$\mathbb{Z}$-Cyclic DTWh(p)/OTWh(p), The Empirical Study Continued For Primes $p \equiv 2^k + 1 \pmod{2^{k+1}}$, $k = 8$

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Abstract

In the past few years several studies have appeared that relate to the existence of $\mathbb{Z}$-cyclic directed-triplewhist tournaments and $\mathbb{Z}$-cyclic ordered-triplewhist tournaments. In these studies the number of players in the tournament is taken to be a prime p of the form $p \equiv 2^k+1 \pmod{2^{k+1}}$, $k \geq 2$. For the cases $k = 2, 3, 4$ it has been shown [6, 4, 5, 12] that $\mathbb{Z}$-cyclic directed-triplewhist tournaments and $\mathbb{Z}$-cyclic ordered-triplewhist tournaments exist for all such primes except for the impossible cases $p = 5, 13, 17$. For the cases $k = 5, 6, 7$ it has been shown [13] that $\mathbb{Z}$-cyclic directed-triplewhist tournaments exist for all such primes less than $3,200,000$ and that $\mathbb{Z}$-cyclic ordered-triplewhist tournaments exist for all such primes less than $3,200,000$ with the exception that existence or non-existence of these designs for $p = 97, 193, 449, 577, 641, 1409$ is an open question. Here the case $k = 8$ is considered. It is established that $\mathbb{Z}$-cyclic directed-triplewhist tournaments and $\mathbb{Z}$-cyclic ordered-triplewhist tournaments exist for all primes $p \equiv 257 \pmod{512}$, $p \leq 6,944,177$ except, possibly, for $p = 257, 769, 3329$. For $p = 3329$ we are able to construct a $\mathbb{Z}$-cyclic directed-triplewhist tournament but the existence of a $\mathbb{Z}$-cyclic ordered-triplewhist tournament remains an open question. Furthermore for each type of design it is conjectured that our basic constructions will produce these designs whenever $p > 5,299,457$.

keywords: $\mathbb{Z}$-cyclic designs, whist tournaments, directedwhist designs, orderedwhist designs, triplewhist designs, directed triplewhist designs, ordered triplewhist designs.

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1 Introduction

A whist tournament on v players is a $(v, 4, 3)$ (near) resolvable BIBD. A whist game is a block, (a, b, c, d), of the BIBD and denotes that the partnership $\{a, c\}$ opposes the partnership $\{b, d\}$. The design is subject to the (whist) conditions that every player is a partner of every other player exactly once and is an opponent of every other player exactly twice. A whist tournament on v players is denoted by Wh(v). Each (near) resolution class of the design is called a round of the tournament. It has been known since the 1970s that Wh(v) exist for all $v \equiv 0, 1 \pmod 4$ [3].

In a whist game (a, b, c, d) the opponent pairs $\{a, b\}$, $\{c, d\}$ are called first kind opponents and the opponent pairs $\{a, d\}$, $\{b, c\}$ are called second kind opponents. A triplewhist tournament on v players, TWh(v), is a whist tournament with the property that every player opposes every other player exactly once as an opponent of the first kind and exactly once as an opponent of the second kind. TWh(v) were introduced by E. H. Moore [18]. It is now known that TWh(v) do not exist for $v = 5, 9, 12, 13$ and do exist for all other $v \equiv 0, 1 \pmod 4$ except, possibly, for $v = 17$ [8].

In a whist game one can define left hand opponents and right hand opponents. These relationships are the obvious ones associated with the players seated at a table with a at the North position, b at the East position, c at the South position and d at the West position. A whist tournament is said to be a directed whist tournament on v players, DWh(v), if every player has every other player exactly once as a left hand opponent and exactly once as a right hand opponent. The directed whist specialization was introduced by R. D. Baker [9]. DWh($4n + 1$) exist for all $n \geq 1$ and DWh($4n$) exist for all but a few values of n [8].

Another specialization of whist tournament designs is an ordered whist tournament, OWh(v). These designs were introduced by Y. Lu [17]. For a Wh(v) to be an OWh(v) each opponent must be played once at North-South, and once at East-West. Necessarily, the number of games a player plays (i.e., $v - 1$) must be even. Abel et al. [2] have shown that OWh($4n + 1$) exist for all $n \geq 1$.

A (directed, ordered, triple)whist design is said to be $\mathbb{Z}$-cyclic if the players are elements in $\mathbb{Z}_m \cup \mathcal{A}$ where $m = v$, $\mathcal{A} = \emptyset$ when $v \equiv 1 \pmod 4$ and $m = v - 1$, $\mathcal{A} = \{\infty\}$ when $v \equiv 0 \pmod 4$. It is also required that the rounds be cyclic. That is to say, the rounds can be labelled, $R_1, R_2, \ldots$, in such a way that R_{j+1} is obtained by adding $+1 \pmod m$ to every element in R_j. Although the existence of TWh(17) is unknown, it is known that a $\mathbb{Z}$-cyclic TWh(17) does not exist.

It is possible to construct whist tournaments that exhibit two or more of the above mentioned specialized structures [1, 2, 4, 5, 6, 7, 11, 12, 13, 19]. In this paper we focus on the combinations of $\mathbb{Z}$-cyclic, directedwhist, triplewhist and orderedwhist. It is shown in [5] that it is impossible for a whist design to exhibit all four of these specializations simultaneously. On the other hand,

$\mathbb{Z}$-cyclic directed triplewhist designs and $\mathbb{Z}$-cyclic ordered triplewhist designs have been shown to exist for certain primes p. Indeed, for primes p of the form $p = 2^k t + 1$ where t is odd, Anderson and Finizio [6] have shown that $\mathbb{Z}$-cyclic directed triplewhist designs, DTWh(p), exist for all such primes with $k = 2$ except the impossible cases $p = 5, 13$. Anderson and Ellison [4] have established the existence of $\mathbb{Z}$-cyclic ordered triplewhist designs, OTWh(p), for all primes with $k = 2$ except $p = 5, 13$. Anderson and Ellison [5] show that both $\mathbb{Z}$-cyclic DTWh(p) and $\mathbb{Z}$-cyclic OTWh(p) exist for all primes with $k = 3$ and Finizio [12] obtained existence of $\mathbb{Z}$-cyclic DTWh(p) and $\mathbb{Z}$-cyclic OTWh(p) for all primes p with $k = 4$, except for the impossible case $p = 17$. In each of these latter studies the methodology consists of the introduction of one or more constructions that generate the desired designs, the determination of an analytic asymptotic bound beyond which the constructions are guaranteed to produce the designs and a demonstration that the desired designs exist for all relevant primes less than the analytic asymptotic bound. Of late, Weil's Theorem [16] and character sum estimates have been popular tools utilized in the production of the analytic asymptotic bound. Unfortunately, for $k \geq 5$ the analytic asymptotic bounds are so large that it is impractical to apply the aforementioned methodology. For $k = 8$, the focus of this study, an approximation of the analytic bound is given below. Of course, analytic asymptotic bounds obtained via character sum arguments depend on the constructions employed to generate the designs. For the cases $3 \leq k \leq 7$ the "generalized Anderson-Ellison" constructions [11] are the primary constructions utilized. It is possible that alternative constructions and/or alternative asymptotic analyses could produce smaller bounds. To counteract the impracticality of the size of the analytic asymptotic bound the concept of an "empirical asymptotic bound" was introduced in [13].

Definition 1.1 *Let C denote a specific construction. Let $S = \{s_i : i \in \mathbb{Z}, s_i < s_{i+1}\}$ denote an infinite set of integers and let $\mathcal{D}$ denote a design whose existence is possible, via C, for each $s_i \in S$. If, for some pre-prescribed integer M there is an integer N such that C produces $\mathcal{D}$ for each $s \in \{s_N, s_{N+1}, \ldots, s_{N+M}\}$ then s_{N-1} is called an* **empirical asymptotic bound** *for the design $\mathcal{D}$ and the construction C.*

Naturally, M is at the discretion of the practitioner but should not be small.

2 Preliminary Materials

For the remainder of this paper we restrict our attention to $\mathbb{Z}$-cyclic Wh(v), $v \equiv 1 \pmod 4$. In this case it is conventional that the initial round is the round that omits 0. Using symmetric differences it follows that a collection of n games (a_i, b_i, c_i, d_i), $i = 1, \ldots, n$ form the initial round of a $\mathbb{Z}$-cyclic triplewhist tournament on $v = 4n + 1$ players if

$$\bigcup_{i=1}^{n}\{a_i, b_i, c_i, d_i\} = Z_{4n+1} \setminus \{0\}, \tag{2.1}$$

$$\bigcup_{i=1}^{n}\{\pm(a_i - c_i), \pm(b_i - d_i)\} = Z_{4n+1} \setminus \{0\}, \tag{2.2}$$

$$\bigcup_{i=1}^{n}\{\pm(a_i - b_i), \pm(c_i - d_i)\} = Z_{4n+1} \setminus \{0\}, \tag{2.3}$$

and

$$\bigcup_{i=1}^{n}\{\pm(a_i - d_i), \pm(c_i - b_i)\} = Z_{4n+1} \setminus \{0\}. \tag{2.4}$$

If, in addition,

$$\bigcup_{i=1}^{n}\{b_i - a_i, c_i - b_i, d_i - c_i, a_i - d_i\} = Z_{4n+1} \setminus \{0\}, \tag{2.5}$$

then these games form the initial round of a $\mathbb{Z}$-cyclic DTWh(v). On the other hand, if, in addition to (2.1) - (2.4), we have that

$$\bigcup_{i=1}^{n}\{a_i - b_i, a_i - d_i, c_i - b_i, c_i - d_i\} = Z_{4n+1} \setminus \{0\}, \tag{2.6}$$

then these games form the initial round of a $\mathbb{Z}$-cyclic OTWh(v) [10].

For convenience of reference the differences (2.2) are called the *partner differences*, (2.3) the *opponents first kind differences*, (2.4) the *opponent second kind differences*, (2.5) the *directed (alt. first forward) differences* and (2.6) the *ordered differences.*

Example 2.1 Using the above materials one can easily verify that the following games (alt. tables) constitute the initial round of a $\mathbb{Z}$-cyclic DTWh(53). These games were constructed using data contained in [7].

(2, 1, 11, 23), (32, 16, 17, 50), (35, 44, 7, 5),
(30, 15, 6, 27), (3, 28, 43, 8), (48, 24, 52, 22),
(26, 13, 37, 34), (45, 49, 9, 14), (31, 42, 38, 12),
(19, 36, 25, 33), (39, 46, 29, 51), (41, 47, 40, 21),
(20, 10, 4, 18).

Example 2.2 The 13 tables listed below constitute the initial round of a $\mathbb{Z}$-cyclic OTWh(53). These games were constructed using data contained in [4].

(1, 14, 9, 39), (16, 12, 38, 41), (44, 33, 25, 20),
(15, 51, 29, 2), (28, 21, 40, 32), (24, 18, 4, 35),
(13, 23, 11, 30), (49, 50, 17, 3), (42, 5, 7, 48),
(36, 27, 6, 26), (46, 8, 43, 45), (47, 22, 52, 31),
(10, 34, 37, 19).

3 The Major Constructions

Let p be a prime of the form $p = 2^k t + 1$ with t odd and $k \geq 3$. Let r denote a primitive root of p. For convenience, set $d = 2^k$, $m = 2^{k-1}$ and $n = 2^{k-2}$. In GF$(p) = Z_p$, the cyclotomic class of order α ($\alpha|(p-1)$) and index j ($0 \leq j < \alpha$), denoted by C_j^α, is the set $\{r^{j+w\alpha} : w = 0, 1, \ldots, [(p-1)/(\alpha - 1)]\}$. Proofs of the theorems presented in this section can be found in [11] and [12].

Construction 1. Let x denote a non-square in $\mathbb{Z}_p$. Consider the following collection of $(p-1)/4$ games.

$$(1, x, x^m, -x) \otimes r^{dj+2i}, \; 0 \leq i \leq n-1, \; 0 \leq j \leq t-1. \tag{3.7}$$

Theorem 3.1 *If there exists an element $x \in \mathbb{Z}_p \setminus \{0\}$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x^{m+1}(\sum_{i=0}^{m-2} x^i) \in C_0^d$,

e. $x(\sum_{i=0}^{m-2}(-x)^i) \in C_0^d$,

then the games (3.7) form the initial round of a $\mathbb{Z}$-cyclic DTWh(p).

Theorem 3.2 *If there exists an element $x \in \mathbb{Z}_p \setminus \{0\}$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x(\sum_{i=0}^{m-2} x^i) \in C_0^d$,

e. $x^{m+1}(\sum_{i=0}^{m-2}(-x)^i) \in C_0^d$, *then the games (3.7) form the initial round of a $\mathbb{Z}$-cyclic OTWh(p).*

Construction 2. Let r denote a primitive root of p and let x denote a non-square in $\mathbb{Z}_p$. Consider the following collection of $(p-1)/4$ games.

$$(1, x^{m-1}, x^m, -x^{m-1}) \otimes r^{dj+2i},\ 0 \le i \le n-1,\ 0 \le j \le t-1. \tag{3.8}$$

Theorem 3.3 *If there exists an element $x \in \mathbb{Z}_p \setminus \{0\}$ such that Conditions a. through e. of Theorem 3.2 are satisfied then the games (3.8) form the initial round of a $\mathbb{Z}$-cyclic DTWh(p).*

Theorem 3.4 *If there exists an element $x \in \mathbb{Z}_p \setminus \{0\}$ such that Conditions a. through e. of Theorem 3.1 are satisfied then the games (3.8) form the initial round of a $\mathbb{Z}$-cyclic OTWh(p).*

Theorem 3.5 below is the basis for our computer studies and for the asymptotic analysis of Section 1.

Theorem 3.5 *If there exists an element $x \in \mathbb{Z}_p \setminus \{0\}$ such that*

a. $x \neq \square$,

b. $x^2 - 1 \neq \square$,

c. $\prod_{i=2}^{k-1}(x^{2^{k-i}} + 1) \neq \square$,

d. $x(\sum_{i=0}^{m-2} x^i) \in C_0^m$,

e. $x^2(\sum_{i=0}^{m-2} x^i)(\sum_{i=0}^{m-2}(-x)^i) \in C_m^d$,

then there exists both a $\mathbb{Z}$-cyclic DTWh(p) and a $\mathbb{Z}$-cyclic OTWh(p).

For the set of sufficient conditions listed in Theorem 3.5 it is possible, for a given k, to find a number $N(k)$, the **analytic asymptotic bound**, such that for all primes $p = 2^k t + 1$ satisfying $p > N(k)$, the sufficient conditions are guaranteed to be satisfied. A popular method used to determine $N(k)$ is based on character sum arguments and utilizes a theorem due to Weil. For a discussion of this methodology and a statement of Weil's theorem see [11]. For a proof of Weil's Theorem see [16].

4 Primes of the form $p \equiv 257 \pmod{512}$

If one applies the character sum methodology mentioned in Section 1, an estimate of the analytic bound, $N(8)$, is

$$6,798,134,420,640,000 \le N(8) \le 6,798,150,745,908,201.$$

There are 2,001 primes of the form $p = 256t + 1$, t odd, that are less than 6,994,177. Assuming that this latter density prevails and that one could process 100 primes per second (the actual average for the primes in this study is 1 prime per second) it would take more than 610 years to establish existence using the analytic asymptotic bound. Consequently the approach taken here was to construct a program based on the sufficient conditions in Theorem 3.5 and conduct a computer search with the criterion that the search would cease as soon as there were 250 consecutive primes (of the required form) for which the sufficient conditions of Theorem 3.5 were satisfied ($M = 250$). With this search criterion our search proceeded up to the prime 6,994,177 resulting in the empirical asymptotic bound 5,299,457. Appendix I gives a sampling of data resulting from this search.

Example 4.1 All entries in Appendix I are of the form (p, r, x, b) wherein p denotes the prime and r, x are, respectively, a primitive root of p and a non-square in $\mathbb{Z}_p$ for which the sufficient conditions of Theorem 3.5 are satisfied. b indicates the construction that yields the $\mathbb{Z}$-cyclic DTWh(p). Thus the first entry in Appendix I, (98561, 3, 57802, 1), indicates that for $r = 3$ and $x = 57802$ Construction 1 produces a $\mathbb{Z}$-cyclic DTWh(98561) and Construction 2 produces a $\mathbb{Z}$-cyclic OTWh(98561). In each case the initial round consists of 24640 tables. We content ourselves here to provide the two initial rounds in skeletal form (see (3.7) and (3.8)). Thus the initial round of the $\mathbb{Z}$-cyclic DTWh(98561) is given by the games $(1, 57802, 53205, 40758) \otimes 3^{256j+2i}$, $0 \le i \le 63$, $0 \le j \le 384$. Similarly, the initial round of a $\mathbb{Z}$-cyclic OTWh(98561) is given by the games $(1, 65569, 53205, 32992) \otimes 3^{256j+2i}$, $0 \le i \le 63$, $0 \le j \le 384$.

There were 310 primes p, $p \le 5,299,457$, for which our search failed. For most of these failures the following construction of Y. S. Liaw [14] proved to be useful. For a discussion of the validity of Liaw's construction and for proofs of Theorems 4.1 and 4.2 see [12].

Liaw's Construction Let r denote a primitive root of p. The following games

$$(1, r, -r, r^{1+a}) \otimes r^{dj+2i}, \quad 0 \le i \le n-1, \quad 0 \le j \le t-1, \tag{4.9}$$

where $a \equiv (m-1) \pmod{d}$ will, under suitable conditions, produce the initial round of a $\mathbb{Z}$-cyclic DTWh(p) or a $\mathbb{Z}$-cyclic OTWh(p).

Theorem 4.1 *Liaw's Construction gives the initial round of a $\mathbb{Z}$-cyclic DTWh(p) if there exists a primitive root r of p for which the following conditions hold:*

a. $r - 1 = \square$;

b. $r^a + 1 = \square$;

c. $\sum_{i=0}^{a} r^i = \square$;

d. $(r+1)\sum_{i=0}^{a-1} r^i = \square$;

e. $\sum_{i=0}^{a} r^i \in C_0^d$;

f. $\frac{r^a+1}{2} \in C_0^d$.

Theorem 4.2 *Liaw's Construction gives the initial round of a* $\mathbb{Z}$*-cyclic OTWh(p) if there exists a primitive root r of p for which the following conditions hold:*

a. $r - 1 = \square$;

b. $r^a + 1 = \square$;

c. $\sum_{i=0}^{a} r^i = \square$;

d. $(r+1)\sum_{i=0}^{a-1} r^i = \square$;

e. $r^m(\sum_{i=0}^{a} r^i) \in C_0^d$;

f. $\frac{r^m(r^a+1)}{2} \in C_0^d$.

Of the 310 primes investigated, Liaw's construction provided solutions for all but the following: (1) no DTWh(p) for $p = 257, 769, 3329, 9473, 14081$ and (2) no OTWh(p) for $p = 257, 769, 3229, 7937, 14081, 36097$. The DTWh data appears in Appendix II and the OTWh data appears in Appendix III. For the remaining failures adaptations of constructions contained in the doctoral dissertation of Y.S. Liaw [15] enabled us to reduce the list of failures as indicated in Theorem 4.3 below. Initial rounds of the DTWh(p) and OTWh(p) obtained via these specialized constructions are given in Appendix IV.

Theorem 4.3 *Let p be a prime of the form* $p = 256t + 1$ *with t odd. Then for all* $p \leq 6,994,177$ *there exist (1)* $\mathbb{Z}$*-cyclic DTWh(p) except, possibly, for* $p = 257, 769$ *and (2)* $\mathbb{Z}$*-cyclic OTWh(p) except, possibly, for* $p = 257, 769, 3329$.

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Appendix I

The data listed in this appendix relates to those primes $p \equiv 257 \pmod{512}$, $p \leq 28001$, for which $\mathbb{Z}$-cyclic DTWh(p), $\mathbb{Z}$-cyclic OTWh(p) are constructable via Construction 1 and Construction 2. The data is presented in the format $\{p, r, x, b\}$ where p is the prime, r is a primitive root of p, x is a non-square in $\mathbb{Z}_p$ for which the conditions of Theorem 3.5 are satisfied and b indicates that Construction b produces the $\mathbb{Z}$-cyclic DTWh(p). It follows that Construction $\{1,2\} \setminus \{b\}$ produces the $\mathbb{Z}$-cyclic OTWh(p). The complete set of data is contained in the file sol257 at ftp://math.uri.edu/pub/finizio.

{98561, 3, 57802, 1}	{107777, 3, 38593, 2}	{143617, 5, 81878, 1}
{168193, 5, 119630, 1}	{179969, 3, 171166, 2}	{184577, 5, 81817, 2}
{186113, 3, 108586, 2}	{206593, 10, 141075, 2}	{231169, 17, 140958, 1}
{237313, 5, 140281, 1}	{239873, 3, 27291, 2}	{247553, 3, 179159, 2}
{274177, 5, 69854, 1}	{303361, 23, 210372, 2}	{311041, 7, 135374, 2}
{314113, 5, 220440, 1}	{318209, 3, 191132, 1}	{379649, 3, 37373, 1}
{381697, 15, 381466, 1}	{384257, 3, 97741, 1}	{399617, 3, 134695, 2}
{414977, 3, 231656, 2}	{420097, 5, 325686, 1}	{421121, 6, 80955, 2}
{421633, 5, 82022, 2}	{428801, 3, 272939, 1}	{436993, 11, 426115, 1}
{438017, 3, 124675, 2}	{448769, 3, 399154, 1}	{453377, 3, 325232, 2}
{453889, 11, 274175, 2}	{464129, 3, 346189, 1}	{467713, 5, 400949, 2}
{468737, 3, 182499, 1}	{484609, 19, 292979, 1}	{487681, 22, 40925, 2}
{489217, 5, 358352, 1}	{493313, 3, 320481, 1}	{495361, 14, 435973, 2}
{496897, 5, 390518, 1}	{505601, 6, 121200, 1}	{506113, 7, 110340, 2}
{507137, 3, 229143, 2}	{530177, 3, 38129, 1}	{544001, 3, 390359, 2}
{544513, 5, 233450, 2}	{549121, 35, 362282, 1}	{550657, 10, 263981, 2}
{552193, 7, 250319, 2}	{560897, 3, 479937, 1}	{570113, 3, 96155, 1}
{572161, 11, 239624, 1}	{581377, 5, 254344, 1}	{596737, 5, 133459, 1}
{597761, 3, 114202, 1}	{605953, 7, 544758, 1}	{613633, 14, 43703, 1}
{614657, 3, 583759, 2}	{622849, 11, 317415, 2}	{630017, 3, 489872, 1}
{630529, 19, 421759, 1}	{636673, 5, 376203, 2}	{646913, 3, 328007, 2}
{648449, 3, 170264, 2}	{648961, 7, 282786, 2}	{653057, 3, 481994, 2}
{654593, 3, 98431, 2}	{656129, 3, 301898, 2}	{659713, 5, 419583, 1}
{665857, 5, 240320, 1}	{668417, 3, 1947, 2}	{679169, 3, 196509, 1}
{699649, 13, 528370, 1}	{702721, 7, 189395, 1}	{708353, 3, 68574, 2}
{712961, 3, 102940, 1}	{716033, 3, 706614, 1}	{727297, 11, 91843, 2}
{737537, 3, 156396, 1}	{742657, 10, 412494, 2}	{748801, 29, 746606, 1}
{763649, 3, 704843, 1}	{775937, 3, 404244, 2}	{776449, 19, 74975, 1}
{784129, 7, 655646, 2}	{785153, 3, 12356, 1}	{805633, 5, 18035, 2}
{806657, 3, 514716, 2}	{809729, 3, 734211, 2}	{814337, 3, 511036, 2}

Appendix II

In this Appendix we list data for which Liaw's Construction provides a DTWh(p), via Theorem 4.1 for those primes $p \equiv 257 \pmod{512}$ not covered by Theorem 3.5. The data is in the form $\{p, r, a\}$.

{7937, 2643, 2687}	{14593, 4172, 10623}	{22273, 1978, 19839}
{23297, 16262, 21631}	{26881, 3427, 9343}	{30977, 3422, 15999}
{31489, 856, 19071}	{36097, 12341, 3455}	{37633, 28562, 3967}
{40193, 2669, 13695}	{41729, 2114, 16511}	{43777, 1983, 20607}
{46337, 117, 33919}	{49409, 2225, 7039}	{49921, 1765, 39295}
{57089, 452, 42623}	{57601, 1750, 5503}	{60161, 226, 13695}
{70913, 352, 22399}	{75521, 5706, 70015}	{77569, 2129, 15743}
{78593, 3245, 23679}	{84737, 1098, 62591}	{88321, 923, 11647}
{91393, 4246, 42879}	{96001, 2821, 17279}	{100609, 1642, 58751}
{103681, 2463, 96383}	{106753, 2085, 83839}	{108289, 1065, 44415}
{109313, 649, 82047}	{110849, 1039, 105343}	{113921, 2730, 5503}
{116993, 1459, 69247}	{118529, 280, 78975}	{120577, 6239, 98687}
{124673, 67, 89727}	{128257, 94, 63871}	{129281, 8109, 51583}
{129793, 1062, 24703}	{130817, 3, 14207}	{134401, 1278, 52863}
{135937, 989, 68991}	{138497, 176, 82303}	{149249, 1193, 53631}
{152833, 924, 151679}	{154369, 2364, 118143}	{160001, 1911, 56447}
{166657, 2230, 64383}	{169217, 24, 83071}	{172801, 35, 37759}
{174337, 397, 122495}	{175361, 506, 136319}	{175873, 2746, 55935}
{177409, 353, 58495}	{183041, 106, 85119}	{185089, 71, 138367}
{189697, 568, 143487}	{193793, 2425, 157567}	{195329, 77, 105599}
{201473, 1389, 34175}	{206081, 1508, 135039}	{208129, 35, 43647}
{215297, 137, 130943}	{221953, 1164, 196991}	{222977, 245, 195711}
{224513, 309, 56447}	{228097, 2493, 57983}	{241921, 258, 228223}
{244481, 139, 241279}	{246017, 760, 116863}	{249089, 397, 97151}
{254209, 543, 75391}	{257281, 452, 135039}	{259841, 2292, 118399}
{264961, 755, 79487}	{267521, 2039, 56191}	{269057, 497, 247679}
{270593, 440, 7039}	{275201, 411, 138111}	{280321, 1251, 244351}
{281857, 1629, 78207}	{282881, 3, 196223}	{285953, 236, 185471}
{293633, 335, 58239}	{299777, 33, 229247}	{304897, 175, 265343}
{307969, 1366, 247679}	{325889, 155, 68479}	{328961, 871, 178559}
{329473, 195, 246399}	{337153, 149, 175743}	{344321, 202, 214143}
{346369, 815, 28543}	{355073, 190, 59263}	{360193, 318, 111999}
{361217, 623, 155263}	{362753, 37, 235647}	{364289, 210, 14975}
{364801, 820, 283007}	{369409, 66, 232575}	{372481, 148, 150143}
{375553, 448, 335231}	{376577, 108, 60031}	{385793, 1318, 171391}
{387329, 147, 239999}	{393473, 108, 54911}	{397057, 2039, 211583}

{415489, 460, 124287}	{416513, 977, 323967}	{422657, 68, 233855}
{436481, 85, 389759}	{447233, 117, 133247}	{450817, 153, 405375}
{459521, 677, 140415}	{461569, 131, 162687}	{473857, 1478, 282751}
{476929, 518, 448127}	{479489, 595, 299647}	{490241, 157, 464255}
{494849, 304, 126079}	{499969, 251, 5759}	{522497, 78, 517759}
{529153, 250, 350591}	{533249, 175, 199807}	{554753, 282, 160383}
{556289, 92, 310143}	{561409, 37, 556927}	{568577, 45, 436863}
{576769, 479, 192383}	{583937, 271, 226687}	{590593, 90, 412287}
{592129, 440, 584063}	{595201, 209, 496255}	{600833, 48, 332159}
{622337, 142, 348543}	{625409, 927, 46975}	{628993, 70, 531583}
{644353, 173, 335743}	{645889, 122, 294527}	{652033, 579, 330623}
{668929, 372, 647039}	{679681, 93, 345471}	{683777, 5, 95359}
{684289, 85, 7039}	{689921, 122, 514687}	{690433, 267, 256639}
{697601, 65, 267903}	{700673, 457, 266623}	{711937, 1078, 24447}
{741121, 364, 27263}	{753409, 174, 110719}	{755969, 449, 552319}
{759553, 222, 330111}	{766721, 116, 260991}	{777473, 33, 555391}
{779521, 59, 688511}	{794881, 539, 114815}	{799489, 140, 671103}
{805121, 260, 17023}	{811777, 319, 20607}	{817409, 215, 169087}
{820481, 84, 103551}	{823553, 153, 758911}	{828673, 137, 193151}
{837377, 48, 465791}	{842497, 446, 697215}	{859393, 5, 522367}
{865537, 82, 579199}	{877313, 325, 499071}	{883969, 104, 616063}
{906497, 213, 602495}	{926977, 65, 430207}	{932609, 34, 288383}
{940801, 571, 706943}	{944897, 275, 51583}	{963841, 718, 209791}
{966401, 387, 585855}	{967937, 472, 785279}	{978689, 163, 965759}
{982273, 478, 458367}	{986369, 48, 360319}	{988417, 235, 555647}
{994561, 226, 337023}	{1000193, 116, 145023}	{1005313, 55, 454271}
{1007873, 47, 221055}	{1021697, 10, 700799}	{1025281, 19, 760191}
{1049857, 37, 354687}	{1053953, 14, 743551}	{1059073, 110, 871295}
{1060097, 33, 524671}	{1066753, 186, 953215}	{1091329, 236, 508799}
{1092353, 83, 560255}	{1093889, 132, 1064575}	{1106177, 147, 1089407}
{1106689, 339, 403327}	{1120513, 437, 862591}	{1125121, 209, 406143}
{1129729, 293, 415103}	{1142017, 73, 622207}	{1155841, 13, 396671}
{1163521, 129, 611199}	{1175297, 51, 497791}	{1186049, 147, 195967}
{1193729, 112, 461695}	{1198849, 138, 179583}	{1203457, 268, 702335}
{1255169, 181, 194943}	{1258241, 114, 660607}	{1272577, 134, 1211007}
{1274113, 263, 1147775}	{1291009, 115, 622975}	{1301249, 19, 363391}
{1303297, 234, 40319}	{1315073, 29, 611967}	{1321729, 76, 504447}
{1328897, 83, 344959}	{1331969, 954, 1226879}	{1347329, 66, 707711}
{1361153, 185, 998527}	{1362689, 56, 621439}	{1365761, 213, 91519}
{1378561, 11, 1287807}	{1383169, 44, 1033087}	{1443073, 10, 575103}
{1447169, 123, 1407}	{1455361, 137, 1257087}	{1457921, 6, 989567}

{1458433, 47, 1159039}	{1486081, 365, 702079}	{1495297, 336, 1431167}
{1516289, 126, 799871}	{1523969, 26, 1009279}	{1527553, 378, 192127}
{1529089, 68, 1477503}	{1544449, 59, 422015}	{1545473, 94, 577151}
{1602817, 114, 968575}	{1622273, 74, 205183}	{1626881, 21, 684159}
{1631489, 350, 716671}	{1637633, 286, 334207}	{1641217, 369, 1433727}
{1646849, 83, 460159}	{1693441, 92, 362111}	{1696001, 246, 1016191}
{1700609, 434, 1615231}	{1745153, 70, 1111679}	{1759489, 66, 152447}
{1774337, 122, 1573247}	{1794817, 7, 1697151}	{1923841, 66, 1607295}
{1928449, 129, 1922175}	{1957121, 6, 523391}	{1977601, 63, 650367}
{2005249, 501, 1833343}	{2020609, 73, 216191}	{2034433, 30, 833407}
{2056961, 12, 241791}	{2069761, 76, 1599871}	{2078977, 205, 1377919}
{2109697, 37, 1506431}	{2146561, 41, 292223}	{2150657, 43, 77183}
{2153729, 24, 2061695}	{2154241, 7, 1749119}	{2175233, 33, 1649791}
{2217217, 37, 1304447}	{2220289, 91, 1675903}	{2258177, 54, 268415}
{2343169, 38, 1803135}	{2344193, 122, 1974911}	{2346241, 92, 808575}
{2394881, 37, 1483903}	{2426113, 259, 1188479}	{2466049, 82, 2223231}
{2517761, 3, 36479}	{2527489, 7, 2200959}	{2538241, 51, 1004415}
{2557697, 37, 1663615}	{2576641, 281, 1538175}	{2643713, 67, 2589823}
{2684161, 17, 2419839}	{2704129, 339, 1848191}	{2726657, 10, 519039}
{2766593, 3, 2656895}	{2786561, 161, 880767}	{2860289, 82, 666239}
{2860801, 183, 2727551}	{2949377, 42, 2765183}	{3021569, 75, 1059199}
{3095809, 17, 651903}	{3410689, 22, 1252223}	{3447553, 137, 2732671}
{3480833, 47, 2386047}	{3518209, 46, 1794175}	{3525377, 5, 182655}
{3623681, 82, 3421055}	{4565761, 26, 688767}	{4619009, 17, 3716479}
{4903681, 70, 4758911}	{5299457, 10, 2306687}	

Appendix III

In this Appendix we list data for which Liaw's Construction provides a $\mathbb{Z}$-cyclic OTWh(p), via Theorem 4.2 for those primes $p \equiv 257 \pmod{512}$ not covered by Theorem 3.5. The data is in the form $\{p, r, a\}$.

{9473, 6423, 4223}	{14593, 8668, 2943}	{22273, 6372, 14975}
{23297, 6495, 21631}	{26881, 1760, 383}	{30977, 921, 29311}
{31489, 1553, 26239}	{37633, 6273, 12671}	{40193, 6370, 27263}
{41729, 2127, 34431}	{43777, 6721, 3455}	{46337, 1489, 30335}
{49409, 735, 29567}	{49921, 1633, 9087}	{57089, 4111, 39039}
{57601, 1155, 53119}	{60161, 4196, 60031}	{70913, 2724, 62591}
{75521, 1061, 11903}	{77569, 1030, 6527}	{78593, 3397, 13439}
{84737, 1266, 43135}	{88321, 10734, 6527}	{91393, 92, 82815}
{96001, 4318, 27007}	{100609, 491, 99967}	{103681, 4102, 127}
{106753, 7298, 55935}	{108289, 764, 28799}	{109313, 1565, 26239}

{110849, 1126, 88191}	{113921, 515, 54911}	{116993, 191, 20095}
{118529, 1308, 76671}	{120577, 103, 24191}	{124673, 1612, 9599}
{128257, 2896, 64895}	{129281, 4007, 102015}	{129793, 489, 73855}
{130817, 1140, 74111}	{134401, 7548, 22399}	{135937, 845, 15231}
{138497, 1817, 84095}	{149249, 1909, 40575}	{152833, 933, 82559}
{154369, 1310, 103807}	{160001, 73, 88959}	{166657, 1084, 142975}
{169217, 75, 24447}	{172801, 2761, 151935}	{174337, 89, 150399}
{175361, 201, 11391}	{175873, 1212, 148863}	{177409, 1208, 122751}
{183041, 3747, 38015}	{185089, 113, 80767}	{189697, 884, 127615}
{193793, 124, 40575}	{195329, 962, 115327}	{201473, 22, 99199}
{206081, 2726, 190847}	{208129, 2272, 207487}	{215297, 1160, 152191}
{221953, 362, 167295}	{222977, 203, 108415}	{224513, 845, 103551}
{228097, 532, 213375}	{241921, 1467, 14463}	{244481, 1106, 46207}
{246017, 573, 81535}	{249089, 3306, 190335}	{254209, 91, 61823}
{257281, 1090, 25215}	{259841, 57, 29311}	{264961, 47, 252543}
{267521, 104, 199807}	{269057, 747, 195967}	{270593, 1132, 265855}
{275201, 159, 216959}	{280321, 19, 170367}	{281857, 2914, 119423}
{282881, 163, 26239}	{285953, 90, 191103}	{293633, 834, 194687}
{299777, 47, 4223}	{304897, 124, 57983}	{307969, 3378, 143487}
{325889, 1022, 13183}	{328961, 48, 100223}	{329473, 506, 329343}
{337153, 217, 90239}	{344321, 1555, 163455}	{346369, 1522, 269439}
{355073, 178, 158591}	{360193, 627, 295551}	{361217, 24, 170367}
{362753, 86, 168831}	{364289, 773, 81279}	{364801, 1408, 55935}
{369409, 589, 93055}	{372481, 728, 62591}	{375553, 636, 225919}
{376577, 125, 18559}	{385793, 119, 265087}	{387329, 197, 268159}
{393473, 19, 236159}	{397057, 204, 281727}	{415489, 1231, 19583}
{416513, 786, 178559}	{422657, 418, 3455}	{436481, 409, 395135}
{447233, 48, 326527}	{450817, 387, 178815}	{459521, 146, 97151}
{461569, 1126, 372351}	{473857, 311, 171903}	{476929, 246, 35711}
{479489, 279, 205695}	{490241, 1043, 355199}	{494849, 1866, 427391}
{499969, 33, 152447}	{522497, 5, 245119}	{529153, 439, 1151}
{533249, 554, 149375}	{554753, 547, 3711}	{556289, 92, 287871}
{561409, 295, 534399}	{568577, 194, 372351}	{576769, 439, 185983}
{583937, 118, 323967}	{590593, 45, 79231}	{592129, 78, 526719}
{595201, 1144, 463487}	{600833, 268, 568703}	{622337, 10, 60799}
{625409, 51, 479359}	{628993, 328, 593279}	{644353, 240, 91007}
{645889, 161, 463999}	{652033, 70, 441727}	{668929, 62, 340863}
{679681, 73, 652927}	{683777, 73, 146303}	{684289, 213, 90239}
{689921, 311, 28287}	{690433, 197, 483455}	{697601, 96, 570239}
{700673, 147, 7295}	{711937, 70, 455039}	{741121, 126, 667007}
{753409, 271, 636543}	{755969, 23, 151167}	{759553, 840, 305791}
{766721, 3, 439935}	{777473, 66, 701311}	{779521, 262, 592511}

{794881, 153, 478847}
{799489, 65, 494975}
{805121, 3, 203647}
{811777, 88, 269439}
{817409, 468, 441983}
{820481, 24, 148607}
{823553, 3, 321151}
{828673, 109, 276607}
{837377, 75, 454783}
{842497, 362, 461183}
{859393, 30, 517759}
{865537, 166, 429183}
{877313, 597, 102783}
{883969, 156, 392319}
{906497, 104, 690303}
{926977, 47, 434559}
{932609, 145, 505471}
{940801, 601, 672639}
{944897, 84, 785791}
{963841, 335, 927359}
{966401, 59, 655743}
{967937, 236, 236671}
{978689, 69, 577919}
{982273, 409, 737919}
{986369, 290, 733823}
{988417, 209, 418943}
{994561, 339, 773759}
{1000193, 238, 74367}
{1005313, 15, 813695}
{1007873, 281, 46719}
{1021697, 152, 764287}
{1025281, 19, 826239}
{1049857, 101, 892287}
{1053953, 3, 725119}
{1059073, 59, 421247}
{1060097, 68, 208767}
{1066753, 17, 379775}
{1091329, 249, 717183}
{1092353, 119, 475007}
{1093889, 135, 1028223}
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{1120513, 245, 1045375}
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{1129729, 109, 690047}
{1142017, 152, 266623}
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{1175297, 51, 317055}
{1186049, 127, 773503}
{1193729, 27, 172159}
{1198849, 104, 1151871}
{1203457, 28, 840063}
{1255169, 432, 1178239}
{1258241, 370, 1169535}
{1272577, 28, 480895}
{1274113, 556, 1079423}
{1291009, 520, 234111}
{1301249, 58, 829311}
{1303297, 52, 63871}
{1315073, 35, 1254527}
{1321729, 130, 937855}
{1328897, 26, 832383}
{1331969, 21, 1273727}
{1347329, 164, 878719}
{1361153, 56, 849023}
{1362689, 66, 425599}
{1365761, 12, 1242751}
{1378561, 709, 319359}
{1383169, 127, 694911}
{1443073, 145, 1212799}
{1447169, 108, 1384831}
{1455361, 42, 176255}
{1457921, 35, 324735}
{1458433, 151, 1319295}
{1486081, 31, 852863}
{1495297, 487, 1269375}
{1516289, 26, 250239}
{1523969, 88, 737151}
{1527553, 37, 17791}
{1529089, 151, 1165951}
{1544449, 194, 576639}
{1545473, 61, 1476991}
{1602817, 59, 995455}
{1622273, 124, 789631}
{1626881, 29, 1408639}
{1631489, 92, 52607}
{1637633, 57, 502399}
{1641217, 111, 1534335}
{1646849, 39, 606847}
{1693441, 138, 1688191}
{1696001, 228, 294015}
{1700609, 59, 514431}
{1745153, 106, 512639}
{1759489, 17, 907647}
{1774337, 216, 700799}
{1794817, 78, 603263}
{1923841, 151, 979327}
{1928449, 33, 423295}
{1957121, 252, 1902207}
{1977601, 33, 198271}
{2005249, 46, 1948799}
{2020609, 278, 653695}
{2034433, 23, 1121919}
{2056961, 27, 443007}
{2069761, 74, 898431}
{2078977, 20, 58751}
{2109697, 212, 969855}
{2146561, 51, 2011775}
{2150657, 104, 42111}
{2153729, 42, 1147519}
{2154241, 126, 972415}
{2175233, 149, 1903999}
{2217217, 37, 574591}
{2220289, 134, 749183}
{2258177, 12, 2118271}
{2343169, 46, 1279615}
{2344193, 33, 1948031}
{2346241, 68, 2194303}
{2394881, 56, 2251903}
{2426113, 15, 368255}
{2466049, 21, 494207}
{2517761, 118, 2225279}
{2527489, 284, 2392703}
{2538241, 109, 2141311}
{2557697, 285, 1653887}

{2576641, 137, 2471551}	{2643713, 40, 1122431}	{2684161, 152, 1645439}
{2704129, 62, 2166655}	{2726657, 3, 2104447}	{2766593, 59, 779647}
{2786561, 257, 331391}	{2860289, 12, 2762879}	{2860801, 46, 2408063}
{2949377, 26, 353151}	{3021569, 104, 2436735}	{3095809, 47, 2235263}
{3410689, 55, 445823}	{3447553, 5, 1213055}	{3480833, 26, 1316735}
{3518209, 39, 3412351}	{3525377, 37, 678015}	{3623681, 163, 2797183}
{4565761, 91, 1508735}	{4619009, 17, 1581951}	{4903681, 399, 931455}
{5299457, 3, 378239}		

Appendix IV

In this Appendix we list data for the miscellaneous cases. This data was obtained by adapting rather general TWh constructions of Y. S. Liaw [15]. In each case listed, the base game (alt. table) is to be multiplied by r^{dj+2i}, $0 \leq i \leq n-1$, $0 \leq j \leq t-1$, where r is a primitive root of $p = 2^k t + 1 = dt + 1 = 4nt + 1$. We abbreviate this multiplication process with the notation $\otimes\{r, n, t\}$. Of course, $n = 64$.

- DTWh(3329), initial round: $(1, 743, 2380, 257) \otimes \{3, 64, 13\}$
- OTWh(7937), initial round: $(1, 5036, 6473, 3564) \otimes \{3, 64, 31\}$
- DTWh(9473), initial round: $(1, 9060, 4188, 274) \otimes \{3, 64, 37\}$
- DTWh(14081), initial round: $(1, 1646, 3054, 1711) \otimes \{3, 64, 55\}$
- OTWh(14081), initial round: $(1, 1646, 6165, 4519) \otimes \{3, 64, 55\}$
- OTWh(36097), initial round: $(1, 125, 8341, 29947) \otimes \{5, 64, 141\}$