Radius and Diameter with respect to Cliques in Graphs

A. P. Santhakumaran[1] S. Arumugam[2]

[1] P. G. and Research Department of Mathematics
St. Xavier's College (Autonomous)
Palayamkottai - 627 002, India.
e-mail: apskumar1953@yahoo.co.in

[2] Core Group Research Facility (CGRF)
National Centre for Advanced Research in
Discrete Mathematics (n-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626 190, INDIA.
e-mail: s_arumugam_akce@yahoo.com

Abstract

Let G be a connected graph. In this paper we introduce the concepts of vertex-to-clique *radius* r_1, vertex-to-clique *diameter* d_1, clique-to-vertex *radius* r_2, clique-to-vertex *diameter* d_2, clique-to-clique *radius* r_3 and clique-to-clique *diameter* d_3 in G. We prove that for any connected graph, $r_i \leq d_i \leq 2r_i + 1$ for $i = 1, 2, 3$. We also find expressions for d_1, d_2 and d_3 for a tree T in terms of r_1, r_2 and r_3 respectively, which determine the cardinality of each $Z_i(T)$, where $Z_i(T)$ is the vertex-to-clique, the clique-to -vertex and the clique-to-clique center respectively of T for $i = 1, 2, 3$. If G is a graph which is not a tree and if $g(G)$ denotes the girth of the graph, then its relation with each of d_1, d_2 and d_3 is discussed. We also characterize the class of graphs G such that G is not a tree, $d_3 \neq 0$ and $g(G) = 2d_3 + 3$.

Key Words: vertex-to-clique radius, vertex-to-clique diameter, clique-to-vertex radius, clique-to-vertex diameter, clique-to-clique radius, clique-to-clique diameter.
AMS Subject Classification: 05C12

1 Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q respectively. For basic graph theoretic terminology, we refer to Buckley and Harary [1]. The distance $d(u, v)$ between two vertices u and v in G is the length of a shortest u–v path in G. It is known that d is a metric on the vertex set V. The *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The set of all vertices for which e is minimized is called the *center* of G and is denoted by $Z(G)$. The concept of the center of a graph arises in the context of selection of a site at which to locate a facility in a graph. Taking into account the situation that the nature of the facility to be constructed could necessitate selecting a structure rather than a vertex to locate a facility, Slater [6] proposed four classes of locational problems, namely, vertex-serves-vertex, vertex-serves-structure, structure-serves-vertex and structure-serves-structure.

For subsets S, $T \subseteq V$ and any vertex v in V, let $d(v, S) = \min\{d(v, u)\colon u \in S\}$ and $d(S, T) = \min\{d(x, y)\colon x \in S,\ y \in T\}$.

Definition 1.1 ([6]) Let $G = (V, E)$ be a connected graph. Let $\zeta = \{C_i\colon i \in I\}$ and $S = \{S_j\colon j \in J\}$ where each of C_i and S_j is a subset of V. Let $e_S(C_i) = \max\{d(C_i, S_j)\colon j \in J\}$; C_i is called a *(ζ, S)-center* if $e_S(C_i) \leq e_S(C_k)$ for all $k \in I$.

Slater [5] investigated the centrality of paths by taking S to be the collection of all paths in G and ζ to be the collection of all single vertex sets in G, leading to the concepts of *path center*, *path centroid* and *path median* of G. A maximal complete subgraph of G is called a *clique* in G. Let ζ denote the set of all cliques in G. Let r and d represent respectively the *radius* and *diameter* of the graph G. For any real number x, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Santhakumaran and Arumugam [4] introduced the concept of *(V, ζ)-center*, *(ζ, V)-center* and *(ζ, ζ)-center* and investigated their properties.

Definition 1.2 ([4]) Let $C \in \zeta$ and $v \in V$. We define the *vertex-to-clique eccentricity* by $e_1(v) = \max\{d(v, C)\colon C \text{ is a clique in } G\}$. The *clique-to-vertex eccentricity* $e_2(C)$ is defined by $e_2(C) = \max\{d(C, v)\colon v \in V\}$.The *clique-to-clique eccentricity* $e_3(C)$ is defined by $e_3(C) = \max\{d(C, C')\colon C' \in \zeta\}$. The set of all vertices for which $e_1(v)$ is minimum is called the *(V, ζ)-center* of G and is denoted by $Z_1(G)$. The set of all cliques C for which $e_2(C)$ is minimum is called the *(ζ, V)-center* of G and is denoted by $Z_2(G)$. The set of all cliques C for which $e_3(C)$ is minimum is called the *(ζ, ζ)- center* of G and is denoted by $Z_3(G)$.

For a clique C in G, a clique C' with $d(C,C') = e_3(C)$ is called an *eccentric clique of* C in G. For a vertex $v \in V$, a clique C with $d(v,C) = e_1(v)$ is called an *eccentric clique* of v in G.

We need the following theorems in the sequel.

Theorem 1.3 ([4]) *For every vertex v of a graph G, $e_1(v) = e(v)$ or $e(v) - 1$. Further $e_1(v) = e(v)$ if and only if every vertex of any (V,ζ)-eccentric clique of v is an eccentric vertex of v.*

Theorem 1.4 ([4]) *If G is a tree, then $e_1(v) = e(v) - 1$ for every vertex v.*

Theorem 1.5 ([4]) *For any tree T, $Z_1(T) = Z(T)$.*

Theorem 1.6 ([4]) *For any clique C of a graph G, $e_3(C) = e_2(C)$ or $e_2(C) - 1$. Further $e_3(C) = e_2(C)$ if and only if every vertex of any eccentric clique of C is a (ζ, V)-eccentric vertex of C.*

Theorem 1.7 ([4]) *If G is a connected block graph, then $Z_2(G) = Z_3(G)$.*

Theorem 1.8 ([4]) *The (ζ, V)-center $Z_2(T)$ of a tree T forms a star.*

Centrality concepts have interesting applications in social networks [2, 3]. In a social network, a clique represents a group of individuals having a "common interest" and hence centrality, radius and diameter with respect to cliques will have useful applications.

2 Radius and diameter with respect to cliques

Definition 2.1 Let $G = (V, E)$ be a connected graph. The *(V,ζ)-radius* r_1 of G and the *(V,ζ)-diameter* d_1 of G are defined by $r_1 = \min\{e_1(v)\colon v \in V\}$ and $d_1 = \max\{e_1(v)\colon v \in V\}$ respectively. The *(ζ, V)-radius* r_2 and the *(ζ, V)-diameter* d_2 are defined by $r_2 = \min\{e_2(C)\colon C \in \zeta\}$ and $d_2 = \max\{e_2(C)\colon C \in \zeta\}$ respectively. The *(ζ,ζ)- radius* r_3 and the *(ζ,ζ)-diameter* d_3 are defined by $r_3 = \min\{e_3(C)\colon C \in \zeta\}$ and $d_3 = \max\{e_3(C)\colon C \in \zeta\}$ respectively.

Remark 2.2 We observe that for any graph G, $d_1 = d_2$. However r_1 and r_2 need not be equal. For the graph G given in Figure 1, $e_1(v_1) = e_1(v_6) = 3$, $e_1(v_2) = e_1(v_3) = e_1(v_5) = 2$ and $e_1(v_4) = 1$; $e_2(v_1v_2) = e_2(v_1v_3) = e_2(v_5v_6) = 3$ and $e_2(v_2v_4) = e_2(v_3v_4) = e_2(v_4v_5) = 2$. Thus $d_1 = d_2 = 3$, $r_1 = 1$ and $r_2 = 2$.

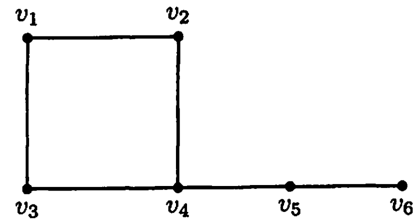

Figure 1: G

Example 2.3 Let G be a connected graph which is not complete such that $\Delta = p - 1$. Then $d = 2$ and $r = 1$. Further every clique of G contains all vertices of degree $p - 1$ and hence $d_1 = 1$, $r_1 = 0$, $d_2 = 1$, $r_2 = 1$, $d_3 = 0$ and $r_3 = 0$.

Theorem 2.4 *Let G be any connected graph and let H be the intersection graph of the family of all cliques in G. For any clique C in G, let $e_H(C)$ denote the eccentricity of the vertex C in H. Let d_H and r_H denote respectively the diameter and radius of H. Then*
(i) $e_3(C) = e_H(C) - 1$.
(ii) $d_3 = d_H - 1$.
(iii) $r_3 = r_H - 1$.

Proof. Let $e_3(C) = n$. Let D be an eccentric clique of C so that $d(C, D) = n$. Let $P = (u_0, u_1, u_2, \dots, u_n)$, where $u_0 \in C$ and $u_n \in D$ be a shortest path in G. Let C_i be a clique containing the edge $u_{i-1}u_i$ $(1 \leq i \leq n)$. Since P is a shortest path in G, the cliques C, C_1, C_2, …, C_n, D are all distinct and $(C, C_1, C_2, \dots, C_n, D)$ is a shortest path joining C and D in H. Further, since $e_3(C) = n$, it follows that D is an eccentric vertex of C in H so that $e_H(C) = n + 1 = e_3(C) + 1$. Thus (i) is proved and now (ii) and (iii) follow from the definitions of d_H, d_3, r_H and r_3. ■

Theorem 2.5 *In any connected graph G, (i) $d_1 = d$ or $d - 1$ and $r_1 = r$ or $r - 1$ and (ii) $d_3 = d_2$ or $d_2 - 1$ and $r_3 = r_2$ or $r_2 - 1$.*

Proof. (i) and (ii) follow from Theorem 1.3 and Theorem 1.6 respectively. ■

A *nonseparable* graph G is connected, nontrivial and has no cut vertices. A *block* of a graph G is a maximal nonseparable subgraph of G. A graph G in which each block is complete is called a *block graph.*

Theorem 2.6 *Let G be a non-complete connected block graph. Then $d_1 = d - 1$, $r_1 = r - 1$, $d_3 = d_2 - 1$ and $r_3 = r_2 - 1$.*

Proof. Let v be any vertex in G and let C be a (V, ζ)-eccentric clique of v. Clearly C is an end-block of G containing exactly one cut vertex say u of G and $e_1(v) = d(v, C) = d(v, u)$. For any vertex w in $C - \{u\}$, $d(v, w) = d(v, u) + 1$ and hence $e_1(v) = e(v) - 1$ so that $d_1 = d - 1$ and $r_1 = r - 1$. Similarly $d_3 = d_2 - 1$ and $r_3 = r_2 - 1$. ∎

Corollary 2.7 *Let G be a connected block graph with $d \geq 2$. Then $d_3 = d - 2$.*

Proof. Since $d_1 = d_2$, the result follows from Theorem 2.6. ∎

The result of Corollary 2.7 is a lower bound for arbitrary connected graphs, as shown in the following theorem.

Theorem 2.8 *Let G be a connected graph with $d \geq 2$. Then $d - 2 \leq d_3 \leq d$.*

Proof. Obviously $d_3 \leq d$. Let u and v be two vertices in G such that $d(u, v) = d$. Let $P = (u = u_0, u_1, \ldots, u_d = v)$ be a shortest $(u$–$v)$ path. Let C be a clique containing the edge u_0u_1 and D be a clique containing the last edge $u_{d-1}u_d$. Then $d(C, D) = d - 2$ and hence $e_3(C) \geq d - 2$ so that $d_3 \geq d - 2$. Thus $d - 2 \leq d_3 \leq d$. ∎

Example 2.9 The bounds in Theorem 2.8 are sharp. For the graph G_1 in Figure 2, $d_3 = 1$ and $d = 2$ so that $d_3 = d - 1$ and for the graph G_2 in Figure 2, $d_3 = 1$ and $d = 3$ so that $d_3 = d - 2$. Also for the Petersen graph, $d_3 = d = 2$.

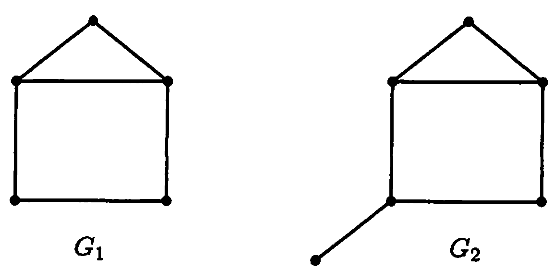

Figure 2:

Problem 2.10 *Characterize the class of graphs for which (i) $d_3 = d - 2$, (ii) $d_3 = d - 1$ and (iii) $d_3 = d$.*

The radius r and diameter d of a connected graph G satisfy the inequality $r \leq d \leq 2r$. We now proceed to prove similar results for r_i and d_i for $i = 1, 2, 3$.

Theorem 2.11 *Let G be a connected graph. Then $r_i \leq d_i \leq 2r_i + 1$, ($i = 1, 2, 3$).*

Proof. We prove the result for $i = 1$. Let $n = \lfloor d_1/2 \rfloor$. Let ($v = v_0$, v_1, $v_2, \ldots, w$) be a (V,ζ)-diameteral path P of length d_1 connecting a vertex v and a clique C, where $w \in C$ so that $d_1 = d(v, C) = d(v, w)$. Let C_0 be a clique in G containing the edge v_0v_1.

We claim that for any vertex u in G, $d(u, C_0) \geq n$ or $d(u, C) \geq n$. Otherwise, there is a vertex u in G such that $d(u, C_0) < n$ and $d(u, C) < n$. Let $P_0 = (u, u_1, u_2, \ldots, u_k)$ be a shortest path connecting u and C_0 and let $Q_0 = (u, w_1, w_2, \ldots, w_l)$ be a shortest path connecting u and C. Hence $k \leq n - 1$, $l \leq n - 1$, $u_k \in C_0$ and $w_l \in C$. Now ($v = v_0$, u_k, $u_{k-1}, \ldots$, u_1, u, w_1, $w_2, \ldots$, w_l) is a walk connecting v and C. Hence $d(v, C) \leq l + k + 1 \leq (n - 1) + (n - 1) + 1 = 2n - 1 < d_1$, which is a contradiction. Thus $d(u, C_0) \geq n$ or $d(u, C) \geq n$ so that $e_1(u) \geq n = \lfloor d_1/2 \rfloor$. Hence $r_1 \geq \lfloor d_1/2 \rfloor$ so that $r_1 \leq d_1 \leq 2r_1 + 1$. The proofs for the cases $i = 2, 3$ are similar. ∎

Example 2.12 The bounds given in Theorem 2.11 are sharp in each case. For any cycle, $d_i = r_i$ for $i = 1, 2, 3$. For the path P_{2m+3} on $(2m + 3)$ vertices, $r_1 = m$ and $d_1 = 2m + 1$ so that $d_1 = 2r_1 + 1$. For the graph G given in Figure 3, $d_2 = 2m + 1$ and $r_2 = m$, where $m \geq 2$ so that $d_2 = 2r_2 + 1$. For the path P_{2m+4} on $(2m + 4)$ vertices, $r_3 = m$ and $d_3 = 2m + 1$ ($m \geq 0$) so that $d_3 = 2r_3 + 1$.

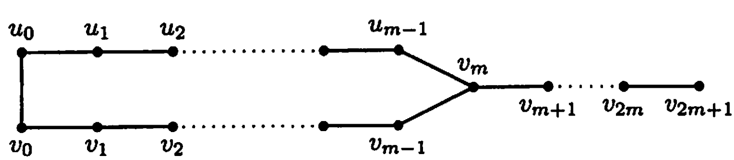

Figure 3: G

Now we proceed to discuss the relation between r_i and d_i ($1 \leq i \leq 3$) for trees.

It is known that for any tree T, $|Z(T)| = 1$ if and only if $d = 2r$, and $|Z(T)| = 2$ if and only if $d = 2r - 1$. Since $|Z(T)| = 1$ or 2 for any tree, we observe that for any tree $d = 2r - 1$ or $2r$.

Theorem 2.13 *For any nontrivial tree T, $|Z_1(T)| = 1$ if and only if $d_1 = 2r_1 + 1$ and $|Z_1(T)| = 2$ if and only if $d_1 = 2r_1$.*

Proof. By Theorem 1.5, $|Z_1(T)| = 1$ if and only if $|Z(T)| = 1$. Also we have $|Z(T)| = 1$ if and only if $d = 2r$. Now, it follows from Theorem 2.6 that $d_1 = d - 1$ and $r_1 = r - 1$. Hence it follows that $|Z_1(T)| = 1$ if and only if $d_1 = 2r_1 + 1$. The proof is similar for the case $|Z_1(T)| = 2$. ∎

Theorem 2.14 *For any tree T, $|Z_2(T)| = 1$ if and only if $d_2 = 2r_2$.*

Proof. Let T be a tree such that $|Z_2(T)| = 1$. If $T = K_2$, then $d_2 = r_2 = 0$. If $T = K_{1,n}$ $(n \geq 2)$, then $|Z_2(T)| = n \geq 2$. Hence T is not a star and $|V(T)| \geq 4$.

Let $Z_2(T) = \{C\}$, where $C = v_0v_1$. Let v be a (ζ, V)-eccentric vertex of C so that $r_2 = e_2(C) = d(C, v) = d(v_1, v)$ (say). Clearly v is a pendant vertex of T. Let $P = (v_1, v_2, \ldots, v_t, v_{t+1} = v)$ be the unique path connecting C and v of length $t = r_2$.

Let $C' = v_1v_2$. As $T \neq K_2$ and $|Z_2(T)| = 1$, $C' \notin Z_2(T)$ so that $e_2(C') > r_2$. Let v' be a (ζ, V)-eccentric vertex of C' so that $r_2 < e_2(C') = d(C', v')$. Clearly v' does not lie on the path P. Let P' be the unique path connecting C' and v'. The first edge on P' must be v_1v_0 and so $P' = (v_1, v_0, u_1, u_2, \ldots, u_l = v')$, where $l \geq r_2$. Since $e_2(C) = r_2$, it follows that $l = r_2$. Since T is a tree, the paths P and P' do not have common vertices except v_0 and v_1. Let $C'' = v_tv_{t+1}$. Now the path $P'' = (v' = u_l, u_{l-1}, \ldots, u_1, v_0, v_1, v_2, \ldots, v_t)$ connecting v' and C'' is such that $d(C'', v') = 2r_2$ and so $e_2(C'') \geq 2r_2$. Hence $d_2 \geq 2r_2$. It follows from Theorem 2.11 that $d_2 = 2r_2$ or $2r_2 + 1$.

Now if $d_2 = 2r_2+1$, then there exists a (ζ, V)- diameteral path $Q = (w_0, w_1, w_2, \ldots, w_m)$ of length $m = 2r_2 + 1$ connecting an edge $E = xw_0$ and a vertex w_m so that $d(E, w_m) = 2r_2 + 1$. Hence it follows that for any edge C in T, $d(C, x) > r_2$ or $d(C, w_m) > r_2$ so that $e_2(C) > r_2$, which is a contradiction. Hence $d_2 = 2r_2$.

Conversely, let $d_2 = 2r_2$. Let $P = (v_1, v_2, \ldots, v_{n-1}, v_n, v_{n+1}, \ldots, v_{2n}, v_{2n+1})$ be a (ζ, V)-diameteral path of length $2n$, where $n = r_2$, connecting an edge $C = v_0v_1$ and a vertex v_{2n+1}. Thus for the edge $C' = v_nv_{n+1}$, we have $d(C', v_{2n+1}) = r_2$ and so $e_2(C') \geq r_2$. If $e_2(C') > r_2$, then we can get an edge uv and a vertex w of T such that $d(uv, w) > 2r_2$, which is a contradiction to $d_2 = 2r_2$. Thus $e_2(C') = r_2$ so that $C' \in Z_2(T)$.

Now, let F be any edge in T such that $F \neq C'$. Then it is clear that

$d(F, v_0) > r_2$ or $d(F, v_{2n+1}) > r_2$ and so $e_2(F) > r_2$. Hence $F \notin Z_2(T)$ and so $Z_2(T) = \{C'\}$. Thus $|Z_2(T)| = 1$. ■

Theorem 2.15 *For any tree T, $|Z_2(T)| \geq 2$ if and only if $d_2 = 2r_2 - 1$.*

Proof. Let T be a tree such that $|Z_2(T)| \geq 2$. Then obviously $T \neq K_2$. If $T = K_{1,n}$ $(n \geq 2)$, then $r_2 = d_2 = 1$ so that $d_2 = 2r_2 - 1$. Suppose $T \neq K_{1,n}$ $(n \geq 2)$ so that $|V(T)| \geq 4$.

Let C_0, $E_0 \in Z_2(T)$. By Theorem 1.8, $Z_2(T)$ forms a star and hence let $C_0 = u_0v_0$ and $E_0 = w_0v_0$. Let v_t be a (ζ, V)-eccentric vertex of C_0 so that $r_2 = e_2(C_0) = d(C_0, v_t)$. Then the unique path P connecting C_0 and v_t has v_0 as its origin, for otherwise $d(E_0, v_t) > r_2$ so that $e_2(E_0) > r_2$, which is a contradiction. Let $P = (v_0, v_1, v_2, \ldots, v_t)$, where $t = r_2$.

Let $F_0 = v_0v_1$. If $F_0 \notin Z_2(T)$, then $e_2(F_0) > r_2$. Let f_l be a (ζ, V)-eccentric vertex of F_0 so that $e_2(F_0) = d(F_0, f_l) > r_2$. The unique path Q of length $l > r_2$ connecting F_0 and f_l has v_0 as its origin, for otherwise both $e_2(C_0) > r_2$ and $e_2(E_0) > r_2$. But the minimum length path from v_0 to f_l shows that $d(C_0, f_l) > r_2$. This final contradiction demonstrates $F_0 \in Z_2(T)$.

Let w_m be a (ζ, V)-eccentric vertex of F_0. Since $d(F_0, v_t) = r_2 - 1$, we have $w_m \neq v_t$. Let $P' = (v_0, w_1, w_2, \ldots, w_m)$ be the path of length $m = r_2$ connecting F_0 and w_m with the origin of P' necessarily v_0. Let $F = w_{m-1}w_m$. Then $Q' = (w_{m-1}, w_{m-2}, \ldots, v_0, v_1, v_2, \ldots, v_t)$ is a path connecting F and v_t so that $d(F, v_t) = 2r_2 - 1$. Hence $e_2(F) \geq 2r_2 - 1$ and so $d_2 \geq 2r_2 - 1$. We claim that $d_2 = 2r_2 - 1$. Suppose $d_2 \neq 2r_2 - 1$. Then we have $d_2 = 2r_2$ or $2r_2 + 1$ since $d_2 \leq 2r_2 + 1$, by Theorem 2.11. Also by Theorem 2.14, $d_2 \neq 2r_2$ and as in the proof of first part of Theorem 2.14, we have $d_2 \neq 2r_2 + 1$. Hence $d_2 = 2r_2 - 1$. The converse follows from Theorem 2.14. ■

Corollary 2.16 *For any tree $T \neq K_2$, $|Z_3(T)| = 1$ if and only if $d_3 = 2r_3+1$ and $|Z_3(T)| \geq 2$ if and only if $d_3 = 2r_3$. Also for $T = K_2$, $|Z_3(T)| = 1$ and $d_3 = 2r_3$.*

Proof. For $T \neq K_2$, this follows from Theorem 1.7, Theorem 2.6, Theorem 2.14 and Theorem 2.15. For $T = K_2$, it is clear that $|Z_3(T)| = 1$ and $d_3 = r_3 = 0$ so that $d_3 = 2r_3$. ■

3 Girth and diameter with respect to cliques

It is known that for a connected graph G which is not a tree, $g(G) \leq 2d+1$, where $g(G)$ denotes the girth of the graph.

We now investigate the relation between the girth $g(G)$ of a connected graph G and the parameters d_i, $i = 1, 2, 3$.

Theorem 3.1 *Let G be a non-complete connected graph which is not a tree. Then $g(G) \leq 2d_1 + 2$ if $g(G)$ is even and $g(G) \leq 2d_1 + 1$ if $g(G)$ is odd.*

Proof. Since G is connected and non-complete, we have $d_1 \geq 1$. Hence the result is obvious if $g(G) = 3$. Suppose $g(G) \geq 4$. Then any clique in G is an edge in G. Let C be a cycle in G of least length. We consider two cases.

Case (i) $C = (v_1, v_2, \ldots, v_{2n}, v_1)$ is an even cycle of length a least 4.

Then $g(G) = 2n$. Let $F = v_n v_{n+1}$. We claim that $d(v_1, F) = n - 1$. Suppose that there exists a path P of length less than $(n-1)$ connecting v_1 and F. Let $v_i \neq v_1$ be the first vertex of the path P that lies on the cycle C. Then at least one $(v_i - v_1)$ section, say Q, of the cycle C has length at most n. Hence the cycle formed by the section of P connecting v_1 and v_i followed by Q has length less than $(n-1)+n = 2n-1 < 2n$, which is a contradiction. Hence $d(v_1, F) = n - 1$. Thus $g(G) = 2n = 2\big(d(v_1, F) + 1\big) \leq 2d_1 + 2$.

Case (ii) $C = (v_1, v_2, \ldots, v_{2n+1}, v_1)$ is an odd cycle of length at least 5.

Then $g(G) = 2n + 1$. Let $F = v_{n+1} v_{n+2}$. Then proceeding as in case (i) we can prove that $d(v_1, F) = n$. Thus $g(G) = 2n + 1 = 2d(v_1, F) + 1 \leq 2d_1 + 1$. ∎

Remark 3.2 The bounds in Theorem 3.1 are sharp. For any even cycle $G = C_{2n}$, $d_1 = n - 1$ and so $g(G) = 2d_1 + 2$. For any odd cycle $G = C_{2n+1}$ of length greater than 3, $d_1 = n$ and so $g(G) = 2d_1 + 1$.

Remark 3.3 If G is a non-complete connected graph which is not a tree, then since $d_1 = d_2$, it follows from Theorem 3.1 that $g(G) \leq 2d_2 + 2$ if $g(G)$ is even and $g(G) \leq 2d_2 + 1$ if $g(G)$ is odd.

Problem 3.4 *Characterize the class of graphs for which $g(G) = 2d_1 + 2$.*

Theorem 3.5 *If G is a connected graph which is not a tree, then $g(G) \leq 2d_3 + 3$.*

Proof. Let $g(G) = n$ and let $C = (v_1, v_2, \ldots, v_n, v_1)$ be a cycle of length n in G. Let $m = \lfloor n/2 \rfloor$. Let $E = v_1 v_2$ and $E' = v_{m+1} v_{m+2}$. Then $d(E, E') = m - 1$. Hence $g(G) \leq 2m+1 = 2\big(d(E, E') + 1\big) + 1 \leq 2d_3 + 3$. ∎

Remark 3.6 The bound in Theorem 3.5 is sharp. For any odd cycle $G = C_{2n+1}$ of length greater than 3, $d_3 = n - 1$ and so $g(G) = 2d_3 + 3$.

For any even cycle $G = C_{2n}$, $d_3 = n - 1$ and so $g(G) = 2d_3 + 2$. If G is a connected graph which is not a tree with $d_3 = 0$, then $g(G) = 3$.

Theorem 3.7 *Let G be a connected graph which is not a tree with $d_3 \neq 0$. Then $g(G) = 2d_3 + 3$ if and only if $G = C_{2d_3+3}$.*

Proof. Since $d_3 \neq 0$, G is not complete. Let $g(G) = 2d_3 + 3$. Since $g(G) \geq 5$, any clique of G is an edge in G. Now let $C = (v_1, v_2, v_3, \ldots, v_{2d_3+3}, v_1)$ be a cycle of length $2d_3 + 3$ in G. If $G \neq C$, then there exists a vertex $v \notin C$ such that v is adjacent to a vertex say v_{d_3+3} in C. Let $E = v_1v_2$ and $E' = v_{d_3+3}v$. We claim that $d(E, E') = d_3 + 1$. Otherwise, there exists a path P of length at most d_3 connecting E and E'. Let $v_i \neq v_1, v_2$ be the first vertex of the path P that lies on C_{2d_3+3}. Then at least one $(v_i - E)$ section, say Q, of the cycle C_{2d_3+3} has length at most $d_3 + 1$. Hence the cycle formed by the section of P connecting E and E' followed by Q has length at most $d_3 + d_3 + 1 + 1$ (the last 1 being the length of the edge $v_{d_3+3}v$, possibly) $= 2d_3 + 2 < 2d_3 + 3$, which is a contradiction. Hence $d(E, E') = d_3 + 1$ so that $e_3(E) \geq d_3 + 1$. It follows that $d_3 \geq d_3 + 1$, which is a contradiction. Thus $G = C_{2d_3+3}$. The converse is obvious. ∎

Remark 3.8 Theorem 3.7 fails if $d_3 = 0$. For the graph G given in Figure 4, $d_3 = 0$ and $g(G) = 3$ so that $g(G) = 2d_3 + 3$.

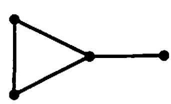

Figure 4: G

Acknowledgments

The authors are thankful to the referee for his valuable suggestions.

References

[1] F. Buckley and F. Harary, Distance in Graphs, Addision- Wesley, Reading MA(1990).

[2] L.C. Freeman, A set of measures of centrality based on betweenness, *Sociometry*, Vol. 40 (1977) 35–41.

[3] L.C. Freeman, Centrality in Social networks: 1. Conceptual clarification, *Social Networks*, Vol.1, 40(1978/79) 215–239.

[4] A.P. Santhakumaran and S. Arumugam, Centrality with respect to cliques, *International Journal of Management and Systems*, Vol. 18 No.3 (2002) 275–280.

[5] P.J. Slater, Centrality of paths and vertices in a graph: Cores and Pits, in "*Theory and applications of graphs*", Ed. G. Chartrand, John Wiley (1981) 529–542.

[6] P.J. Slater, Some definitions of Central Structures, Preprint.