

Zero-Sum Magic and Null Sets of Planar Graphs

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Abstract

For any $h \in \mathbb{N}$, a graph $G = (V, E)$ is said to be h -magic if there exists a labeling $l : E(G) \rightarrow \mathbb{Z}_h - \{0\}$ such that the induced vertex labeling $l^+ : V(G) \rightarrow \mathbb{Z}_h$ defined by

$$l^+(v) = \sum_{uv \in E(G)} l(uv)$$

is a constant map. When this constant is 0 we call G a zero-sum h -magic graph. The null set of G is the set of all natural numbers $h \in \mathbb{N}$ for which G admits a zero-sum h -magic labeling. A graph G is said to be uniformly null if every magic labeling of G induces zero sum. In this paper we will identify the null sets of certain planar graphs such as wheels and fans.

Key Words: magic, non-magic, zero-sum, null set.

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1 Introduction

In this paper all graphs are connected, finite, simple, and undirected. For graph theory notations and terminology not described in this paper, we refer the readers to [2]. For an abelian group A , written additively, any mapping $l : E(G) \rightarrow A - \{0\}$ is called a *labeling*. Given a labeling on the edge set of G one can introduce a vertex labeling $l^+ : V(G) \rightarrow A$ by

$$l^+(v) = \sum_{uv \in E(G)} l(uv).$$

A graph G is said to be A -magic if there is a labeling $l : E(G) \rightarrow A - \{0\}$ such that for every vertex $v \in V(G)$, the sum of the labels of the edges incident with v is equal to the same constant; that is, $l^+(v) = c$ for some fixed $c \in A$. In general, a graph G may admit more than one labeling to become A -magic; for example,

if $|A| > 2$ and $l : E(G) \rightarrow A - \{0\}$ is a magic labeling of G with sum c , then $\lambda : E(G) \rightarrow A - \{0\}$, the *inverse labeling* of l , defined by $\lambda(uv) = -l(uv)$ is another magic labeling of G with sum $-c$. A graph $G = (V, E)$ is called *fully magic* if it is A -magic for every abelian group A . For example, every regular graph is fully magic. A graph $G = (V, E)$ is called *non-magic* if for every abelian group A , the graph is not A -magic. The most obvious example of a non-magic graph is P_n ($n \geq 3$), the path of order n . As a result, any graph with a P_3 pendant would be non-magic. Here is another example of a non-magic graph: Consider the graph H in Figure 1. Given any abelian group A , a potential magic labeling of H is illustrated in that figure. The condition $l^+(u) = l^+(v)$ implies that $6x + y = 7x + y$ or $x = 0$, which is not an acceptable magic labeling. Thus H is not A -magic.

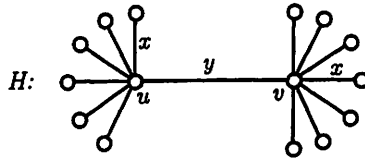


Figure 1: An example of a non-magic graph.

Certain classes of non-magic graphs are presented in [1]. The original concept of A -magic graph is due to J. Sedlacek [16, 17], who defined it to be a graph with a real-valued edge labeling such that

1. distinct edges have distinct nonnegative labels; and
2. the sum of the labels of the edges incident to a particular vertex is the same for all vertices.

Jenzy and Trenkler [4] proved that a graph G is magic if and only if every edge of G is contained in a $(1-2)$ -factor. \mathbb{Z} -magic graphs were considered by Stanley [18, 19], who pointed out that the theory of magic labeling can be put into the more general context of linear homogeneous diophantine equations. Recently, there has been considerable research articles in graph labeling, interested readers are directed to [3, 20]. For convenience, the notation 1-magic will be used to indicate \mathbb{Z} -magic and \mathbb{Z}_h -magic graphs will be referred to as h -magic graphs. Clearly, if a graph is h -magic, it is not necessarily k -magic ($h \neq k$).

Definition 1.1. For a given graph G the set of all positive integers h for which G is h -magic is called the integer-magic spectrum of G and is denoted by $IM(G)$.

Since any regular graph is fully magic, then it is h -magic for all positive integers $h \geq 2$; therefore, $IM(G) = \mathbb{N}$. On the other hand, the graph H , Figure 1, is non-magic, hence $IM(H) = \emptyset$. The integer-magic spectra of certain classes of graphs resulted by the amalgamation of cycles and stars have already been identified [6], and in [7] the integer-magic spectra of the trees of diameter at most four have been completely characterized. Also, the integer-magic spectra of some other graphs have been studied in [5, 8, 9, 10, 13, 14, 15].

2 Zero-sum magic graphs

Definition 2.1. An h -magic graph G is said to be h -zero-sum (or just zero-sum) if there is a magic labeling of G in \mathbb{Z}_h that induces a vertex labeling with sum 0. The null set of a graph G , denoted by $N(G)$, is the set of all natural numbers $h \in \mathbb{N}$ such that G is h -magic and admits a zero-sum labeling in \mathbb{Z}_h .

Clearly, a graph that has an edge pendant is not zero-sum. Also, the null set of a graph is contained in its integer-magic spectrum. The idea of the null set of a graph was introduced in [11] and the following results are established in [11, 12]:

Theorem 2.2. If $n \geq 4$, then $N(K_n) = \mathbb{N} - \{1 + (-1)^n\}$.

Theorem 2.3. If $m, n \geq 2$, then $N(K(m, n)) = \mathbb{N} - \{1 - (-1)^{m+n}\}$.

Null Sets of Cycle Related Graphs. There are different classes of cycle related graphs that have been studied for variety of labeling purposes. J. Gallian [3] has a nice collection of such graphs. Here are some results concerning the null sets of some of the cycle related graphs [11, 12].

Theorem 2.4. If $n \geq 3$, then $N(C_n) = \begin{cases} \mathbb{N} & \text{if } n \text{ is even;} \\ 2\mathbb{N} & \text{if } n \text{ is odd.} \end{cases}$

For any three positive integers $\alpha < \beta \leq \gamma$, the *theta graph* $\theta_{\alpha, \beta, \gamma}$ consists of three edge disjoint paths of length α, β and γ having the same endpoints, as illustrated in Figure 2. Theta graphs are also known as cycles with a P_k chord.

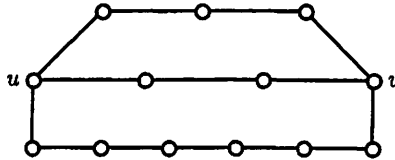


Figure 2: The graph $\theta_{3,4,7}$.

Theorem 2.5. $N(\theta_{\alpha, \beta, \gamma}) = \begin{cases} \mathbb{N} - \{2\} & \text{if } \alpha, \beta, \gamma \text{ have the same parity;} \\ 2\mathbb{N} - \{2\} & \text{otherwise.} \end{cases}$

When k copies of C_n ($n \geq 3$) share a common edge, it will form an n -gon book of k pages and is denoted by $B(n, k)$. When k cycles of order n_1, n_2, \dots, n_k share a common edge the result is known as the generalized book of k pages.

Theorem 2.6. $N(B(n, k)) = \begin{cases} \mathbb{N} - \{1 + (-1)^k\} & \text{if } n \text{ is even;} \\ 2\mathbb{N} - \{1 + (-1)^k\} & \text{if } n \text{ is odd.} \end{cases}$

Generalized Theta Graphs. Given $k \geq 2$ and positive integers $a_1 < a_2 \leq a_3 \leq \dots \leq a_k$, the generalized theta graph $\theta(a_1, a_2, \dots, a_k)$ consists of k edge disjoint paths of lengths a_1, a_2, \dots, a_k having the same initial and terminal points. The following observation will be useful in determining the null sets of graphs.

Lemma 2.7. (Alternating label) *Let v_1, v_2, v_3 and v_4 be four vertices of a graph G that are adjacent ($v_1 \sim v_2 \sim v_3 \sim v_4$) and $\deg v_2 = \deg v_3 = 2$. Then in any magic labeling of G the edges u_1u_2 and u_3u_4 have the same label.*

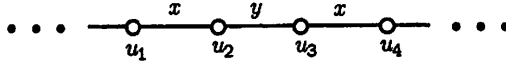


Figure 3: Alternating label in a magic labeling.

Proof. Let $l : E(G) \rightarrow A - \{0\}$ be any magic labeling of G by the nonzero elements of an abelian group A . Then the condition $l^+(u_2) = l^+(u_3)$ implies that $l(u_1u_2) + l(u_2u_3) = l(u_2u_3) + l(u_3u_4)$ or $l(u_1u_2) = l(u_3u_4)$. \square

When discussing magic labeling of a generalized theta graph $\theta(a_1, a_2, \dots, a_k)$, the alternating label lemma (2.7), allows us to assume that $a_i = 2$ or 3 . For convenience, we will use $\theta(2^m, 3^n)$ to denote the generalized theta graph which consists of m paths of even lengths and n paths of odd lengths. If the generalized theta graph consists of just paths of even (or odd) lengths, then we will use the notation $\theta(2^m)$ (or $\theta(3^n)$) and will require $m > 1$ (or $n > 1$). We observe that any magic labeling of $\theta(2^m, 3)$ is similar to that of $\theta(2^m, 1)$, the generalized m -page book all of its pages are odd cycles. The null set of the latter is the same as that of $B(3, m)$, the 3-gon book of m pages. By 2.6, we have $N(\theta(2^m, 3)) = N(B(3, m)) = 2\mathbb{N} - \{1 + (-1)^m\}$. Similarly, by the alternating label lemma, any magic labeling of $\theta(3^{n+1})$ is similar to that of $\theta(3^n, 1)$, the generalized n -page book all of its pages are even cycles. The null set of the latter is the same as that of $B(4, n)$, the 4-gon book of n pages. By 2.6, we have $N(\theta(3^{n+1})) = N(B(4, n)) = \mathbb{N} - \{1 + (-1)^n\}$. Finally, we note that the magic labeling of $\theta(2^m)$ is similar to that of $K(2, m)$. Therefore, by 2.3, $N(\theta(2^m)) = N(K(2, m)) = \mathbb{N} - \{1 - (-1)^m\}$. The following theorem determines the null sets of the generalized theta graphs [12]:

Theorem 2.8. *Following the above notations, for any two non-negative integers m, n*

$$N(2^m, 3^n) = \begin{cases} 2\mathbb{N} - \{1 - (-1)^{m+n}\} & \text{if } m = 1 \text{ or } n = 1; \\ \mathbb{N} - \{1 - (-1)^{m+n}\} & \text{otherwise.} \end{cases}$$

Definition 2.9. *An h -magic graph G is said to be uniformly null if every h -magic labeling of G induces 0 sum.*

The following theorem identifies a class of uniformly null graphs [12]:

Theorem 2.10. *The bipartite graph $K(m, n)$ is uniformly null if and only if $|m - n| = 1$.*

3 Null sets of Wheels

For $n \geq 3$, wheels are defined to be $W_n = C_n + K_1$, where C_n is the cycle of order n . The integer-magic spectra of wheels are determined in [10].

Theorem 3.1. *If $n \geq 3$, then $IM(W_n) = \mathbb{N} - \{1 + (-1)^n\}$.*

In this section we determine the null sets of wheels. Since the degree set of the W_n is $\{3, n\}$, then W_n cannot have a zero sum magic labeling in \mathbb{Z}_2 . Therefore, for any $n \geq 3$, $2 \notin N(W_n)$. Let $u_1 \sim u_2 \sim \dots \sim u_n \sim u_1$ be the vertices of the cycle C_n and u the center vertex of the wheel. In some cases, for convenient, we may use u_{n+1} for u_1 and u_{-1}, u_0 for u_{n-1}, u_n , respectively. The following observation will be useful in finding the null sets of wheels.

Observation 3.2. *If $l : E(W_n) \rightarrow \mathbb{Z}_h$ is a zero sum magic labeling, then*

$$2 \left(l(u_1 u_2) + l(u_2 u_3) + \dots + l(u_{n-1} u_n) + l(u_n u_1) \right) \equiv 0 \pmod{h}.$$

Proof. Let $l : E(W_n) \rightarrow \mathbb{Z}_h$ be the edge labeling that provides zero sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all spokes is 0. Also, $l^*(u_k) = 0$ ($1 \leq k \leq n$). Therefore,

$$\begin{aligned} \sum_{k=1}^n l^*(u_k) &= 2 \sum l(u_i u_{i+1}) + l^*(u) \\ &= 2 \left(l(u_1 u_2) + l(u_2 u_3) + \dots + l(u_{n-1} u_n) + l(u_n u_1) \right) \equiv 0. \end{aligned}$$

□

Observation 3.3. *For $n \geq 3$, $3 \in N(W_n)$ if and only if $n \equiv 0 \pmod{3}$.*

Proof. If $n \equiv 0 \pmod{3}$, then we label all the edges of W_n by 1 and this provides a zero sum in \mathbb{Z}_3 . Now suppose $n \not\equiv 0 \pmod{3}$ and let $l : E(W_n) \rightarrow \mathbb{Z}_3$ be any magic labeling of W_n with zero sum. Then by Observation 3.2, the sum of the labels of the outer edges is 0. Since the outer edges cannot all be labeled 1 (or 2), two adjacent outer edges would have labels 1 and 2. This implies that the spoke adjacent to these two outer edges must have label 0, which is not an acceptable label. □

Observation 3.4. *If W_n is zero sum h -magic, so is W_{kn} for every $k \in \mathbb{N}$.*

Proof. Following the notations used above, let $u_1 \sim u_2 \sim \dots \sim u_n \sim u_1$ be the vertices of the cycle C_n and u the center vertex of W_n . For W_{kn} , let $v_1 \sim v_2 \sim \dots \sim v_{kn} \sim v_1$ be the vertices of C_{kn} and v be its center vertex. Also, assume that $f : E(W_n) \rightarrow \mathbb{Z}_h$ is a magic labeling of W_n with 0 sum. Now define $g : E(W_{kn}) \rightarrow \mathbb{Z}_h$ by $g(vv_m) = f(uu_i)$ whenever $m \equiv i \pmod{n}$ and $g(v_m v_{m+1}) = f(u_i u_{i+1})$ whenever $m \equiv i \pmod{n}$. Then for the induced vertex labeling $g^* : V(W_{kn}) \rightarrow \mathbb{Z}_h$ we have $g^*(v) = kf^*(u) = 0$. Moreover, given any v_m let $m = qn + r$ ($0 \leq r \leq n-1$). Then $g^*(v_m) = g(v_{m-1} v_m) + g(v_m v_{m+1}) + g(vv_m) = f(u_{r-1} u_r) + f(u_r u_{r+1}) + f(uu_r) = f^*(u_r) = 0$. Therefore, g is a magic labeling of W_{kn} with 0 sum. □

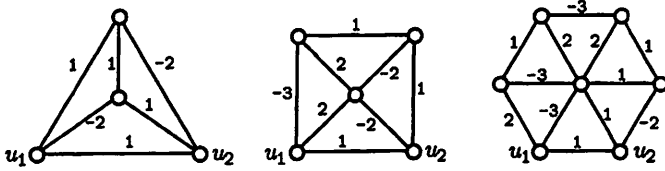


Figure 4: A zero-sum labeling of W_3, W_4 and W_6 .

Corollary 3.5. For any $n \geq 1$, $N(W_{3n}) = \mathbb{N} - \{2\}$.

Proof. Note that $W_3 \cong K_4$, for which we have $N(W_3) = \mathbb{N} - \{2\}$. Therefore, by 3.4, $N(W_{3n}) = \mathbb{N} - \{2\}$. \square

Lemma 3.6. For any $n \geq 3$, $\mathbb{N} - \{2, 3\} \subset N(W_n)$.

Proof. To prove the lemma we consider the following four cases:

Case 1. Suppose $n \equiv 0 \pmod{4}$ or $n = 4p$ for some $p \in \mathbb{N}$.

A zero sum magic labeling of W_4 is provided in Figure 4, which indicates that for every $h > 3$, the graph W_4 admits a zero sum magic labeling in \mathbb{Z}_h . Therefore, by Observation 3.4, W_{4p} has a zero sum magic labeling in \mathbb{Z}_h . That is $\mathbb{N} - \{2, 3\} \subset N(W_{4p})$.

Case 2. Suppose $n \equiv 1 \pmod{4}$ or $n = 4p + 1$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

“for any p , there is a zero sum magic labeling for W_{4p+1} . Moreover, in this labeling at least one of the outer edges has label 1.”

Let $p = 1$. In Figure 5(A), a zero sum magic labeling of W_5 in \mathbb{Z}_4 is provided. Also, Figure 5(B) indicates that W_5 admits a zero sum magic labeling in \mathbb{Z}_h for all $h \geq 5$.

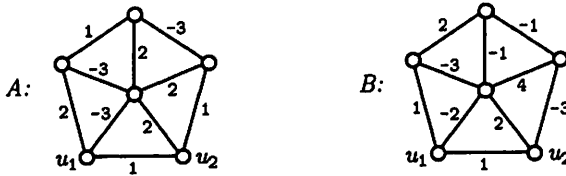


Figure 5: The four-spoke extension of a wheel.

Now, assume that the statement is true for W_{4p+1} and let $u_1 u_2$ be the outer edge of W_{4p+1} that has label 1. Then we eliminate this edge and insert the four-spoke extension, which is given in Figure 6, in such a way that the vertices z, v and w of this extension be identified with the central vertex u and vertices u_1, u_2 of W_{4p+1} , respectively. This provides the desired zero sum magic labeling for W_{4p+5} .

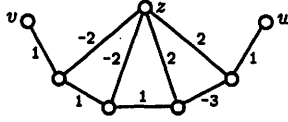


Figure 6: The four-spoke extension of a wheel.

An argument similar to the one presented in case 2, will also work for the remaining two cases:

Case 3. Suppose $n \equiv 2 \pmod{4}$ or $n = 4p + 2$ for some $p \in \mathbb{N}$.

Case 4. Suppose $n \equiv 3 \pmod{4}$ or $n = 4p + 3$ for some $p \in \mathbb{N}$. □

We summarize the main result of this section in the following theorem:

Theorem 3.7. For any $n \geq 3$, $N(W_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } 3|n; \\ \mathbb{N} - \{2, 3\} & \text{if } 3 \nmid n. \end{cases}$

4 Null sets of Fans

For $n \geq 2$, Fans are defined to be $F_n = P_n + K_1$, where P_n is the path of order n . In this section we determine the null sets of Fans. Since the degree set of the F_n is $\{2, 3, n\}$, it cannot have a magic labeling in \mathbb{Z}_2 . Therefore, for any $n \geq 3$, $2 \notin N(F_n)$. Note that $F_2 \cong C_3$, and we know that $N(F_2) = 2\mathbb{N}$. Also, a typical magic labeling of $F_3 \cong K_4 - e$ is illustrated in Figure 7(A), for which we require that $a + b - z = a + b + z$ or $2z \equiv 0 \pmod{h}$; that is, h has to be even. On the other hand, if $h = 2r$, then F_3 admits a zero sum magic labeling in \mathbb{Z}_h , as indicated in Figure 7(B). Therefore, $N(F_3) = 2\mathbb{N} - \{2\}$. For the general case, let $u_1 \sim u_2 \sim \dots \sim u_n$ be the vertices of the path P_n and u the central vertex of the fan. We call the edges uu_i ($1 \leq i \leq n$) blades of the fan F_n . The following observation will be useful in finding the null sets of fans.

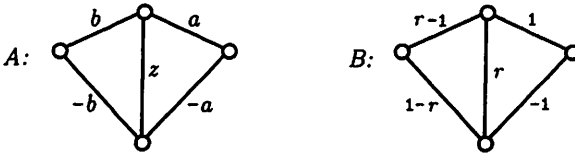


Figure 7: A typical magic labeling of F_3 .

Observation 4.1. If $l : E(F_n) \rightarrow \mathbb{Z}_h$ is a zero sum magic labeling, then

$$2 \left(l(u_1 u_2) + l(u_2 u_3) + \dots + l(u_{n-1} u_n) \right) \equiv 0 \pmod{h}.$$

Proof. Let $l : E(W_n) \rightarrow \mathbb{Z}_h$ be the edge labeling that provides zero sum. Clearly, $l^*(u) = 0$ implies that sum of the labels of all blades is 0. Also, $l^*(u_k) = 0$ ($1 \leq$

$k \leq n$). Therefore,

$$\begin{aligned} \sum_{k=1}^n l^*(u_k) &= 2 \sum l(u_i u_{i+1}) + l^*(v) \\ &= 2 \left(l(u_1 u_2) + l(u_2 u_3) + \dots + l(u_{n-1} u_n) \right) \equiv 0. \end{aligned}$$

Observation 4.2. For any $n \geq 2$, $3 \in N(F_n)$ if and only if $n \equiv 1 \pmod{3}$.

Proof. The statement is true for $n = 2, 3$. Suppose $n \geq 4$ and $n \equiv 1 \pmod{3}$. Then we label all the edges of P_n by 2, the two outer blades by 1 and all other blades by 2, as illustrated in Figure 8. This is a zero sum magic labeling of F_n .

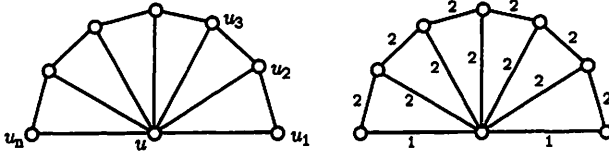


Figure 8: The fan F_n ($n = 7$).

Next, suppose $n \not\equiv 1 \pmod{3}$ and let $l : E(F_n) \rightarrow \mathbb{Z}_3$ be a zero sum magic labeling. By Observation 4.1, we require that $l(u_1 u_2) + l(u_2 u_3) + \dots + l(u_{n-1} u_n) \equiv 0 \pmod{3}$, which implies that at least two adjacent edges of P_n are labeled 1 and 2. But this will force the label of the blade adjacent to these edges be 0, which is not an acceptable label. Therefore, such a zero sum magic labeling does not exist. \square

Lemma 4.3. For any $n \geq 4$, $\mathbb{N} - \{2, 3\} \subset N(F_n)$.

Proof. To prove the lemma we consider the following three cases:

Case 1. Suppose $n \equiv 1 \pmod{3}$ or $n = 3p + 1$ for some $p \in \mathbb{N}$. We proceed by induction on p and show that

“for any p , there is a zero sum magic labeling for F_{3p+1} . Moreover, in this labeling at least one of the edges of P_n has label 2.”

Let $p = 1$. In Figure 9, a zero sum magic labeling of F_4 is provided in \mathbb{Z}_h for all $h \geq 4$.

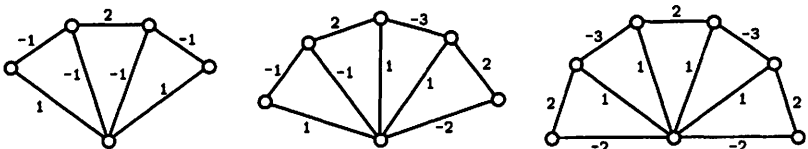


Figure 9: A zero sum magic labelings of F_4, F_5 and F_6 .

Now, assume that the statement is true for F_{3p+1} and let $u_i u_{i+1}$ be the edge of F_{3p+1} that has label 2. Then we eliminate this edge and insert the three-blade extension, which is given in Figure 10, in such a way that the vertices z, v and w of this extension be identified with vertices u, u_i, u_{i+1} of F_{3p+1} , respectively. This provides the desired zero sum magic labeling for F_{3p+4} .

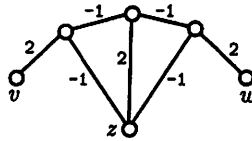


Figure 10: The three-blade extension of a fan.

An argument similar to the one presented in case 1, will also work for the remaining two cases:

Case 2. Suppose $n \equiv 2 \pmod{3}$ or $n = 3p + 2$ for some $p \in \mathbb{N}$.

Case 3. Suppose $n \equiv 0 \pmod{3}$ or $n = 3p$ for some $p \in 1 + \mathbb{N}$. □

We summarize the main result of this section in the following theorem:

Theorem 4.4. $N(F_2) = 2\mathbb{N}$, $N(F_3) = 2\mathbb{N} - \{2\}$ and for any $n \geq 4$,

$$N(F_n) = \begin{cases} \mathbb{N} - \{2\} & \text{if } n \equiv 1 \pmod{3}; \\ \mathbb{N} - \{2, 3\} & \text{otherwise.} \end{cases}$$

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