On Local Metric Dimensions of Graphs

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ABSTRACT

For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of k distinct vertices in a nontrivial connected graph G, the metric code of a vertex v of G with respect to W is the k-vector

$$\text{code}(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

where $d(v, w_i)$ is the distance between v and w_i for $1 \leq i \leq k$. The set W is a local metric set of G if $\text{code}(u) \neq \text{code}(v)$ for every pair u, v of adjacent vertices of G. The minimum positive integer k for which G has a local metric set of cardinality k is the local metric dimension $\text{lmd}(G)$ of G. We determine the local metric dimensions of joins and compositions of some well-known classes of graphs, namely complete graphs, cycles, and paths. For a nontrivial connected graph G, a vertex v of G, and an edge e of G, where v is not a cut-vertex and e is not a bridge, it is shown that $\text{lmd}(G - v) \leq \text{lmd}(G) + \deg v$ and $\text{lmd}(G - e) \leq \text{lmd}(G) + 2$. The sharpness of these two bounds are studied. We also present several open questions in this area of research.

Key Words: distance, local metric set, local metric dimension.
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1 Introduction

We refer to the book [2] for graph theory notation and terminology not described in this paper. The *distance* $d(u, v)$ between two vertices u and v in a nontrivial connected graph G is the length of a shortest path between these two vertices. For a vertex v of G, the *eccentricity* $e(v)$ of v is the distance between v and a vertex farthest from v and the *diameter* $\text{diam}(G)$ of G is the largest eccentricity among all vertices of G. Suppose that $W = \{w_1, w_2, \ldots, w_k\}$ is an ordered set of vertices of a nontrivial connected graph G. For each vertex v of G, there is associated a k-vector called the *metric code*, or simply the *code* of v (with respect to W), which is denoted by $\text{code}_W(v)$ and defined by

$$\text{code}_W(v) = (d(v, w_1), d(v, w_2), \cdots, d(v, w_k))$$

(or simply $\text{code}(v)$ if the set W under consideration is clear). If $\text{code}(u) \neq \text{code}(v)$ for every pair u, v of distinct vertices of G, then W is called a *metric set* or a *resolving set*. The minimum k for which G has a metric k-set is the *global metric dimension*, or simply the *metric dimension* of G, which is denoted by $\dim(G)$. This concept has been considered in a number of papers (see [1, 3, 5, 6, 7], for example). In [4] a local version of this concept was considered based on the property mentioned above required of proper vertex colorings. In this case, we consider those ordered sets W of vertices of G for which two vertices of G may have the same code with respect to W provided that each set of vertices having the same code is independent in G. If $\text{code}(u) \neq \text{code}(v)$ for every pair u, v of *adjacent* vertices of G, then W is called a *local metric set* of G. The minimum k for which G has a local metric k-set is the *local metric dimension* of G, which is denoted by $\text{lmd}(G)$. A local metric set of cardinality $\text{lmd}(G)$ in G is a *local metric basis* of G. While each metric set of a nontrivial connected graph G is *vertex-distinguishing* (since every two vertices of G have distinct codes), each local metric set is *neighbor-distinguishing* (since every two *adjacent* vertices of G have distinct codes). Thus every metric set is also a local metric set and so if G is a nontrivial connected graph of order n, then

$$1 \leq \text{lmd}(G) \leq \dim(G) \leq n - 1. \tag{1}$$

We define $\text{lmd}(G) = 0$ if G is the trivial graph K_1. To illustrate these concepts, consider the graph G of Figure 1. In this case, $W_1 = \{v_1, v_4\}$ is a local metric 2-set and $W_2 = \{v_1, v_3, v_5\}$ is a metric 3-set. The corresponding codes for the vertices of G with respect to the sets W_1 and W_2, respectively, are shown in Figure 1. In fact, $\text{lmd}(G) = 2$ and $\dim(G) = 3$.

The following three results have been established in [4]. The *clique number* $\omega(G)$ of a graph G is the order of a largest complete subgraph (clique) in G.

Theorem 1.1 [4] *Let G be a nontrivial connected graph of order n. Then*

(a) $\text{lmd}(G) = n - 1$ *if and only if* $G = K_n$.

(b) $\text{lmd}(G) = n - 2$ *if and only if* $\omega(G) = n - 1$ *and* $n \geq 3$.

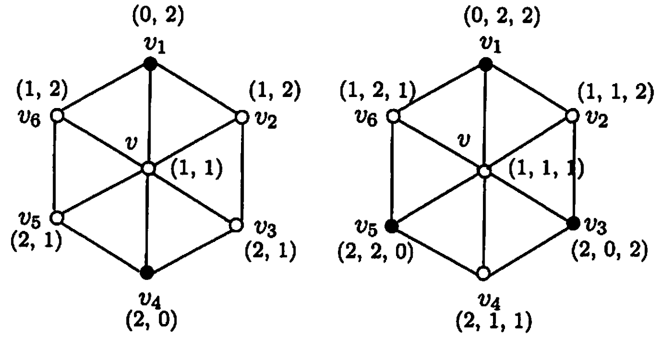

Figure 1: A graph G with $\text{lmd}(G) = 2$ and $\dim(G) = 3$

(c) $\text{lmd}(G) = 1$ *if and only if G is bipartite.*

Furthermore, for each pair k, n of positive integers with $1 \le k \le n-1$, there exists a connected graph G of order n with $\text{lmd}(G) = k$.

Theorem 1.2 [4] *If G is a nontrivial connected graph of order n with clique number ω, then*

$$\text{lmd}(G) \ge \max\{\lceil \log_2 \omega \rceil, n - 2^{n-\omega}\}. \tag{2}$$

Furthermore,

(a) *for each integer $\omega \ge 2$, there exists a connected graph G_ω with clique number ω such that* $\text{lmd}(G_\omega) = \lceil \log_2 \omega \rceil$.

(b) *for each pair n, ω of positive integers with $2^{n-\omega} \le \omega \le n$, there exists a connected graph G of order n whose clique number is ω such that* $\text{lmd}(G) = n - 2^{n-\omega}$.

Theorem 1.3 [4] *For each pair a, b of positive integers with $a \le b$, there is a nontrivial connected graph G with* $\text{lmd}(G) = a$ *and* $\dim(G) = b$.

2 Joins of Graphs

In this section, we study the local metric dimension of the join $G + H$ of two connected graphs G and H. In order to do this, we first present some additional definitions and preliminary information. Two vertices u and v in a connected graph G are *distance similar* if $d(u, x) = d(v, x)$ for all $x \in V(G) - \{u, v\}$. Therefore, if u and v are distance similar vertices, then $0 \le d(u, v) \le 2$. In particular, two nonadjacent vertices u and v are distance similar if and only if $N(u) = N(v)$ (where $N(x)$ denotes the neighborhood of the vertex x or the set of vertices adjacent to x in G), while two adjacent vertices u and v are distance similar if and only if $N(u)-\{v\} = N(v)-\{u\}$. A relation R is defined on $V(G)$ by $u\ R\ v$ if either (i) $u = v$ or (ii) u and v are adjacent and u and v are distance similar. Then R is an equivalence relation on $V(G)$. The equivalence classes resulted from the relation R are called the *distance equivalence classes* of G. If G is a nontrivial connected graph of order n having ℓ distance equivalence classes $U_1, U_2, \ldots, U_\ell$, then every local metric set of G must contain at least $|U_i| - 1$ vertices from U_i for each i with $1 \leq i \leq \ell$. The following three lemmas will be useful to us, the first two of which appear in [4].

Lemma 2.1 [4] *Let G be a nontrivial connected graph of order n and having ℓ distance equivalence classes. If p of these ℓ distance equivalence classes consist of a single vertex, then*

$$n - \ell \leq \text{lmd}(G) \leq n - \ell + p. \tag{3}$$

In particular, if $p = 0$, then $\text{lmd}(G) = n - \ell$.

Lemma 2.2 [4] *For each complete k-partite graph G, where $k \geq 2$,*

$$\text{lmd}(G) = k - 1.$$

Lemma 2.3 *For each integer $n \geq 3$,*

$$\text{lmd}(C_n) = \begin{cases} 1 & \textit{if } n \textit{ is even} \\ 2 & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof. By Theorem 1.1, $\text{lmd}(C_n) = 1$ if n is even and $\text{lmd}(C_n) \geq 2$ if n is odd. On the other hand, any set consisting of two adjacent vertices of C_n is a local metric set and so $\text{lmd}(C_n) = 2$ if n is odd. ■

Theorem 2.4 *For every two connected graphs G and H,*

$$\text{lmd}(G + H) \geq \text{lmd}(G) + \text{lmd}(H).$$

Proof. If $G = H = K_1$, then the result is true trivially. Thus we may assume that G is a nontrivial connected graph. Let W be a local metric basis of $G+H$ and $W_G = W \cap V(G)$ and $W_H = W \cap V(H)$. We claim that $W_G \neq \emptyset$; for otherwise, let x and y be two adjacent vertices of G. Then $\text{code}_W(x) = \text{code}_W(y) = (1, 1, \cdots, 1)$, which is impossible. Thus $W_G \neq \emptyset$. Similarly, if H is a nontrivial connected graph, then $W_H \neq \emptyset$.

Next we show that W_G is a local metric set of G. If this is not the case, then there are adjacent vertices x and y such that $\text{code}_{W_G}(x) = \text{code}_{W_G}(y)$ in G. Let $W_G = \{w_1, w_2, \ldots, w_k\} \subseteq V(G)$. Thus $d_G(x, w_i) = d_G(y, w_i)$ for $1 \leq i \leq k$. If $d_G(x, w_i) = d_G(y, w_i) = 1$, where $1 \leq i \leq k$, then

$d_{G+H}(x, w_i) = d_{G+H}(y, w_i) = 1$, while if $d_G(x, w_i) = d_G(y, w_i) \geq 2$, then $d_{G+H}(x, w_i) = d_{G+H}(y, w_i) = 2$. This implies that $\text{code}_W(x) = \text{code}_W(y)$ in $G + H$, which is impossible. Similarly, if H is a nontrivial connected graph, then W_H is a local metric set of H. Thus regardless of H being trivial or nontrivial, $\text{lmd}(G) + \text{lmd}(H) \leq |W_G| + |W_H| = |W| = \text{lmd}(G + H)$. ■

There are graphs G and H for which $\text{lmd}(G + H) = \text{lmd}(G) + \text{lmd}(H)$. For example, $\text{lmd}(C_5 + C_5) = 4 = 2\,\text{lmd}(C_5)$, which we will see later in Theorem 2.8. Also, there are graphs G and H for which $\text{lmd}(G + H) > \text{lmd}(G) + \text{lmd}(H)$. For example, $\text{lmd}(K_{n_1} + K_{n_2}) = \text{lmd}(K_{n_1}) + \text{lmd}(K_{n_2}) + 1$ for all integers $n_1, n_2 \geq 1$. In fact, the difference between $\text{lmd}(G + H)$ and $\text{lmd}(G) + \text{lmd}(H)$ can be arbitrarily large, as we will see soon. We now determine $\text{lmd}(G + H)$ for several well-known classes of graphs G and H, namely complete graphs, cycles, and paths. In order to do this, we present a useful lemma.

Lemma 2.5 *Let G and H be graphs and let W be a local metric set of G.*

(a) *If $G = K_n + H$ where $n \geq 1$, then $|W \cap V(K_n)| \geq n - 1$.*

(b) *If $G = C_n + H$ where $n \geq 4$, then $|W \cap V(C_n)| \geq \lceil n/4 \rceil$.*

(c) *If $G = P_n + H$ where $n \geq 3$, then $|W \cap V(P_n)| \geq \lceil (n-1)/4 \rceil$.*

Proof. The result in (a) follows by the proof of Theorem 2.4. For (b), let $C_n : u_1, u_2, \ldots, u_n, u_1$ and construct $G = C_n + H$. Consider an arbitrary local metric set W of G and let X be a set of four consecutive vertices in C_n, say $X = \{u_1, u_2, u_3, u_4\}$. If $X \cap W = \emptyset$, then observe that $\text{code}_W(u_2) = \text{code}_W(u_3)$, which is a contradiction. This implies that $|W \cap V(C_n)| \geq \lceil n/4 \rceil$.

For (c), let $P_n : u_1, u_2, \ldots, u_n$ and construct $G = P_n + H$. Let W be an arbitrary local metric set of G. Observe that $\{u_i, u_{i+1}, u_{i+2}, u_{i+3}\} \cap W \neq \emptyset$ for $1 \leq i \leq n - 3$ and furthermore, $\{u_1, u_2, u_3\} \cap W \neq \emptyset$ and $\{u_{n-2}, u_{n-1}, u_n\} \cap W \neq \emptyset$. Hence, $|W \cap V(P_n)| \geq \lceil (n-1)/4 \rceil$. ■

Theorem 2.6 *For integers $n_1 \geq 4$ and $n_2 \geq 1$,*

$$\text{lmd}(C_{n_1} + K_{n_2}) = \max\{2, \lceil \tfrac{n_1}{4} \rceil\} + n_2 - 1.$$

Proof. Let $C_{n_1} : u_1, u_2, \ldots, u_{n_1}, u_1$ and $V(K_{n_2}) = \{v_1, v_2, \ldots, v_{n_2}\}$ and construct $G = C_{n_1} + K_{n_2}$. By Lemma 2.5, $\text{lmd}(G) \geq \lceil n_1/4 \rceil + n_2 - 1$. Consider the set $U \subseteq V(C_{n_1})$ defined by

$$U = \begin{cases} \{u_1, u_4\} & \text{if } n_1 = 4, 6 \\ \{u_1, u_5, \ldots, u_{4\lceil n_1/4 \rceil - 3}\} & \text{if } n_1 = 5 \text{ or } n_1 \geq 7 \end{cases}$$

and let $W = U \cup [V(K_{n_2}) - \{v_1\}]$. Then W is a local metric set of G and so $\text{lmd}(G) \leq |W| = \max\{2, \lceil n_1/4 \rceil\} + n_2 - 1$.

If $n_1 = 4$, then $\text{lmd}(G) \in \{n_2, n_2+1\}$. Assume, to the contrary, that there exists a local metric set W' of G containing n_2 vertices. Then $|W' \cap V(C_4)| = 1$ and $|W' \cap V(K_{n_2})| = n_2 - 1$. Without loss of generality, assume that $u_1 \in W'$ and $v_1 \notin W'$. However, this implies that $\text{code}_{W'}(u_2) = \text{code}_{W'}(v_1)$, which is impossible since $u_2v_1 \in E(G)$. Therefore, $\text{lmd}(G) = n_2 + 1$ for $n_1 = 4$. This establishes the desired result. ∎

Theorem 2.7 *For integers $n_1 \geq 3$ and $n_2 \geq 1$,*

$$\text{lmd}(P_{n_1} + K_{n_2}) = \max\left\{2, \left\lceil \tfrac{n_1-1}{4} \right\rceil\right\} + n_2 - 1.$$

Proof. Let $P_{n_1}: u_1, u_2, \ldots, u_{n_1}$ and $V(K_{n_2}) = \{v_1, v_2, \ldots, v_{n_2}\}$ and construct $G = P_{n_1} + K_{n_2}$. By Lemma 2.5, $\text{lmd}(G) \geq \lceil (n_1-1)/4 \rceil + n_2 - 1$. Let $U \subseteq V(P_{n_1})$ be the set defined by

$$U = \begin{cases} \{u_2, u_3\} & \text{if } 3 \leq n_1 \leq 5 \\ \{u_3, u_7, \ldots, u_{4\lfloor n_1/4 \rfloor - 1}\} & \text{if } n_1 \geq 6 \text{ and } n_1 \equiv 0, 1 \pmod 4 \\ \{u_3, u_7, \ldots, u_{4\lfloor n_1/4 \rfloor - 1}\} \cup \{u_{4\lfloor n_1/4 \rfloor + 2}\} & \text{if } n_1 \geq 6 \text{ and } n_1 \equiv 2, 3 \pmod 4 \end{cases}$$

and let $W = U \cup [V(K_{n_2}) - \{v_1\}]$. Then W is a local metric set of G and so $\text{lmd}(G) \leq |W| = \max\{2, \lceil (n_1-1)/4 \rceil\} + n_2 - 1$.

For $n_1 \leq 5$, observe that $\text{lmd}(G) \in \{n_2, n_2+1\}$. By a similar argument used in the proof of Theorem 2.6, there is no local metric set containing n_2 vertices. Therefore, $\text{lmd}(G) = n_2 + 1$ if $n_1 \leq 5$. ∎

Theorem 2.8 *For integers $n_1, n_2 \geq 4$,*

$$\text{lmd}(C_{n_1} + C_{n_2}) = \max\left\{3, \left\lceil \tfrac{n_1}{4} \right\rceil + \left\lceil \tfrac{n_2}{4} \right\rceil\right\}.$$

Proof. Suppose that $n_1 \leq n_2$. Let

$$C_{n_1}: u_1, u_2, \ldots, u_{n_1}, u_1 \text{ and } C_{n_2}: v_1, v_2, \ldots, v_{n_2}, v_1$$

be disjoint cycles and construct $G = C_{n_1} + C_{n_2}$. By Lemma 2.5, $\text{lmd}(G) \geq \lceil n_1/4 \rceil + \lceil n_2/4 \rceil$. Consider the sets $W_1 \subseteq V(C_{n_1})$ and $W_2 \subseteq V(C_{n_2})$ defined by

$$W_1 = \begin{cases} \{u_1, u_4\} & \text{if } n_1 = 6 \\ \{u_1, u_5, \ldots, u_{4\lceil n_1/4 \rceil - 3}\} & \text{if } n_1 \neq 6 \end{cases}$$

$$W_2 = \begin{cases} \{v_1, v_4\} & \text{if } n_2 = 4, 6 \\ \{v_1, v_5, \ldots, v_{4\lceil n_2/4 \rceil - 3}\} & \text{if } n_2 = 5 \text{ or } n_2 \geq 7. \end{cases}$$

Then $W = W_1 \cup W_2$ is a local metric set of G and so

$$\text{lmd}(G) \leq |W| = \max\{3, \lceil \tfrac{n_1}{4} \rceil + \lceil \tfrac{n_2}{4} \rceil\}.$$

If $n_2 = 4$, then $\text{lmd}(G) = 2$ or $\text{lmd}(G) = 3$. By a similar argument used in the proof of Theorem 2.6, there is no local metric set containing two vertices. Therefore, $\text{lmd}(G) = 3$ in this case. ∎

Theorem 2.9 *For integers $n_1, n_2 \geq 3$,*

$$\operatorname{lmd}(P_{n_1} + P_{n_2}) = \max\left\{3, \left\lceil \tfrac{n_1-1}{4} \right\rceil + \left\lceil \tfrac{n_2-1}{4} \right\rceil \right\}.$$

Proof. Suppose that $n_1 \leq n_2$. Let

$$P_{n_1} : u_1, u_2, \ldots, u_{n_1} \quad \text{and} \quad P_{n_2} : v_1, v_2, \ldots, v_{n_2}$$

be disjoint paths and construct $G = P_{n_1} + P_{n_2}$. By Lemma 2.5, $\operatorname{lmd}(G) \geq \lceil (n_1 - 1)/4 \rceil + \lceil (n_2 - 1)/4 \rceil$. Consider the sets $W_1 \subseteq V(P_{n_1})$ and $W_2 \subseteq V(P_{n_2})$ defined by

$$W_1 = \begin{cases} \{u_3\} & \text{if } 3 \leq n_1 \leq 5 \\ \{u_3, u_7, \ldots, u_{4\lfloor n_1/4 \rfloor - 1}\} & \text{if } n_1 \geq 6,\ n_1 \equiv 0, 1 \pmod 4 \\ \{u_3, u_7, \ldots, u_{4\lfloor n_1/4 \rfloor - 1}\} \cup \{u_{4\lfloor n_1/4 \rfloor + 2}\} & \text{if } n_1 \geq 6,\ n_1 \equiv 2, 3 \pmod 4 \end{cases}$$

$$W_2 = \begin{cases} \{v_2, v_3\} & \text{if } 3 \leq n_2 \leq 5 \\ \{v_3, v_7, \ldots, v_{4\lfloor n_2/4 \rfloor - 1}\} & \text{if } n_2 \geq 6,\ n_2 \equiv 0, 1 \pmod 4 \\ \{v_3, v_7, \ldots, v_{4\lfloor n_2/4 \rfloor - 1}\} \cup \{v_{4\lfloor n_2/4 \rfloor + 2}\} & \text{if } n_2 \geq 6,\ n_2 \equiv 2, 3 \pmod 4. \end{cases}$$

Then $W = W_1 \cup W_2$ is a local metric set of G and so

$$\operatorname{lmd}(G) \leq |W| = \max\{3, \lceil \tfrac{n_1-1}{4} \rceil + \lceil \tfrac{n_2-1}{4} \rceil\}.$$

If $n_2 \leq 5$, then $\operatorname{lmd}(G) = 2$ or $\operatorname{lmd}(G) = 3$. By a similar argument used in the proof of Theorem 2.6, $\operatorname{lmd}(G) \neq 2$ and so $\operatorname{lmd}(G) = 3$ if $n_2 \leq 5$. ∎

Theorem 2.10 *For integers $n_1 \geq 4$ and $n_2 \geq 3$,*

$$\operatorname{lmd}(C_{n_1} + P_{n_2}) = \max\left\{3, \left\lceil \tfrac{n_1}{4} \right\rceil + \left\lceil \tfrac{n_2-1}{4} \right\rceil \right\}.$$

Proof. Let $C_{n_1} : u_1, u_2, \ldots, u_{n_1}, u_1$ and $P_{n_2} : v_1, v_2, \ldots, v_{n_2}$ and construct $G = C_{n_1} + P_{n_2}$. By Lemma 2.5, $\operatorname{lmd}(G) \geq \lceil n_1/4 \rceil + \lceil (n_2 - 1)/4 \rceil$. If $n_1 = 4$ and $n_2 \leq 5$, then let $W = \{u_1, v_2, v_3\}$. Otherwise, consider the sets $W_1 \subseteq V(P_{n_1})$ and $W_2 \subseteq V(P_{n_2})$ defined by

$$W_1 = \begin{cases} \{u_1, u_4\} & \text{if } n_1 = 6 \\ \{u_1, u_5, \ldots, u_{4\lceil n_1/4 \rceil - 3}\} & \text{if } n_1 \neq 6 \end{cases}$$

$$W_2 = \begin{cases} \{v_3\} & \text{if } 3 \leq n_2 \leq 5 \\ \{v_3, v_7, \ldots, v_{4\lfloor n_2/4 \rfloor - 1}\} & \text{if } n_2 \geq 6,\ n_2 \equiv 0, 1 \pmod 4 \\ \{v_3, v_7, \ldots, v_{4\lfloor n_2/4 \rfloor - 1}\} \cup \{v_{4\lfloor n_2/4 \rfloor + 2}\} & \text{if } n_2 \geq 6,\ n_2 \equiv 2, 3 \pmod 4 \end{cases}$$

and let $W = W_1 \cup W_2$. Then W is a local metric set of G and so

$$\operatorname{lmd}(G) \leq |W| = \max\{3, \lceil \tfrac{n_1}{4} \rceil + \lceil \tfrac{n_2-1}{4} \rceil\}.$$

If $n_1 = 4$ and $n_2 \leq 5$, then $\operatorname{lmd}(G) = 2$ or $\operatorname{lmd}(G) = 3$. Again by a similar argument used in the proof of Theorem 2.6, $\operatorname{lmd}(G) \neq 2$ and so $\operatorname{lmd}(G) = 3$ in this case. ∎

3 Compositions of Graphs

For graphs G and H, the *composition* $G[H]$ is the graph with vertex set $V(G) \times V(H)$ such that (v_1, u_1) is adjacent to (v_2, u_2) if either (i) $v_1v_2 \in E(G)$ or (ii) $v_1 = v_2$ and $u_1u_2 \in E(H)$. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then $G[H]$ is constructed by replacing each vertex v_i by a copy H_i of H and joining each vertex in H_i to every vertex in H_j if and only if $v_iv_j \in V(G)$ with $1 \leq i, j \leq n$. Let us always assume that $V(G[H]) = V_1 \cup V_2 \cup \cdots \cup V_n$, where $V_i = \{(v_i, u) : u \in V(H)\}$ for $1 \leq i \leq n$.

If $G = K_1$, then of course $G[H] = H$. Otherwise, consider two distinct vertices $(v_i, u) \in V_i$ and $(v_j, u') \in V_j$ in $G[H]$ where $1 \leq i, j \leq n$. Observe that

$$d_{G[H]}((v_i, u), (v_j, u')) = \begin{cases} \min\{d_H(u, u'), 2\} & \text{if } i = j \\ d_G(v_i, v_j) & \text{if } i \neq j. \end{cases}$$

Theorem 3.1 *If G is a connected graph of order n and H is a graph, then*

$$\text{lmd}(G[H]) \geq n \cdot \text{lmd}(H).$$

Proof. The result is trivially true if $G = K_1$ or $H = K_1$. Hence suppose that both G and H are nontrivial graphs. Suppose that $\text{lmd}(H) = k$ and consider an arbitrary local metric set W of $G[H]$. Assume, to the contrary, that $|W| \leq nk - 1$. Let $W_i = W \cap V_i$ for $1 \leq i \leq n$ and without loss of generality, suppose that $|W_1| \leq k - 1$. Let $W_1 = \{(v_1, w_1), (v_1, w_2), \ldots, (v_1, w_{|W_1|})\}$. Since $W_H = \{w_1, w_2, \ldots, w_{|W_1|}\}$ is not a local metric set of H, there exist adjacent vertices $x, y \in V(H)$ such that $d_H(x, w) = d_H(y, w)$ for every $w \in W_H$. Consider the two vertices (v_1, x) and (v_1, y) in V_1 and observe that they are adjacent in $G[H]$. Then

$$\begin{aligned} d_{G[H]}((v_1, x), (v_1, w)) &= \min\{d_H(x, w), 2\} = \min\{d_H(y, w), 2\} \\ &= d_{G[H]}((v_1, y), (v_1, w)) \end{aligned}$$

for every $(v_1, w) \in W_1$, while

$$d_{G[H]}((v_1, x), (v, u)) = d_G(v_1, v) = d_{G[H]}((v_1, y), (v, u))$$

for every $(v, u) \in V(G[H]) - V_1$. Therefore, $\text{code}((v_1, x)) = \text{code}((v_1, y))$, which is a contradiction. Therefore, $|W| \geq nk$ and so $\text{lmd}(G[H]) \geq nk$. ■

The following is a consequence of Theorem 3.1.

Corollary 3.2 *If G is a connected graph of order n each of whose distance equivalence classes is a singleton set and H is a nontrivial connected graph with* $\text{diam}(H) \leq 2$, *then* $\text{lmd}(G[H]) = n \cdot \text{lmd}(H)$.

Proof. If $n = 1$, then $G[H] = H$ and the result is obvious. Suppose that $n \geq 2$. Let $\text{lmd}(H) = k$ and suppose that $W_H = \{w_1, w_2, \ldots, w_k\}$ is a local metric basis of H. Let $W_i = \{(v_i, w_1), (v_i, w_2), \ldots, (v_i, w_k)\} \subseteq V_i$ for $1 \leq i \leq n$ and $W = W_1 \cup W_2 \cup \cdots \cup W_n$. Hence $|W| = nk$. We show that W is a local metric set of $G[H]$.

Suppose that $(v_i, x), (v_j, y) \in V(G[H]) - W$ and $(v_i, x)(v_j, y) \in E(G[H])$. Hence, $x, y \notin W_H$. We consider two cases.

Case 1. $i = j$, *say* $i = j = 1$. Since W_H is a local metric set of H, there exists a vertex $w \in W_H$ such that $d_H(x, w) \neq d_H(y, w)$. Since $\text{diam}(H) \leq 2$, it follows that $\{d_H(x, w), d_H(y, w)\} = \{1, 2\}$. Then observe that $(v_1, w) \in W$ and

$$\begin{aligned} d_{G[H]}((v_1, x), (v_1, w)) &= \min\{d_H(x, w), 2\} \neq \min\{d_H(y, w), 2\} \\ &= d_{G[H]}((v_1, y), (v_1, w)). \end{aligned}$$

Case 2. $i \neq j$, *say* $i = 1$ *and* $j = 2$. Hence $v_1 v_2 \in E(G)$. Since v_1 and v_2 belong to different distance equivalence classes in G, there exists a vertex $v \in V(G) - \{v_1, v_2\}$ such that $d_G(v_1, v) \neq d_G(v_2, v)$. Observe then that $(v, w_1) \in W$ and

$$\begin{aligned} d_{G[H]}((v_1, x), (v, w_1)) &= d_G(v_1, v) \neq d_G(v_2, v) \\ &= d_{G[H]}((v_2, y), (v, w_1)). \end{aligned}$$

Thus, $\text{code}_W((v_i, x)) \neq \text{code}_W((v_j, y))$ in each case and so W is a local metric set of $G[H]$. Consequently, $\text{lmd}(G[H]) \leq nk$ and the result now follows by Theorem 3.1. ∎

The converse of Corollary 3.2 is false. To see this, consider a 7-cycle $C_7 : u_1, u_2, \ldots, u_7, u_1$ and let $H = C_7 + u_2 u_7$. Then $\text{diam}(H) = 3$ and the set $W_H = \{u_3, u_6\}$ is a local metric set and so $\text{lmd}(H) = 2$. Let $G = K_3$ with $V(G) = \{v_1, v_2, v_3\}$ and construct $G[H]$. Of course, G has only one distance equivalence class, which is not a singleton set. Let $W = \{(v_i, u_3), (v_i, u_6) : 1 \leq i \leq 3\}$ and observe that W is a local metric set of $G[H]$. Therefore, $\text{lmd}(G[H]) = 6 = 3 \cdot \text{lmd}(H)$ in this case. In fact, with this particular graph H, we have $\text{lmd}(G[H]) = n \cdot \text{lmd}(H)$ for every connected graph G of order n.

The key to observe is the following: The graph H described above has the property that there exists a local metric basis W such that

- for every vertex $x \in V(H) - W$,

  $$\max\{d(x, w) : w \in W\} \geq 2,$$

  that is, there is no vertex in $V(H) - W$ that is adjacent to all vertices in W, and

- for every two adjacent vertices $x, y \in V(H) - W$, there exists a vertex $w \in W$ such that

$$\{d(x,w), d(y,w)\} = \{1,2\}.$$

Note that for every graph H, the join $H + H$ equals $K_2[H]$. By Theorems 2.8 and 2.9 then,

$$\begin{aligned}
\text{lmd}(K_2[C_n]) &= \begin{cases} 3 & \text{if } n = 4 \\ 2\left\lceil \frac{n}{4} \right\rceil & \text{if } n \geq 5 \end{cases} \\
\text{lmd}(K_2[P_n]) &= \begin{cases} 3 & \text{if } 3 \leq n \leq 5 \\ 2\left\lceil \frac{n-1}{4} \right\rceil & \text{if } n \geq 6. \end{cases}
\end{aligned}$$

We next generalize these to obtain the local metric dimension of $G[H]$, where one of G and H is a complete graph and the other is either a cycle or a path. First, we present a useful lemma, whose proof is similar to the proof of Lemma 2.5.

Lemma 3.3 *Let G be a connected graph of order $n_1 \geq 2$. Let H be a graph and W a local metric set of $G[H]$.*

(a) *If $H = K_{n_2}$ where $n_2 \geq 2$, then $|W \cap V_i| \geq n_2 - 1$ for $1 \leq i \leq n_1$.*

(b) *If $H = C_{n_2}$ where $n_2 \geq 4$, then $|W \cap V_i| \geq \lceil n_2/4 \rceil$ for $1 \leq i \leq n_1$.*

(c) *If $H = P_{n_2}$ where $n_2 \geq 3$, then $|W \cap V_i| \geq \lceil (n_2 - 1)/4 \rceil$ for $1 \leq i \leq n_1$.*

Theorem 3.4 *Suppose that $n_1 \geq 2$ is an integer.*

(a) *For $n_2 \geq 4$,*

$$\begin{aligned}
\text{lmd}(K_{n_1}[C_{n_2}]) &= \begin{cases} 2n_1 - 1 & \text{if } n_2 = 4 \\ n_1\left\lceil \frac{n_2}{4} \right\rceil & \text{if } n_2 \geq 5 \end{cases} \\
\text{lmd}(C_{n_2}[K_{n_1}]) &= n_2(n_1 - 1).
\end{aligned}$$

(b) *For $n_2 \geq 3$,*

$$\begin{aligned}
\text{lmd}(K_{n_1}[P_{n_2}]) &= \begin{cases} 2n_1 - 1 & \text{if } 3 \leq n_2 \leq 5 \\ n_1\left\lceil \frac{n_2-1}{4} \right\rceil & \text{if } n_2 \geq 6 \end{cases} \\
\text{lmd}(P_{n_2}[K_{n_1}]) &= n_2(n_1 - 1).
\end{aligned}$$

Proof. Let $G = K_{n_1}$ with $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$. We first verify (a). Let $H = C_{n_2} : u_1, u_2, \ldots, u_{n_2}, u_1$ and construct $G[H]$. For $n_2 \geq 5$, consider $W \subseteq V(G[H])$ such that

$$W \cap V_i = \begin{cases} \{(v_i, u_1), (v_i, u_4)\} & \text{if } n_2 = 6 \\ \{(v_i, u_1), (v_i, u_5), \ldots, (v_i, u_{4\lceil n_2/4\rceil - 3})\} & \text{if } n_2 = 5 \text{ or } n_2 \geq 7 \end{cases}$$

for $1 \leq i \leq n_1$. Then W is a local metric set with $|W| = n_1\lceil n_2/4\rceil$. By Lemma 3.3, $\text{lmd}(G[H]) = n_1\lceil n_2/4\rceil$.

If $n_2 = 4$, then let $W \subseteq V(G[H])$ such that $W \cap V_1 = \{(v_1, u_1)\}$ and $W \cap V_i = \{(v_i, u_1), (v_i, u_4)\}$ for $2 \leq i \leq n_1$. Then W is a local metric set whose cardinality is $2n_1 - 1$.

If $W' \subseteq V(K_{n_1}[C_4])$ is a set containing $2n_1 - 2$ vertices, then we may assume, without loss of generality, that $|W' \cap V_1| = |W' \cap V_2| = 1$. Suppose that $W' \cap V_i = \{(v_i, u_1)\}$ for $i = 1, 2$. Then $\text{code}_{W'}((v_1, u_2)) = \text{code}_{W'}((v_2, u_2))$ and so W' is not a local metric set of $K_{n_1}[C_4]$. Therefore, $\text{lmd}(K_{n_1}[C_4]) = 2n_1 - 1$.

For $C_{n_2}[K_{n_1}] = H[G]$, consider the set $W = V(H[G]) - \{(u_i, v_1) : 1 \leq i \leq n_2\}$ and observe that W is a local metric set of $H[G]$. By Lemma 3.3,

$$n_2(n_1 - 1) \leq \text{lmd}(H[G]) \leq |W| = n_2(n_1 - 1)$$

and the result now follows.

To verify (b), let $H = P_{n_2} : u_1, u_2, \ldots, u_{n_2}$ and construct $G[H]$. For $n_2 \geq 6$, consider $W \subseteq V(G[H])$ such that

$$W \cap V_i = \{(v_i, u_3), (v_i, u_7), \ldots, (v_i, u_{4\lfloor n_2/4\rfloor - 1})\}$$

if $n_2 \equiv 0, 1 \pmod 4$; while

$$W \cap V_i = \{(v_i, u_3), (v_i, u_7), \ldots, (v_i, u_{4\lfloor n_2/4\rfloor - 1})\} \cup \{(v_i, u_{4\lfloor n_2/4\rfloor + 2})\}$$

if $n_2 \equiv 2, 3 \pmod 4$ (where $1 \leq i \leq n_1$). Then W is a local metric set with $|W| = n_1\lceil (n_2 - 1)/4\rceil$. By Lemma 3.3, $\text{lmd}(G[H]) = n_1\lceil (n_2 - 1)/4\rceil$.

If $3 \leq n_2 \leq 5$, then let $W \subseteq V(G[H])$ such that $W \cap V_1 = \{(v_1, u_3)\}$ and $W \cap V_i = \{(v_i, u_2), (v_i, u_3)\}$ for $2 \leq i \leq n_1$. Then W is a local metric set containing $2n_1 - 1$ vertices. By a similar argument used for verifying (a), $|W \cap V_i| = 1$ for at most one i for any local metric set W of $G[H]$, implying that $|W| \geq 2n_1 - 1$. Therefore, $\text{lmd}(G[H]) = 2n_1 - 1$.

To prove that $\text{lmd}(H[G]) = n_2(n_1 - 1)$ is straightforward. ∎

Corollary 3.5 *Let G be a connected graph of order $n_1 \geq 2$. Then*

(a) $n_1(n_2 - 1) \leq \text{lmd}(G[K_{n_2}]) \leq n_1(n_2 - 1) + \text{lmd}(G)$ for $n_2 \geq 1$;

(b) $\text{lmd}(G[C_{n_2}]) = n_1\lceil n_2/4\rceil$ for $n_2 \geq 5$;

(c) $\mathrm{lmd}(G[P_{n_2}]) = n_1\lceil (n_2-1)/4\rceil$ for $n_2 \geq 6$.

Proof. We first verify (a). Since the result is trivial for $n_2 = 1$, suppose that $n_2 \geq 2$. By Lemma 3.3, $n_1(n_2-1) \leq \mathrm{lmd}(G[K_{n_2}])$ and so we only show that $\mathrm{lmd}(G[K_{n_2}]) \leq n_1(n_2-1) + \mathrm{lmd}(G)$. Let $\mathrm{lmd}(G) = k$ and $V(G) = \{v_1, v_2, \ldots, v_{n_1}\}$ such that $W_G = \{v_1, v_2, \ldots, v_k\}$ is a local metric basis of G. Let $W \subseteq V(G[K_{n_2}])$ such that

$$W \cap V_i = \begin{cases} V_i & \text{if } 1 \leq i \leq k \\ V_i - \{(v_i, u_1)\} & \text{if } k+1 \leq i \leq n_1. \end{cases}$$

Then W is a local metric set of $G[K_{n_2}]$ whose cardinality is $n_1(n_2-1)+k$.

For (b), we need only show that $\mathrm{lmd}(G[C_{n_2}]) \leq n_1\lceil n_2/4\rceil$ by Lemma 3.3. Let $C_{n_2} : u_1, u_2, \ldots, u_{n_2}, u_1$ and consider the set $W \subseteq V(G[C_{n_2}])$ such that

$$W \cap V_i = \begin{cases} \{(v_i,u_1),(v_i,u_4)\} & \text{if } n_2 = 6 \\ \{(v_i,u_1),(v_i,u_5),\ldots,(v_i,u_{4\lceil n_2/4\rceil-3})\} & \text{if } n_2 = 5 \text{ or } n_2 \geq 7 \end{cases}$$

for $1 \leq i \leq n_1$. Consider two adjacent vertices (v_i, x) and (v_j, y) belonging to $V(G[C_{n_2}]) - W$. If $i = j$, then there exists $w \in W \cap V_i$ such that $\{d((v_i,x),w), d((v_j,y),w)\} = \{1,2\}$, that is, $d((v_i,x),w) \neq d((v_j,y),w)$. On the other hand, if $i \neq j$, then $v_iv_j \in E(G)$ and so $|N((v_i,x)) \cap W_i| < |W_i| = |N((v_j,y)) \cap W_i|$, where $W_i = W \cap V_i$. Therefore, $\mathrm{code}_W((v_i,x)) \neq \mathrm{code}_W((v_j,y))$. Hence, W is a local metric set of $G[C_{n_2}]$ containing $n_1\lceil n_2/4\rceil$ vertices and we obtain the desired result.

For (c), we again only show that $\mathrm{lmd}(G[C_{n_2}]) \leq n_1\lceil (n_2-1)/4\rceil$. Let $P_{n_2} : u_1, u_2, \ldots, u_{n_2}$ and consider the set $W \subseteq V(G[P_{n_2}])$ such that

$$W \cap V_i = \{(v_i,u_3),(v_i,u_7),\ldots,(v_i,u_{4\lfloor n_2/4\rfloor-1})\}$$

if $n_2 \equiv 0, 1 \pmod 4$; while

$$W \cap V_i = \{(v_i,u_3),(v_i,u_7),\ldots,(v_i,u_{4\lfloor n_2/4\rfloor-1})\} \cup \{(v_i,u_{4\lfloor n_2/4\rfloor+2})\}$$

if $n_2 \equiv 2, 3 \pmod 4$ (where $1 \leq i \leq n_1$). The result now follows since W is a local metric set of $G[C_{n_2}]$ containing $n_1\lceil (n_2-1)/4\rceil$ vertices. ■

Note that the upper and lower bounds in (a) are both sharp. For example, $\mathrm{lmd}(C_{n_1}[K_{n_2}]) = n_1(n_2-1)$ for $n_1 \geq 4$, while $\mathrm{lmd}(K_{n_1}[K_{n_2}]) = n_1n_2 - 1 = n_1(n_2-1) + \mathrm{lmd}(K_{n_1})$ for $n_1, n_2 \geq 1$.

4 Vertex or Edge Deletions

A common question in graph theory concerns how the value of a parameter is affected by making a small change in the graph. In this section, we study how the local metric dimension of a connected graph is affected by deleting a vertex or an edge from the graph, beginning with deleting a vertex. Observe that for the wheel $W_n = C_n + K_1$ of order $n+1 \geq 4$ and a vertex $v \in V(W_n)$,

$$W_n - v \in \{C_n, P_{n-1} + K_1\}.$$

For $n \geq 3$ we have seen that

$$\operatorname{lmd}(W_n) = \begin{cases} 3 & \text{if } n = 3 \\ 2 & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil & \text{if } n \geq 5. \end{cases}$$

$$\operatorname{lmd}(C_n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 2 & \text{if } n \text{ is odd.} \end{cases}$$

$$\operatorname{lmd}(P_{n-1} + K_1) = \begin{cases} 2 & \text{if } 3 \leq n \leq 6 \\ \lceil \frac{n-2}{4} \rceil & \text{if } n \geq 7. \end{cases}$$

Thus for a connected graph G and a vertex v of G, it is possible that $\operatorname{lmd}(G - v) = \operatorname{lmd}(G)$ or $\operatorname{lmd}(G - v) < \operatorname{lmd}(G)$. In general, we have the following. For a vertex v in a nontrivial graph G, let $N[v] = N(v) \cup \{v\}$ be the *closed neighborhood* of v.

Theorem 4.1 *If v is a non-cut-vertex of a nontrivial connected graph G, then*

$$\operatorname{lmd}(G - v) \leq \operatorname{lmd}(G) + \deg v.$$

Proof. Let W be a local metric basis of a connected graph G and let $W' = [W \cup N[v]] - \{v\}$. Thus $|W'| \leq |W| + \deg v = \operatorname{lmd}(G) + \deg v$. We show that W' is a local metric set of $G - v$, for otherwise, there exists a pair $x, y \in V(G - v) - W'$ of adjacent vertices such that $\operatorname{code}_{W'}(x) = \operatorname{code}_{W'}(y)$ in $G - v$. Since W is a local metric set of G, there exists a vertex $w \in W$ such that $d_G(x, w) \neq d_G(y, w)$, say $a = d_G(x, w) < d_G(y, w)$. Then since $w \in W'$ and $\operatorname{code}_{W'}(x) = \operatorname{code}_{W'}(y)$, it follows that $d_{G-v}(x, w) = d_{G-v}(y, w)$. Hence

$$a = d_G(x, w) < d_G(y, w) \leq d_{G-v}(y, w) = d_{G-v}(x, w),$$

implying that every $x - w$ geodesic in G contains the vertex v. Let $P : x = u_0, u_1, \ldots, u_b, u_{b+1} = v, u_{b+2}, \ldots, u_a = w$ be an $x - w$ geodesic in G. Thus, $d_{G-v}(x, u_b) = d_G(x, u_b) = b$. Furthermore, since $d_G(y, w) \geq a + 1$, the path $Q : y, x$ followed by P is a $y - w$ geodesic in G. This implies that

$$d_{G-v}(y, u_b) = d_G(y, u_b) = b + 1 > d_{G-v}(x, u_b),$$

which is a contradiction since $u_b \in N(v) \subseteq W'$ and $\operatorname{code}_{W'}(x) = \operatorname{code}_{W'}(y)$. Therefore, W' is a local metric set of $G - v$ and so $\operatorname{lmd}(G - v) \leq |W'| \leq \operatorname{lmd}(G) + \deg v$. ∎

By the proof of Theorem 4.1, if there exists a local metric basis W of a connected graph G such that $W \cap N[v] \neq \emptyset$, then $W' = [W \cup N[v]] - \{v\}$ is a local metric set of $G - v$ and $|W'| \leq \text{lmd}(G) + \deg v - 1$. Thus

$$\text{lmd}(G - v) \leq |W'| \leq \text{lmd}(G) + \deg v - 1.$$

In fact, for every local metric basis W of G,

$$\text{lmd}(G - v) \leq \text{lmd}(G) + \deg v - |W \cap N[v]|.$$

Although it is not known whether there exists a connected graph H containing a non-cut-vertex v for which $\text{lmd}(H - v) = \text{lmd}(H) + \deg v$, there are infinitely many connected graphs G containing a non-cut-vertex v for which $\text{lmd}(G - v) = \text{lmd}(G) + \deg v - 1$.

Theorem 4.2 *For every positive integer k, there exists a connected graph G containing a non-cut-vertex v such that G has local metric dimension k and*

$$\text{lmd}(G - v) = \text{lmd}(G) + \deg v - 1.$$

Proof. For $k = 1$, let G be a tree of order at least 3 and v an end-vertex. Hence, assume that $k \geq 2$ and consider the set $A = \{1, 2, \ldots, k-2\}$ for $k \geq 3$ while $A = \emptyset$ if $k = 2$, and let $\mathcal{P}(A) = \{S_1, S_2, \ldots, S_{2^{k-2}}\}$ be the power set of A. Define the sets $S_{2^{k-2}+1}, S_{2^{k-2}+2}, \ldots, S_{2^k}$ by

$$S_{i+2^{k-2}} = S_i \cup \{k-1\}, \quad S_{i+2^{k-1}} = S_i \cup \{k\}, \quad S_{i+2^{k-1}+2^{k-2}} = S_i \cup \{k-1, k\}$$

for $1 \leq i \leq 2^{k-2}$ and observe that $\{S_1, S_2, \ldots, S_{2^k}\}$ is the power set of $A \cup \{k-1, k\} = \{1, 2, \ldots, k\}$. Let $H = K_{2^k}$ be a complete graph of order 2^k with $V(H) = \{u_1, u_2, \ldots, u_{2^k}\}$. We construct G from H by adding k new vertices in the set $W = \{w_1, w_2, \ldots, w_k\}$ and joining u_i to w_j if and only if $j \in S_i$. Hence W is a local metric basis and $\text{lmd}(G) = k$ (see [4]). Furthermore, $\deg w_i = 2^{k-1}$ for $1 \leq i \leq k$.

We show that $\text{lmd}(G - w_i) = k + 2^{k-1} - 1$ for $1 \leq i \leq k$. By symmetry, it suffices to show that $\text{lmd}(G - w_k) = k + 2^{k-1} - 1$. Since the set

$$W' = (W \cup N[w_k]) - \{w_k\} = (W - \{w_k\}) \cup \{u_{2^{k-1}+1}, u_{2^{k-1}+2}, \ldots, u_{2^k}\}$$

is a local metric set of $G - w_k$ containing $k + 2^{k-1} - 1$ vertices,

$$\text{lmd}(G - w_k) \leq k + 2^{k-1} - 1.$$

Observe that each set $U_i = \{u_i, u_{i+2^{k-1}}\}$ is a distance equivalence class in $G - w_k$ for $1 \leq i \leq 2^{k-1}$. Thus, if there exists a local metric set W^* containing at most $k + 2^{k-1} - 2$ vertices, then we may assume that

$$\{u_1, u_2, \ldots, u_{2^{k-1}}\} \subseteq W^* \text{ and } w_{k-1} \notin W^*.$$

On the other hand,

$$d_{G-w_k}(u_{i+2^{k-1}}, v) \neq d_{G-w_k}(u_{i+2^{k-1}+2^{k-2}}, v) \text{ if and only if } v = w_{k-1}$$

for $1 \leq i \leq 2^{k-2}$. Since $w_{k-1} \notin W^*$, we further assume that

$$\{u_{2^{k-1}+1}, u_{2^{k-1}+2}, \ldots, u_{2^{k-1}+2^{k-2}}\} \subseteq W^*.$$

However then, $2^{k-1} + 2^{k-2} \leq |W^*| < 2^{k-1} + k - 1$, which is impossible. Therefore, $\text{lmd}(G - w_k) = k + 2^{k-1} - 1$ as claimed. ∎

The following result provides a sufficient condition for a connected graph G containing a vertex v to have $\text{lmd}(G - v) \leq \text{lmd}(G) + \deg v - 1$.

Theorem 4.3 *Let v be a vertex with $\deg v \geq 2$ that is not a cut-vertex in a connected graph G. If there exists a vertex $v_1 \in N(v)$ such that $d_{G-v}(z, v_1) \leq 2$ for every $z \in N(v) - \{v_1\}$, then $\text{lmd}(G - v) \leq \text{lmd}(G) + \deg v - 1$.*

Proof. Let W be a local metric basis of G. We may assume that $W \cap N[v] = \emptyset$. Let $W' = W \cup N(v)$ and $W_1' = W' - \{v_1\}$. We show that W_1' is a local metric set of $G - v$.

Suppose that this is not the case. Since W' is a local metric set of $G-v$, it follows that there exists a pair $x, y \in V(G - v) - W_1'$ of adjacent vertices such that

$$d_{G-v}(x, w) \neq d_{G-v}(y, w) \text{ if and only if } w = v_1$$

for each vertex $w \in W'$. Also, since W is a local metric set of G, there exists a vertex $w^* \in W \subseteq W'$ such that $d_G(x, w^*) \neq d_G(y, w^*)$, say $a = d_G(x, w^*) < d_G(y, w^*)$. Observe that $d_{G-v}(x, w^*) = d_{G-v}(y, w^*)$ by assumption since $w^* \neq v_1$. Then

$$d_G(x, w^*) < d_G(y, w^*) \leq d_{G-v}(y, w^*) = d_{G-v}(x, w^*),$$

implying that every $x - w^*$ geodesic in G contains v. Let $P : x = u_0, u_1, \ldots, u_b, u_{b+1} = v, u_{b+2}, \ldots, u_a = w^*$ be an $x - w^*$ geodesic in G. Observe that $u_b, u_{b+2} \in N(v) \subseteq W'$. Since $d_G(x, w^*) < d_G(y, w^*)$, it follows that $d_{G-v}(x, u_b) \neq d_{G-v}(y, u_b)$ and so $u_b = v_1$. However then, $d_{G-v}(u_b, u_{b+2}) = d_{G-v}(v_1, u_{b+2}) \leq 2$, implying that there exists an $x - w^*$ path in $G - v$ having length at most a. This is a contradiction.

Therefore, no such pair x, y exists and W_1' is a local metric set of $G - v$. Consequently, $\text{lmd}(G - v) \leq |W_1'| = \text{lmd}(G) + \deg v - 1$. ∎

It is not known whether there is a connected graph G containing a non-cut-vertex v of G such that $\text{lmd}(G - v) = \text{lmd}(G) + \deg v$. On the other hand, there are many connected graphs G with a vertex v such that $\text{lmd}(G - v) = \text{lmd}(G)$. The following observation will be useful to us.

Observation 4.4 *If G is a nontrivial connected graph and v is an end-vertex of G, then G contains a local metric basis not containing v.*

Proposition 4.5 *If v is an end-vertex in a connected graph G, then* $\text{lmd}(G - v) = \text{lmd}(G)$.

Proof. We first show that $\text{lmd}(G - v) \leq \text{lmd}(G)$. By Observation 4.4 there is a local metric basis W of G such that $v \notin W$. Since $d_{G-v}(x, y) = d_G(x, y)$ for every two vertices $x, y \in V(G - v)$, it follows that W is a local metric set of $G - v$.

Next we verify that $\text{lmd}(G) \leq \text{lmd}(G-v)$. Let v_1 be the vertex adjacent to v in G and suppose that W' is a local metric basis of $G - v$. Consider a pair x, y of adjacent vertices in $V(G) - W'$. If $v \in \{x, y\}$, then $\{x, y\} = \{v, v_1\}$. Since $d_G(v, u) = d_G(v_1, u) + 1$ for every $u \in V(G) - \{v\}$, it follows that $\text{code}_{W'}(v) \neq \text{code}_{W'}(v_1)$. Hence, assume that $v \notin \{x, y\}$. Since W' is a local metric set of $G - v$, it follows that there exists a vertex $w \in W'$ such that $d_{G-v}(x, w) \neq d_{G-v}(y, w)$. On the other hand, $d_G(x, w) = d_{G-v}(x, w)$ as well as $d_G(y, w) = d_{G-v}(y, w)$ and so

$$d_G(x, w) = d_{G-v}(x, w) \neq d_{G-v}(y, w) = d_G(y, w),$$

that is, $d_G(x, w) \neq d_G(y, w)$. Hence, W' is a local metric set of G. ■

Next, we investigate how the local metric dimension of a connected graph is affected by deleting an edge from the graph.

Theorem 4.6 *If e is an edge that is not a bridge of a connected graph G, then*

$$\text{lmd}(G - e) \leq \text{lmd}(G) + 2.$$

Proof. Let W be a local metric basis of G, $e = v_1v_2$, and $W' = W \cup \{v_1, v_2\}$. Then $|W'| \leq |W| + 2 = \text{lmd}(G) + 2$. We show that W' is a local metric set of $G - e$. If this is not the case, then there exists a pair $x, y \in V(G-e) - W'$ of adjacent vertices such that $\text{code}_{W'}(x) = \text{code}_{W'}(y)$ in $G-e$. Since W is a local metric set of G, there exists a vertex $w \in W$ such that $d_G(x, w) \neq d_G(y, w)$, say $a = d_G(x, w) < d_G(y, w)$. Also, since $w \in W'$ and $\text{code}_{W'}(x) = \text{code}_{W'}(y)$, it follows that $d_{G-e}(x, w) = d_{G-e}(y, w)$. Hence

$$a = d_G(x, w) < d_G(y, w) \leq d_{G-e}(x, w) = d_{G-e}(y, w),$$

implying that every $x - w$ geodesic in G contains the edge e. We may assume, therefore, that $P : x = u_0, u_1, \ldots, u_b = v_1, u_{b+1} = v_2, \ldots, u_a = w$ is an $x - w$ geodesic in G. Observe that $d_{G-e}(x, v_1) = d_G(x, v_1) = b$. Furthermore, since $d_G(y, w) \geq a + 1$, the path $Q : y, x$ followed by P is a $y - w$ geodesic in G. Therefore, $d_{G-e}(y, v_1) = d_G(y, v_1) = b + 1 > d_{G-e}(x, v_1)$, which contradicts the fact that $\text{code}_{W'}(x) = \text{code}_{W'}(y)$ and

$v_1 \in W'$. Therefore, W' is a local metric set of $G - e$ and $\text{lmd}(G - e) \leq |W'| \leq \text{lmd}(G) + 1$. ■

By the proof of Theorem 4.6 if $e = v_1v_2$ is not a bridge of G and there exists a local metric basis W of G such that $W \cap \{v_1, v_2\} \neq \emptyset$, then

$$\text{lmd}(G - e) \leq \text{lmd}(G) + 1.$$

In fact, for every local metric basis W of G,

$$\text{lmd}(G - e) \leq \text{lmd}(G) + 2 - |W \cap \{v_1, v_2\}|.$$

We conclude this paper with the following two conjectures.

Conjecture 4.7 *If v is a vertex that is not a cut-vertex of a connected graph G, then* $\text{lmd}(G - v) \geq \text{lmd}(G) - \deg v$.

Conjecture 4.8 *If e is an edge that is not a bridge of a connected graph G, then* $\text{lmd}(G - e) \geq \text{lmd}(G) - 2$.

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