \le 4m$ and $v_{j,2} \rightarrow v_{i,1}$ otherwise, if $v_{j,2} \in V_2''$ and $v_{i,1} \in V_1''$. Furthermore, let $R_2 \rightsquigarrow (X = \{v_{i,j} \mid 4m+1 \le i \le 7m, 1 \le i \le 3\}) \rightsquigarrow R_1$. If $X_i = X \cap V_i$, then finally let $X_1 \rightarrow \{v_{i,3} \mid 1 \le i \le 2m\} \rightarrow X_2 \rightarrow \{v_{i,3} \mid 2m+1 \le i \le 4m\} \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow X_1$. The resulting digraph B_m (see Figure 4 for $m = 1$) is a $7m$-regular 3-partite tournament with the property that $|X_i| = 3m$ ($1 \le i \le 3$) and $pc(D - X) = 4m$.

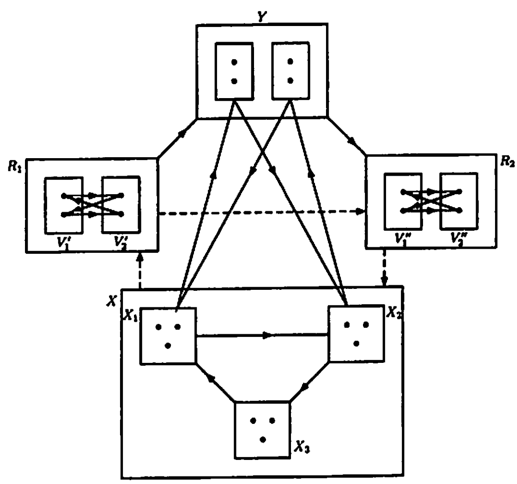

Figure 4: The regular 3-partite tournament B_1 with the property that $pc(D - X) = 4$.

To determine $f(s, 3)$ the Examples 5.1, 5.2 and 5.3 are best possible as we can see in the following theorem.

Theorem 5.4 *Let D be a regular 3-partite tournament with exactly r vertices in each partite set. If $f(s,3)$ and s are defined as in Problem 1.5, then it follows that*

$$f(3m-1,3)=4m-2, \quad f(3m-2,3)=4m-3 \quad \text{and} \quad f(3m,3)=4m$$

for all integers $m \geq 1$.

Proof. Let D be a regular 3-partite tournament with the partite sets V_1, V_2, V_3 such that $|V_1| = |V_2| = |V_3| = r$. If X is an arbitrary subset of $V(D)$ such that $|X \cap V_i| = s < r$ for all $1 \leq i \leq 3$, then we define $D' := D - X$. D' has the partite sets V_1', V_2', V_3' such that $|V_1'| = |V_2'| = |V_3'| = r - s$. Since D is regular, we obviously have $i_g(D') \leq 2s$. Suppose that $pc(D') > k > \frac{4s}{3} - 1$. Then Corollary 2.5 yields the contradiction

$$2s \geq i_g(D') \geq \frac{|V(D')| - |V_2'| - 2|V_3'| + 3k + 3}{2} = \frac{3k+3}{2} > 2s.$$

Hence, we have

$$pc(D') \leq \begin{Bmatrix} 4m-2 \\ 4m-3 \\ 4m \end{Bmatrix} \text{ and thus } \begin{Bmatrix} f(3m-1,3) \leq 4m-2 \\ f(3m-2,3) \leq 4m-3 \\ f(3m,3) \leq 4m \end{Bmatrix},$$

$$\text{if } \begin{Bmatrix} s = 3m-1 \\ s = 3m-2 \\ s = 3m \end{Bmatrix}.$$

According to the Examples 5.1, 5.2 and 5.3, we obtain $f(3m-1,3) = 4m-2$, $f(3m-2,3) = 4m-3$ and $f(3m) = 4m$ for all integers $m \geq 1$, the desired result. □

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