# On the basis number and the minimum cycle bases of the wreath product of some graphs II.

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**Abstract**

A construction of a minimum cycle bases for the wreath product of a star by a path, two stars and a star by a wheel is given. Moreover, the basis numbers of these products are determined.

**Keywords:** Cycle space; Basis number; Cycle basis; Wreath product.
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## 1 Introduction.

The basis number of a graph is one of the numbers which give rise to a better understanding and interpretations of geometric properties of a graph (see [20]). Minimum cycle bases (MCBs) of a cycle spaces have a variety of applications in sciences and engineering, for example, in structural flexibility analysis, electrical networks, and in chemical structure storage and retrieval systems (see [9], [10] and [18]).

In general, required cycle bases, and minimum cycle bases are not very well behaved under graph operations. Neither the basis number $b(G)$ of a graph $G$ is monotonic (see [3] and [22]), nor the total length $l(G)$ and the length of the longest cycle in a minimum cycle basis $\lambda(G)$ are minor monotone (see[12]). Hence, there does not seem to be a general way of extending required cycle bases and minimum cycle bases of a certain collection of partial graphs of $G$ to a required cycle basis and to a minimum cycle basis of $G$, respectively. Global upper bounds $b(G) \leq 2\gamma(G) + 2$ and $l(G) \leq \dim \mathcal{C}(G) + \kappa(T(G))$ where $\gamma(G)$ is the genus of $G$ and $\kappa(T(G))$ is the connectivity of the tree graph of $G$ are proven in [22] and [19], respectively.

In this paper, we continue what we started in [17] by investigating the basis number for some classes of graphs and we construct minimum cycle bases for same, also, we give their total lengths and the length of longest cycles.

# 2 Definitions and preliminaries.

The graphs considered in this paper are finite, undirected, simple and connected. Most of the notations that follow can be found in [6]. For a given graph $G$, we denote the vertex set of $G$ by $V(G)$ and the edge set by $E(G)$.

## 2.1 Cycle bases.

Given a graph $G$, let $e_1, e_2, \ldots, e_{|E(G)|}$ be an ordering of its edges. Then a subset $S$ of $E(G)$ corresponds to a $(0,1)$-vector $(b_1, b_2, \ldots, b_{|E(G)|})$ in the usual way with $b_i = 1$ if $e_i \in S$, and $b_i = 0$ if $e_i \notin S$. These vectors form an $|E(G)|$-dimensional vector space, denoted by $(Z_2)^{|E(G)|}$, over the field of integers modulo 2. The vectors in $(Z_2)^{|E(G)|}$ which correspond to the cycles in $G$ generate a subspace called the *cycle space* of $G$ and denoted by $\mathcal{C}(G)$. We shall say that the cycles themselves, rather than the vectors corresponding to them, generate $\mathcal{C}(G)$. It is known that for a connected graph $G$ $\dim \mathcal{C}(G) = |E(G)| - |V(G)| + 1$ (see [7]).

A basis $\mathcal{B}$ for $\mathcal{C}(G)$ is called a *cycle basis* of $G$. A cycle basis $\mathcal{B}$ of $G$ is called a $d$-fold if each edge of $G$ occurs in at most $d$ of the cycles in $\mathcal{B}$. *The basis number*, $b(G)$, of $G$ is the least non-negative integer $d$ such that $\mathcal{C}(G)$ has a *$d$-fold* basis. A *required basis* of $\mathcal{C}(G)$ is a basis with $b(G)$-fold. The *length*, $|C|$, *of the element* $C$ of the cycle space $\mathcal{C}(G)$ is the number of its edges. The *length* $l(\mathcal{B})$ *of a cycles basis* $\mathcal{B}$ is the sum of the lengths of its elements: $l(\mathcal{B}) = \sum_{C \in \mathcal{B}} |C|$. $\lambda(G)$ is defined to be the minimum length of the longest element in an arbitrary cycle basis of $G$. A *minimum cycle basis* (MCB) is a cycle basis with minimum length. Since the cycle space $\mathcal{C}(G)$ is a matroid in which an element $C$ has weight $|C|$, the greedy algorithm can be used to extract a MCB (see [24]). The following results will be used frequently in the sequel.

**Theorem 2.1.1.**(MacLane). *The Graph $G$ is planar if and only if $b(G) \leq 2$.*

A cycle is *relevant* if it is contained in some MCB (see [23]).

**Proposition 2.1.2.**(Plotkin). *A cycle $C$ is relevant if and only if it cannot be written as a linear combinations modulo 2 of shorter cycles.*

Chickering, Geiger and Heckerman [8], showed that $\lambda(G)$ is the length of the longest element in a MCB.

## 2.2 Products.

Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two graphs. (1) The cartesian product $G\Box H$ has the vertex set $V(G\Box H) = V(G) \times V(H)$ and the edge set $E(G\Box H) = \{(u_1, v_1)(u_2, v_2)|u_1u_2 \in E(G)$ and $v_1 = v_2$, or $v_1v_2 \in E(H)$ and $u_1 = u_2\}$. (2) The direct product $G \times H$ is the graph with the vertex set $V(G \times H) = V(G) \times V(H)$ and the edge set $E(G \times H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)\}$. (3) The strong product $G \boxtimes H$ is the graph with the vertex set $V(G \boxtimes H) = V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)$ and $u_2 = v_2\}$. (4) The semi-strong product $G_1 \bullet G_2$ is the graph with the vertex set $V(G \bullet H) = V(G) \times V(H)$ and the edge set $E(G \bullet H) = \{(u_1, u_2)(v_1, v_2)|u_1v_1 \in E(G)$ and $u_2v_2 \in E(H)$ or $u_1 = v_1$ and $u_2v_2 \in E(H)$ $\}$. (5) The Lexicographic product $G_1[G_2]$ is the graph with vertex set $V(G[H]) = V(G) \times V(H)$ and the edge set $E(G[H]) = \{(u_1, u_2)(v_1, v_2)|u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_1v_1 \in E(G)\}$. (6) The wreath product $G\rho H$ has the vertex set $V(G\rho H) = V(G)\times V(H)$ and the edge set $E(G\rho H) = \{(u_1, v_1)(u_2, v_2)|u_1 = u_2$ and $v_1v_2 \in H$, or $u_1u_2 \in G$ and there is $\alpha \in$Aut$(H)$ such that $\alpha(v_1) = v_2\}$ (See [1] and [11]).

Many authors studied the basis number and the minimum cycle bases of graph products. The cartesian product of any two graphs was studied by Ali and Marougi [4] and Imrich and Stadler [12].

**Theorem 2.2.1.** *(Ali and Marougi) If $G$ and $H$ are two connected disjoint graphs, then* $b(G\Box H) \leq \max\{ b(G) + \Delta(T_H),\ b(H) + \Delta(T_G)\}$ *where $T_H$ and $T_G$ are spanning trees of $H$ and $G$, respectively, such that the maximum degrees $\Delta(T_H)$ and $\Delta(T_G)$ are minimum with respect to all spanning trees of $H$ and $G$.*

**Theorem 2.2.2.** *(Imrich and Stadler) If $G$ and $H$ are triangle free, then* $l(G\Box H) = l(G) + l(H) + 4[|E(G)|(|V(H)| - 1) + |E(H)|(|V(G)| - 1) - (|V(H)| - 1)(|V(G)| - 1)]$ *and* $\lambda(G\Box H) = \max\{4, \lambda(G), \lambda(H)\}$.

Schmeichel [22], Ali [2], [3] and Jaradat [13] gave an upper bound for the basis number of the semi-strong and the direct products of some special graphs. They proved the following results:

**Theorem 2.2.3.** *(Schmeichel) For each $n \geq 7$, $b(K_n \bullet P_2) = 4$.*

**Theorem 2.2.4.** *(Ali) For each integers $n, m$, $b(K_m \bullet K_n) \leq 9$.*

**Theorem 2.2.5.** *(Ali) For any two cycles $C_n$ and $C_m$* with $n, m \geq 3$, $b(C_n \times C_m) = 3$.

**Theorem 2.2.6.** *(Jaradat) For each bipartite graphs $G$ and $H$, $b(G \times H) \leq 5 + b(G) + b(H)$.*

**Theorem 2.2.7.** *(Jaradat) For each bipartite graph $G$ and cycle $C$, $b(G \times C) \leq 3 + b(G)$.*

The strong product was studied by Imrich and Stadler [12] and Jaradat [15]. They gave the following results:

**Theorem 2.2.8.** *(Imrich and Stadler) For any two graphs $G$ and $H$, $l(G \boxtimes H) = l(G) + l(H) + 3[\dim C(G \boxtimes H) - \dim C(G) - \dim C(H)]$ and $\lambda(G \boxtimes H) = \max\{3, \lambda(G), \lambda(H)\}$.*

**Theorem 2.2.9.** *(Jaradat) Let $G$ be a bipartite graph and $H$ be a graph. Then $b(G \boxtimes H) \leq \max\{b(H)+1, 2\Delta(H)+b(G)-1, \left\lfloor \frac{3\Delta(T_G)+1}{2} \right\rfloor, b(G)+2\}$.*

Jaradat [17] investigate the basis number and the minimal cycle bases of the wreath product of two paths, a cycle with a path, a path with a star, a cycle with a star, a path with a wheel and a cycle with a wheel.

In this paper, we continue the study initiated in [17] by constructed a minimum cycle basis for the wreath products of a star by a path, two stars and a star by a wheel. Additionally, we determine the basis number of the above products.

In the rest of this paper, $f_B(e)$ stand for the number of elements of $B$ containing the edge $e$ where $B \subseteq \mathcal{C}(G)$.

# 3 The basis number of the wreath product of graphs.

In this chapter we study the required bases and investigate the basis number of the wreath product of a star with a path, two stars, a star with a wheel.

## 3.1 The basis number of $S_n \rho P_m$

Let $\{v_1, v_2, \ldots, v_m\}$ be a set of vertices and $ab$ be an edge. Throughout this work we use the notations $\mathcal{K}_{ab}$ and $\mathcal{R}_{ab}$ which were introduced by Jaradat [30] and a new notation $\mathcal{Z}_{ab}$ as follows:

$$\mathcal{K}_{ab} = \left\{\mathcal{K}_{ab}^{(j)} = (a, v_j)(b, v_j)(b, v_{j+1})(a, v_{j+1})(a, v_j) \mid j = 1, 2, \ldots, m-1\right\},$$

$$\mathcal{R}_{ab} = \left\{\mathcal{R}_{ab}^{(j)} = (a,v_j)(b,v_{m-j+1})(b,v_{m-j})(a,v_{j+1})(a,v_j) \mid j = 1,2, \ldots, \lfloor\frac{m}{2}\rfloor\right\},$$

$$\mathcal{Z}_{ab} = \left\{\begin{array}{c} \mathcal{Z}_{ab}^{(1)} = (a,v_{\lfloor\frac{m}{2}\rfloor})(b,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor+2})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor}), \\ \mathcal{Z}_{ab}^{(2)} = (a,v_{\lfloor\frac{m}{2}\rfloor})(b,v_{\lfloor\frac{m}{2}\rfloor+2})(a,v_{\lfloor\frac{m}{2}\rfloor+2})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor}), \\ \mathcal{Z}_{ab}^{(3)} = (a,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor}), \\ \mathcal{Z}_{ab}^{(4)} = (a,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor}). \end{array}\right\}$$

**Lemma 3.1.1.** For any odd integer $m$, every linear combination of cycles of $\mathcal{R}_{ab}$ contains at least one edge of the form $(a,v_j)(b,v_{m-j+1}), 1 \le j < \lfloor\frac{m}{2}\rfloor$ or $(b,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+2})$. Moreover, every linear combination of cycles of $\mathcal{R}_{ba}$ contains at least one edge of the form $(b,v_j)(a,v_{m-j+1}), 1 \le j < \lfloor\frac{m}{2}\rfloor$ or $(b,v_{\lfloor\frac{m}{2}\rfloor})(b,v_{\lfloor\frac{m}{2}\rfloor+1})$.

**Proof.** Let $\mathcal{R}$ be a linear combination of the cycles of $S = \left\{\mathcal{R}_{ab}^{(j_1)}, \mathcal{R}_{ab}^{(j_2)}, \ldots, \mathcal{R}_{ab}^{(j_k)}\right\} \subseteq \mathcal{R}_{ab}$ where $j_1 < j_2 < \cdots < j_k$. Then by the definition of $\mathcal{R}_{ab}$, $\mathcal{R}_{ab}^{(j_1)}$ contains the edge $(a,v_{j_1})(b,v_{m-j_1+1})$. Since $j_1 < j_2 < \cdots < j_k$, as a result $\mathcal{R}$ contains $(a,v_{j_1})(b,v_{m-j_1+1})$ if $j_1 \ne \lfloor\frac{m}{2}\rfloor$ otherwise $S = \left\{\mathcal{R}_{ab}^{(\lfloor\frac{m}{2}\rfloor)}\right\}$ and so $\mathcal{R} = \mathcal{R}_{ab}^{(\lfloor\frac{m}{2}\rfloor)}$ which contains the edge $(b,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+2})$. Similarly for $\mathcal{R}_{ba}$. □

By the same argument as in the above lemma, we have the following results:

**Lemma 3.1.2.** For any even integer $m$, every linear combination of cycles of $\mathcal{R}_{ab}$ contains at least one edge of the form $(a,v_j)(b,v_{m-j+1}), 1 \le j < \lfloor\frac{m}{2}\rfloor$ or $(a,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor+1})$.

**Lemma 3.1.3.** Every linear combination of cycles of $\mathcal{K}_{ab} - \{\mathcal{K}_{ab}^{(k)}, \mathcal{K}_{ab}^{(k+1)}, \ldots, \mathcal{K}_{ab}^{(k+s)}\}$ contains at least one edge of the form $(a,v_j)(b,v_j)$, $j \le k-1$ or $j \ge k+s+2$.

Let $P_m = v_1v_2\ldots v_m$. Then the automorphism group of the path $P_m$ consists of two elements the identity, $I$, and the automorphism $\alpha$ which is defined as follows:

$$\alpha(v_j) = v_{m-j+1}, j = 1,2,\ldots,m.$$

Therefore, $ab\rho P_m$ is decomposable into $ab\Box P_m \cup M_{ab}$ where $M_{ab}$ is the graph with edge set

$$E(M_{ab}) = \{(a,v_j)(b,v_{m-j+1}),(a,v_{m-j+1})(b,v_j)|j = 1,2,\ldots,\lfloor\frac{m}{2}\rfloor\}. \quad (1)$$

**Lemma 3.1.4.** Let $m$ be an odd integer. Then $\mathcal{A}_{ab} = \mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\} - \left\{\mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor}, \mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor+1}\right\}$ is linearly independent subset of $\mathcal{C}(ab\rho P_m)$.
**Proof.** By Lemma 3.1 of [17] each of $\mathcal{K}_{ab}, \mathcal{R}_{ab}$ and $\mathcal{R}_{ba}$ is linearly independent. Since $\mathcal{Z}_{ab}^{(1)} \neq \mathcal{Z}_{ab}^{(2)}$, $\left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$ is linearly independent. By Lemma 3.1.1, any linear combination of cycles of $\mathcal{R}_{ab}$ contains an edge of the form $(a,v_j)(b,v_{m-j+1}), j < \lfloor\frac{m}{2}\rfloor$ or $(b,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+2})$ which is not in any cycle of $\left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$. Thus $\mathcal{R}_{ab} \cup \left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$ is linearly independent. Similarly, each linear combination of cycles of $\mathcal{R}_{ba}$ contains an edge of the form $(b,v_j)(a,v_{m-j+1}), j < \lfloor\frac{m}{2}\rfloor$ or $(b,v_{\lfloor\frac{m}{2}\rfloor})(b,v_{\lfloor\frac{m}{2}\rfloor+1})$ which is not in any cycle of $\mathcal{R}_{ab} \cup \left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$. Thus, $\mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$ is linearly independent. Since $\mathcal{K}_{ab}$ is linearly independent, $\mathcal{K}_{ab} - \left\{\mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor}, \mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor+1}\right\}$ is linearly independent. Now, By Lemma 3.1.3, any linear combination of cycles of $\mathcal{K}_{ab} - \left\{\mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor}, \mathcal{K}_{ab}^{\lfloor\frac{m}{2}\rfloor+1}\right\}$ contains an edge of the form $(a,v_j)(b,v_j)$, $j < \lfloor\frac{m}{2}\rfloor$ or $(a,v_{j+1})(b,v_{j+1})$, $j > \lfloor\frac{m}{2}\rfloor + 1$, which is not in any cycle of $\mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$. Thus $\mathcal{A}_{ab}$ is linearly independent. □

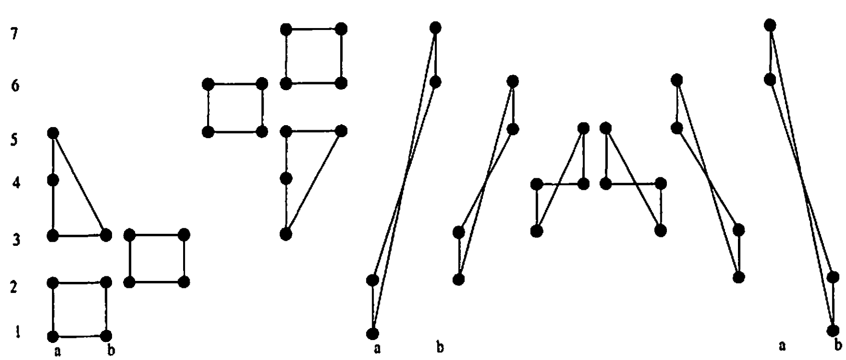


Figure 1: Cycles of $\mathcal{A}_{ab}$ for $m = 7$

**Remark 3.1.5.** Let $m$ be an odd integer and $e \in E(ab\rho P_m)$. Then by the aid of Figure 1, we have

(1) If $e = (a, v_j)(b, v_{m-j+1})$ or $(b, v_j)(a, v_{m-j+1})$ such that $j \neq \lfloor \frac{m}{2} \rfloor$, then $f_{\mathcal{A}_{ab}}(e) \leq 2$. (2) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor + 2})$ or $(a, v_{\lfloor \frac{m}{2} \rfloor + 2})(b, v_{\lfloor \frac{m}{2} \rfloor})$, then $f_{\mathcal{A}_{ab}}(e) = 3$. (3) If $e = (a, v_1)(b, v_m)$ or $(b, v_1)(a, v_m)$, then $f_{\mathcal{A}_{ab}}(e) = 1$. (4) If $e = (a, v_j)(b, v_j)$ such that $j \notin \{1, m\}$, then $f_{\mathcal{A}_{ab}}(e) = 2$. (5) If $e = (a, v_j)(b, v_j)$ such that $j \in \{1, m\}$, then $f_{\mathcal{A}_{ab}}(e) = 1$. (6) If $e = (b, v_j)(b, v_{j+1})$ or $(a, v_j)(a, v_{j+1})$ such that $j \notin \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1\}$, then $f_{\mathcal{A}_{ab}}(e) = 2$. (7) If $e = (a, v_j)(a, v_{j+1})$ such that $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1\}$, then $f_{\mathcal{A}_{ab}}(e) = 3$. (8) If $e = (b, v_j)(b, v_{j+1})$ such that $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1\}$, then $f_{\mathcal{A}_{ab}}(e) = 1$.

By a similar argument as in Lemma 3.1.4 after replacing $\left\{\mathcal{Z}_{ab}^{(1)}, \mathcal{Z}_{ab}^{(2)}\right\}$ by $\left\{\mathcal{Z}_{ab}^{(3)}, \mathcal{Z}_{ab}^{(4)}\right\}$, $\mathcal{K}_{ab} - \left\{\mathcal{K}_{ab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{K}_{ab}^{\lfloor \frac{m}{2} \rfloor + 1}\right\}$ by $\mathcal{K}_{ab} - \left\{\mathcal{K}_{ab}^{\lfloor \frac{m}{2} \rfloor}\right\}$ and $\mathcal{R}_{ab}$ by $\mathcal{R}_{ab} - \left\{\mathcal{R}_{ab}^{\lfloor \frac{m}{2} \rfloor}\right\}$ and by the aid of Lemmas 3.1.2 and 3.1.3, we have the following result:

**Lemma 3.1.6.** Let $m$ be an even integer. Then $\mathcal{T}_{ab} = \mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}_{ab}^{(3)}, \mathcal{Z}_{ab}^{(4)}\right\} - \left\{\mathcal{R}_{ab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{K}_{ab}^{\lfloor \frac{m}{2} \rfloor}\right\}$ is linearly independent subset of $\mathcal{C}(ab\rho P_m)$.

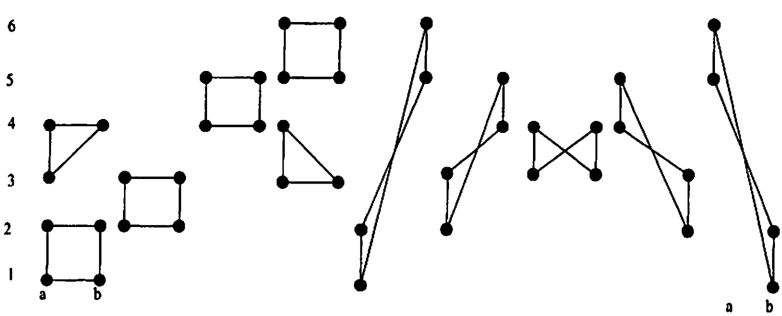


Figure 2: Cycles of $\mathcal{T}_{ab}$ for $m = 6$

**Remark 3.1.7.** Let $m$ be an even integer and $e \in E(ab\rho P_m)$. Then by the aid of Figure 2, we have

(1) If $e = (a, v_j)(b, v_{m-j+1})$ such that $j \notin \{1, \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1, m\}$, then $f_{\mathcal{T}_{ab}}(e) = 2$. (2) If $e = (a, v_1)(b, v_m)$ or $(b, v_1)(a, v_m)$, then $f_{\mathcal{T}_{ab}}(e) = 1$. (3) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor + 1})$ or $(b, v_{\lfloor \frac{m}{2} \rfloor})(a, v_{\lfloor \frac{m}{2} \rfloor + 1})$, then $f_{\mathcal{T}_{ab}}(e) = 3$. (4) If $e = (a, v_j)(a, v_{j+1})$ or $(b, v_j)(b, v_{j+1})$ such that $j \neq \lfloor \frac{m}{2} \rfloor$, then $f_{\mathcal{T}_{ab}}(e) = 2$. (5) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(a, v_{\lfloor \frac{m}{2} \rfloor + 1})$, then $f_{\mathcal{T}_{ab}}(e) = 3$. (6) If $e = (b, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor + 1})$, then $f_{\mathcal{T}_{ab}}(e) = 1$. (7) If $e = (a, v_j)(b, v_j)$ such that $j \notin \{1, m\}$, then $f_{\mathcal{T}_{ab}}(e) = 2$. (8) If $e = (a, v_j)(b, v_j)$ such that $j \in \{1, m\}$, then $f_{\mathcal{T}_{ab}}(e) = 1$.

Let $cab$ be any path of order 3. We define the following sets of cycles:

$$\begin{aligned}\mathcal{N}_{cab} &= \{\mathcal{N}_{cab}^{(j)} = (c,v_j)(a,v_j)(b,v_j)(b,v_{j+1})(a,v_{j+1})(c,v_{j+1})(c,v_j) \mid j = \\ &\quad 1,2,3,\dots,m-1\},\end{aligned}$$

$$\begin{aligned}\mathcal{Q}_{cab} &= \{\mathcal{Q}_{cab}^{(j)} = (c,v_j)(c,v_{j+1})(a,v_{m-j})(b,v_{j+1})(b,v_j)(a,v_{m-j+1}) \\ (c,v_j)|j &= 1,2,3,\dots,m-1\},\end{aligned}$$

$$\mathcal{M}_{cab} = \left\{\begin{array}{c} \mathcal{M}_{cab}^{(1)} = (c,v_{\lfloor\frac{m}{2}\rfloor})(c,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor})(b,v_{\lfloor\frac{m}{2}\rfloor}) \\ (b,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(c,v_{\lfloor\frac{m}{2}\rfloor}), \\ \mathcal{M}_{cab}^{(2)} = (c,v_{\lfloor\frac{m}{2}\rfloor+2})(c,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+1}) \\ (b,v_{\lfloor\frac{m}{2}\rfloor})(a,v_{\lfloor\frac{m}{2}\rfloor})(c,v_{\lfloor\frac{m}{2}\rfloor+2}), \\ \mathcal{M}_{cab}^{(3)} = (c,v_{\lfloor\frac{m}{2}\rfloor})(c,v_{\lfloor\frac{m}{2}\rfloor+1})(a,v_{\lfloor\frac{m}{2}\rfloor+1})(b,v_{\lfloor\frac{m}{2}\rfloor+1}) \\ (b,v_{\lfloor\frac{m}{2}\rfloor+2})(a,v_{\lfloor\frac{m}{2}\rfloor+2})(c,v_{\lfloor\frac{m}{2}\rfloor}). \end{array}\right\}.$$

**Lemma 3.1.8.** Every linear combination of cycles of $\mathcal{Q}_{cab} - \{\mathcal{Q}_{cab}^{(j)}|i \leq j \leq k\}$ contains at least one edge of the form $(c,v_j)(a,v_{m-j+1})$ such that (1) $j \leq m$ and $j \notin \{i,i+1,\dots,k+1\}$ if $m$ is even, (2) $j \leq m$ and $j \notin \{i,i+1,\dots,k+1,\lfloor\frac{m}{2}\rfloor+1\}$ if $m$ is odd.

**Proof.** Let $\mathcal{Q}$ be a linear combination of the cycles $\mathcal{Q}_{cab}^{(j_1)}, \mathcal{Q}_{cab}^{(j_2)}, \dots, \mathcal{Q}_{cab}^{(j_k)}$ of $\mathcal{Q}_{cab} - \{\mathcal{Q}_{cab}^{(j)}|i \leq j \leq k\}$. Let $j_1 < j_2 < \dots < j_k$. Then we split our work into two subcases:

**Case 1:** $m$ is odd. Then we split this case into two cases:

**Subcase 1:** $j_1 \leq \lfloor\frac{m}{2}\rfloor$. Then, by the definition of $\mathcal{Q}_{cab}$, $\mathcal{Q}_{cab}^{(j_1)}$ contains the edge $(c,v_{j_1})(a,v_{m-j_1+1})$. Since $j_1 < j_2 < \dots < j_k$, as a result non of $\mathcal{Q}_{cab}^{(j_2)}, \mathcal{Q}_{cab}^{(j_3)}, \dots, \mathcal{Q}_{cab}^{(j_k)}$ contains this edge. Thus, $\mathcal{Q}$ contains $(c,v_{j_1})(a,v_{m-j_1+1})$.

**Subcase 2:** $j_1 \geq \lfloor\frac{m}{2}\rfloor+1$. Then by the definition of $\mathcal{Q}_{cab}$, $\mathcal{Q}_{cab}^{(j_k)}$ contains the edge $(c,v_{j_k+1})(a,v_{m-(j_k+1)+1})$. Since $j_1 < j_2 < \dots < j_k$, it implies that no cycle of $\mathcal{Q}_{cab}^{(j_1)}, \mathcal{Q}_{cab}^{(j_2)}, \dots, \mathcal{Q}_{cab}^{(j_{k-1})}$ contains such edge. Hence, $\mathcal{Q}$ contains $(c,v_{j_k+1})(a,v_{m-(j_k+1)+1})$.

**Case 2:** $m$ is even. Then we argue as in Case 1 taking into account only Subcase 1 and for each $j_1$. □

By using the same idea as in the first subcase of the first case of Lemma 3.1.6, we have the following result:

**Lemma 3.1.9.** Every linear combination of cycles of $\mathcal{N}_{cab} - \{\mathcal{N}_{cab}^{(j)}|i \leq j \leq k\}$ contains at least one edge of the form $(c,v_j)(a,v_j)$, $j \leq i-1$ or $j \geq k+2$.

Note that $cab\rho P_m$ is decomposable into $cab\Box P_m \cup M_{ab} \cup M_{ac}$ where $M_{ab}, M_{ac}$ are as defined in (1).

**Lemma 3.1.10.** Let $m$ be an odd integer. Then $\mathcal{X}_{cab} = \mathcal{N}_{cab} \cup \mathcal{Q}_{cab} \cup \{\mathcal{M}_{cab}^{(2)}, \mathcal{M}_{cab}^{(3)}, \mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\} - \left\{\mathcal{N}_{cab}^{(k)}, \mathcal{Q}_{cab}^{(k)} \mid k = \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1\right\}$ is linearly independent subset of $\mathcal{C}(cab\rho P_m)$.

**Proof.** We prove that $\mathcal{N}_{cab}$ is linearly independent using the mathematical induction on $m$ for any $m$. If $m = 2$, then $\mathcal{N}_{cab}$ consists only of one cycle $\mathcal{N}_{cab}^{(1)}$, thus $\mathcal{N}_{cab}$ is linearly independent. Note that $\mathcal{N}_{cab} = \left(\cup_{j=1}^{m-2} \mathcal{N}_{cab}^{(j)}\right) \cup \mathcal{N}_{cab}^{(m-1)}$. Assume that $m$ is grater than 2 and it is true for less than $m$. Since $\mathcal{N}_{cab}^{(m-1)}$ contains the edge $(c, v_m)(a, v_m)$ which is not in any cycle of $\cup_{j=1}^{m-2} \mathcal{N}_{cab}^{(j)}$, as a result $\mathcal{N}_{cab}$ is linearly independent. By a similar way, we can show that $\mathcal{Q}_{cab}$ is linearly independent. Then we have $\mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$ and $\mathcal{N}_{cab} - \left\{\mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$ are linearly independent. Now, the cycle $\mathcal{Z}_{ca}^{(1)}$ contains the edge $(a, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor+2})$ which is not in the cycle $\mathcal{Z}_{ca}^{(2)}$, thus $\left\{\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\right\}$ is linearly independent. The cycle $\mathcal{M}_{cab}^{(2)}$ contains the edge $(a, v_{\lfloor \frac{m}{2} \rfloor+1})(b, v_{\lfloor \frac{m}{2} \rfloor+1})$ which is not in any cycle of $\{\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\}$, thus $\left\{\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}, \mathcal{M}_{cab}^{(2)}\right\}$ is linearly independent. Also the cycle $\mathcal{M}_{cab}^{(3)}$ contains the edge $(b, v_{\lfloor \frac{m}{2} \rfloor+1})(b, v_{\lfloor \frac{m}{2} \rfloor+2})$ which is not in any of the cycles $\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}, \mathcal{M}_{cab}^{(2)}$. Hence $\left\{\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}, \mathcal{M}_{cab}^{(2)}, \mathcal{M}_{cab}^{(3)}\right\}$ is linearly independent. By Lemma 3.1.6, any linear combination of cycles of $\mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$ contains an edge of the form $(c, v_j)(a, v_{m-j+1}), j \notin \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor+1, \lfloor \frac{m}{2} \rfloor+2\}$ which is not in any cycle of $\left\{\mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}, \mathcal{M}_{cab}^{(2)}, \mathcal{M}_{cab}^{(3)}\right\}$. Thus $\mathcal{Q}_{cab} \cup \left\{\mathcal{M}_{cab}^{(2)}, \mathcal{M}_{cab}^{(3)}, \mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\right\} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$ is linearly independent. By Lemma 3.1.9, any linear combination of cycles of $\mathcal{N}_{cab} - \left\{\mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$ contains an edge of the form $(c, v_j)(a, v_j), j \le \lfloor \frac{m}{2} \rfloor - 1$ or $j \ge \lfloor \frac{m}{2} \rfloor + 3$, which is not in any cycle of $\mathcal{Q}_{cab} \cup \left\{\mathcal{M}_{cab}^{(2)}, \mathcal{M}_{cab}^{(3)}, \mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\right\} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$. Thus $\mathcal{X}_{cab}$ is linearly independent. □

**Remark 3.1.11.** Let $m$ be an odd integer and $e \in E(cab\rho P_m)$. Then by the aid of Figure 3, we have that

(1) If $e = (a, v_j)(c, v_{m-j+1})$ or $(a, v_j)(b, v_{m-j+1})$ such that $j \neq \lfloor \frac{m}{2} \rfloor$, then $f_{\mathcal{X}_{cab}}(e) \le 2$. (2) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor+2})$ or $(a, v_{\lfloor \frac{m}{2} \rfloor+2})(b, v_{\lfloor \frac{m}{2} \rfloor})$, then $f_{\mathcal{X}_{cab}}(e) = 1$. (3) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor+2})$ or $(a, v_{\lfloor \frac{m}{2} \rfloor+2})(c, v_{\lfloor \frac{m}{2} \rfloor})$, then $f_{\mathcal{X}_{cab}}(e) = 3$. (4) If $e = (a, v_j)(b, v_j)$ or $(a, v_j)(c, v_j)$ such that $j \notin \{1, m\}$, then $f_{\mathcal{X}_{cab}}(e) = 2$. (5) If $e = (b, v_j)(b, v_{j+1})$ or $(c, v_j)(c, v_{j+1})$ such that

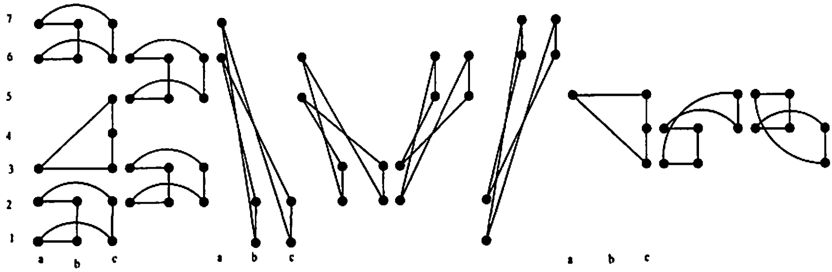

Figure 3: Cycles of $\mathcal{X}_{cab}$ for $m = 7$

$j \notin \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1\}$, then $f_{\mathcal{X}_{cab}}(e) = 2$. (6) If $e = (c, v_j)(c, v_{j+1})$ such that $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1\}$, then $f_{\mathcal{X}_{cab}}(e) \leq 3$. (7) If $e = (b, v_j)(b, v_{j+1})$ such that $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1\}$, then $f_{\mathcal{X}_{cab}}(e) \leq 1$. (8) If $e = (a, v_j)(a, v_{j+1})$, then $f_{\mathcal{X}_{cab}}(e) = 0$.

**Lemma 3.1.12.** Let $m$ be an even integer. Then $\mathcal{J}_{cab} = \mathcal{N}_{cab} \cup \mathcal{Q}_{cab} \cup \{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}, \mathcal{Z}_{ca}^{(4)}\} - \left\{\mathcal{N}_{cab}^{\lfloor\frac{m}{2}\rfloor}, \mathcal{Q}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ is linearly independent subset of $\mathcal{C}(cab\, \rho P_m)$.

**Proof.** Using the same argument as in Lemma 3.1.10, we have that $\mathcal{N}_{cab} - \left\{\mathcal{N}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ and $\mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ are linearly independent. The cycle $\mathcal{Z}_{ca}^{(3)}$ contains the edge $(c, v_{\lfloor\frac{m}{2}\rfloor+1})(a, v_{\lfloor\frac{m}{2}\rfloor+1})$ which is not in the cycle $\mathcal{M}_{cab}^{(1)}$, thus $\left\{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}\right\}$ is linearly independent. Also the cycle $\mathcal{Z}_{ca}^{(4)}$ contains the edge of the form $(c, v_{\lfloor\frac{m}{2}\rfloor})(a, v_{\lfloor\frac{m}{2}\rfloor})$ which is not in the cycles $\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}$. Hence, $\left\{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}, \mathcal{Z}_{ca}^{(4)}\right\}$ is linearly independent. Any linear combination of cycles of $\mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ contains an edge of the form $(c, v_j)(a, v_{m-j+1})$, $j \neq \lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1$, which is not in any cycle of $\left\{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}, \mathcal{Z}_{ca}^{(4)}\right\}$. Thus $\mathcal{Q}_{cab} \cup \left\{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}, \mathcal{Z}_{ca}^{(4)}\right\} - \left\{\mathcal{Q}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ is linearly independent. Similarly, any linear combination of cycles of $\mathcal{N}_{cab} - \left\{\mathcal{N}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$ contains an edge of the form $(c, v_j)(a, v_j)$, $j \leq \lfloor\frac{m}{2}\rfloor - 1$ or $j \geq \lfloor\frac{m}{2}\rfloor + 2$, which is not in any cycle of $\mathcal{Q}_{cab} \cup \left\{\mathcal{M}_{cab}^{(1)}, \mathcal{Z}_{ca}^{(3)}, \mathcal{Z}_{ca}^{(4)}\right\} - \left\{\mathcal{Q}_{cab}^{\lfloor\frac{m}{2}\rfloor}\right\}$. Thus $\mathcal{J}_{cab}$ is linearly independent. □

**Remark 3.1.13.** Let $m$ be an even integer and $e \in E(cab\rho P_m)$. Then by the aid of Figure 4, we have

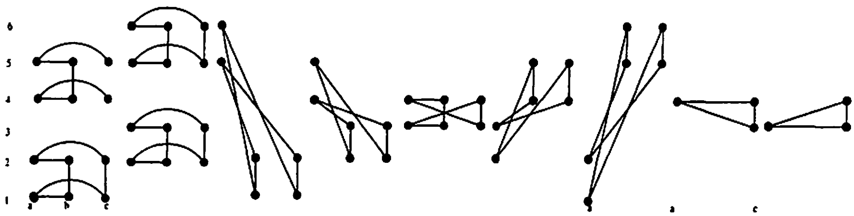

Figure 4: Cycles of $\mathcal{J}_{cab}$ for $m = 6$

(1) If $e = (a, v_j)(c, v_{m-j+1})$ or $(a, v_j)(b, v_{m-j+1})$ such that $j \notin \{1, \lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor +1, m\}$, then $f_{\mathcal{J}_{cab}}(e) = 2$. (2) If $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor+1})$ or $(a, v_{\lfloor \frac{m}{2} \rfloor+1})(c, v_{\lfloor \frac{m}{2} \rfloor})$, then $f_{\mathcal{J}_{cab}}(e) = 3$. (3) If $e = (a, v_j)(c, v_{m-j+1})$ or $(a, v_j)(b, v_{m-j+1})$ such that $j \in \{1, m\}$, then $f_{\mathcal{J}_{cab}}(e) = 1$. (4) If $e = (a, v_j)(b, v_j)$ or $(a, v_j)(c, v_j)$ such that $j \notin \{1, m\}$, then $f_{\mathcal{J}_{cab}}(e) = 2$. (5) If $e = (a, v_j)(b, v_j)$ or $(a, v_j)(c, v_j)$ such that $j \in \{1, m\}$, then $f_{\mathcal{J}_{cab}}(e) = 1$. (6) If $e = (a, v_j)(b, v_{m-j+1})$ such that $j \in \{\lfloor \frac{m}{2} \rfloor, \lfloor \frac{m}{2} \rfloor + 1\}$, then $f_{\mathcal{J}_{cab}}(e) = 1$. (7) If $e = (b, v_j)(b, v_{j+1})$ or $(c, v_j)(c, v_{j+1})$ such that $j \neq \lfloor \frac{m}{2} \rfloor$, then $f_{\mathcal{J}_{cab}}(e) = 2$. (8) If $e = (c, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor+1})$, then $f_{\mathcal{J}_{cab}}(e) = 3$. (1) If $e = (b, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor+1})$, then $f_{\mathcal{J}_{cab}}(e) = 2$. If $e = (a, v_j)(a, v_{j+1})$, then $f_{\mathcal{J}_{cab}}(e) = 0$.

**Lemma 3.1.14.** Any linear combination of cycles of $\mathcal{X}_{cab}$ or of $\mathcal{J}_{cab}$ contains an edge of the form $(c, v_j)(c, v_{j+1})$ or $(c, v_j)(a, v_l)$.

**Proof.** We will prove the case for $\mathcal{X}_{cab}$ and similarly we can prove it for $\mathcal{J}_{cab}$. Let $\mathcal{X}$ be the linear combination of cycles of $\mathcal{X}^* = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k\} \subseteq \mathcal{X}_{cab}$. Then $\mathcal{X}^*$ can be partitioned into three pairwise disjoint subsets $\mathcal{X}_1^* = \{\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_{n_1}\}$, $\mathcal{X}_2^* = \{\mathcal{X}_{n_1+1}, \mathcal{X}_{n_1+2}, \ldots, \mathcal{X}_{n_2}\}$, and $\mathcal{X}_3^* = \{\mathcal{X}_{n_2+1}, \mathcal{X}_{n_2+2}, \ldots, \mathcal{X}_k\}$ such that $\mathcal{X}_1^* \subseteq \mathcal{N}_{cab} - \left\{\mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$, $\mathcal{X}_2^* \subseteq \mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$, and $\mathcal{X}_3^* \subseteq \{\mathcal{M}_{cab}^{(1)}, \mathcal{M}_{cab}^{(2)}, \mathcal{Z}_{ca}^{(1)}, \mathcal{Z}_{ca}^{(2)}\}$.

**Case 1:** $n_1 \geq 1$. Then by Lemma 3.1.9, and since $\mathcal{X}_{cab}$ does not contain any of $\left\{\mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{N}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\}$, as a result

$$\mathcal{X}_1 \oplus \mathcal{X}_2 \oplus \cdots \oplus \mathcal{X}_{n_1}$$

contains an edge of the form $e = (a, v_j)(c, v_j)$, $j \leq \lfloor \frac{m}{2} \rfloor - 2$ or $j \geq \lfloor \frac{m}{2} \rfloor + 3$ which does not occur in any cycle of $\mathcal{X}_{cab} - \mathcal{N}_{cab}$. Thus, $e \in \mathcal{X}$.

**Case 2:** $n_1 = 0$ and $\mathcal{X}_3^* \neq \emptyset$. Then we split our work into two subcases:

**Subcase 1:** One of $\mathcal{Z}_{ca}^{(1)}$ and $\mathcal{Z}_{ca}^{(2)} \in \mathcal{X}_3^*$, say $\mathcal{Z}_{ca}^{(1)} \in \mathcal{X}_3^*$. Then the edge $e = (a, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor}) \in \mathcal{Z}_{ca}^{(1)}$ which belongs to no cycle of $\mathcal{Q}_{cab} -$

$\left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\} \cup \left\{\mathcal{M}_{cab}^{(1)}, \mathcal{M}_{cab}^{(2)}\right\}$. Hence, $e$ does not belong to any cycle of $\mathcal{X}_2^* \cup \mathcal{X}_3^* - \left\{\mathcal{Z}_{ca}^{(1)}\right\}$. Thus, $e \in \mathcal{X}$.

**Subcase 2:** Non of $\mathcal{Z}_{ca}^{(1)}$ and $\mathcal{Z}_{ca}^{(2)} \in \mathcal{X}_3^*$. Then at least one of $\mathcal{M}_{cab}^{(1)}$ and $\mathcal{M}_{cab}^{(2)}$ belongs to $\mathcal{X}_3^*$, say $\mathcal{M}_{cab}^{(1)}$. By the definition of $\mathcal{M}_{cab}^{(1)}$, $e = (c, v_{\lfloor \frac{m}{2} \rfloor})(c, v_{\lfloor \frac{m}{2} \rfloor+1}) \in \mathcal{M}_{cab}^{(1)}$ which belongs to no other cycle of $\mathcal{Q}_{cab} - \left\{\mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor}, \mathcal{Q}_{cab}^{\lfloor \frac{m}{2} \rfloor+1}\right\} \cup \left\{\mathcal{M}_{cab}^{(2)}\right\}$. Hence, $e$ does not belong to any other cycle of $\mathcal{X}_2^* \cup \mathcal{X}_3^*$. Thus, $e \in \mathcal{X}$.

**Case 3:** $n_1 = 0$ and $\mathcal{X}_3^* = \emptyset$. Then $\mathcal{X}$ is the linear combination of cycles of $\{\mathcal{X}_{n_1+1}, \mathcal{X}_{n_1+2}, \ldots, \mathcal{X}_{n_2}\}$. By Lemma 3.1.8, $\mathcal{X}$ contains at least one edge of the form $(c, v_j)(a, v_{m-j+1})$. □

**Lemma 3.1.15.** Let $m$ be an odd integer. Then $\mathcal{A}_{ab} \cup \mathcal{X}_{cab}$ is a linearly independent subset of $\mathcal{C}(cab\rho P_m)$.

**Proof.** We know that each of $\mathcal{A}_{ab}$ and $\mathcal{X}_{cab}$ is linearly independent. By Lemma 3.1.14, any linear combination of cycles of $\mathcal{X}_{cab}$ contains an edge of the form $(c, v_j)(c, v_{j+1})$ or $(c, v_j)(a, v_l)$ which is not in any cycle of $\mathcal{A}_{ab}$. Thus $\mathcal{A}_{ab} \cup \mathcal{X}_{cab}$ is linearly independent. □

By using the same argument as in Lemma 3.1.15 after replacing $\mathcal{A}_{ab}$ by $\mathcal{T}_{ab}$ and $\mathcal{X}_{cab}$ by $\mathcal{J}_{cab}$, we have the following result.

**Lemma 3.1.16.** Let $m$ be an even integer. Then $\mathcal{T}_{ab} \cup \mathcal{J}_{cab}$ is linearly independent subset of $\mathcal{C}(cab\rho P_m)$.

Throughout the rest of this work, consider

$$\mathcal{B}_{ab} = \begin{cases} \mathcal{A}_{ab}, & \text{if } m \text{ is odd} \\ \mathcal{T}_{ab}, & \text{if } m \text{ is even} \end{cases} \tag{2}$$

and

$$\mathcal{B}_{cab} = \begin{cases} \mathcal{X}_{cab}, & \text{if } m \text{ is odd} \\ \mathcal{J}_{cab}, & \text{if } m \text{ is even} \end{cases}. \tag{3}$$

Let $V(S_n) = \{u_1, u_2, \ldots, u_n\}$ with $d_{S_n}(u_1) = n-1$ and $P_m = v_1 v_2 \ldots v_m$. Then the graph $S_n \rho P_m$ is decomposable into $S_n \Box P_m \cup (\cup_{i=2}^{n} M_{u_1 u_i})$ where $M_{u_1 u_i}$ is the graph defined as in (1). Hence, $|E(S_n \rho P_m)| = n(m-1) + m(n-1) + 2(n-1)\lfloor m/2 \rfloor$. Therefore,

$$\dim \mathcal{C}(S_n \rho P_m) = mn - n - m + 2(n-1)\lfloor m/2 \rfloor + 1. \tag{4}$$

**Theorem 3.1.17.** For any star $S_n$ of order $n \geq 4$ and path $P_m$ of order $m \geq 3$, $b(S_n \rho P_m) \leq 4$.

**Proof.** Define $\mathcal{B}(S_n\rho P_m) = \mathcal{B}_{u_1u_2} \cup (\cup_{j=2}^{n-1}\mathcal{B}_{u_{j+1}u_1u_j})$. We now show that $\mathcal{B}(S_n\rho P_m)$ is linearly independent using the mathematical induction on $n$. If $n = 2$, then $\mathcal{B}(S_n\rho P_m) = \mathcal{B}_{u_1u_2}$ and is linearly independent by Lemmas 3.1.4 and 3.1.6. If $n = 3$, then $\mathcal{B}(S_n\rho P_m) = \mathcal{B}_{u_1u_2} \cup \mathcal{B}_{u_3u_1u_2}$ and it is linearly independent by Lemmas 3.1.15 and 3.1.16. Assume $n \geq 4$, and it is true for less than or equal to $n-2$. Note that $\mathcal{B}(S_n\rho P_m) = (\mathcal{B}_{u_1u_2} \cup (\cup_{j=2}^{n-2}\ \mathcal{B}_{u_{j+1}u_1u_j})) \cup \mathcal{B}_{u_nu_1u_{n-1}}$. By induction steps and Lemmas 3.1.10 and 3.1.12, each of $\mathcal{B}_{u_1u_2} \cup (\cup_{j=2}^{n-2}\ \mathcal{B}_{u_{j+1}u_1u_j})$ and $\mathcal{B}_{u_nu_1u_{n-1}}$ is linearly independent. By Lemma 3.1.14 and (2), any linear combination of cycles of $\mathcal{B}_{u_nu_1u_{n-1}}$ contains an edge of the form $(u_n, v_j)(u_n, v_{j+1})$ or $(u_n, v_j)(u_1, v_l)$ which are not in any cycle of $\mathcal{B}_{u_1u_2} \cup (\cup_{j=2}^{n-2}\ \mathcal{B}_{u_{j+1}u_1u_j})$. Thus $\mathcal{B}(S_n\rho P_m)$ is linearly independent. Now, from (2)

$$|\mathcal{B}_{u_iu_{i+1}}| = |\mathcal{A}_{ab}| = (m-1) + 2\lfloor m/2 \rfloor$$

if $m$ is odd, and

$$|\mathcal{B}_{u_iu_{i+1}}| = |\mathcal{T}_{ab}| = (m-1) + 2\lfloor m/2 \rfloor$$

if $m$ is even. Also, from (3)

$$|\mathcal{B}_{u_{j+1}u_1u_j}| = |\mathcal{X}_{cab}| = 2(m-1) = (m-1) + 2\lfloor m/2 \rfloor$$

if $m$ is odd, and

$$\begin{aligned} |\mathcal{B}_{u_{j+1}u_1u_j}| &= |\mathcal{J}_{cab}| = (m-1) + (m-1) + 1 \\ &= 2(m-1) + 1 \\ &= (m-1) + 2\lfloor m/2 \rfloor \end{aligned}$$

if $m$ is even. Thus

$$\begin{aligned} |\mathcal{B}(S_n\rho P_m)| &= |\mathcal{B}_{u_1u_2}| + \sum_{i=2}^{n-1} |\mathcal{B}_{u_{j+1}u_1u_j}| \\ &= (m-1) + 2\lfloor m/2 \rfloor + (n-2)\left((m-1) + 2\lfloor m/2 \rfloor\right) \\ &= mn - m - n + 2(n-1)\lfloor m/2 \rfloor + 1 \\ &= \dim C(S_n\rho P_m) \end{aligned}$$

where the last equality follows from (4). Therefore, $\mathcal{B}(S_n\rho P_m)$ is a basis for $C(S_n\rho P_m)$. To complete the proof, we have to show that $\mathcal{B}(S_n\rho P_m)$ is of fold 4. Note that

$$\begin{aligned} E(\mathcal{B}_{u_{j+1}u_1u_j}) \cap E(\mathcal{B}_{u_{k+1}u_1u_k}) &= \emptyset \text{ if } |k-j| > 1 \text{ and} \\ E(\mathcal{B}_{u_1u_2}) \cap E(\mathcal{B}_{u_{k+1}u_1u_k}) &= \emptyset \text{ if } k > 2. \end{aligned}$$

Now we consider two cases.

**Case 1.** $m$ is odd. Then by Remarks 3.1.5 and 3.1.11, we have the following: (1) If $e = (u_1, v_j)(u_1, v_{j+1})$ such that $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1\}$, then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_1u_2}} + f_{\mathcal{B}_{u_3u_1u_2}} \leq 3 + 0$. (2) If $e = (u_1, v_{\lfloor\frac{m}{2}\rfloor})(u_2, v_{\lfloor\frac{m}{2}\rfloor+2})$ or $(u_1, v_{\lfloor\frac{m}{2}\rfloor+2})(u_2, v_{\lfloor\frac{m}{2}\rfloor})$, then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_1u_2}} + f_{\mathcal{B}_{u_3u_1u_2}} \leq 3 + 1$. (3) If $e \in u_1u_2\rho P_m$ which is not as in (1)or (2), then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_1u_2}} + f_{\mathcal{B}_{u_3u_1u_2}} \leq 2 + 2 = 4$. (4) If $e = (u_i, v_j)(u_i, v_{j+1})$ for $i \geq 3$ such that $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor + 1\}$, then $f_{\mathcal{B}(S_n\rho P_m)}(e) = f_{\mathcal{B}_{u_{i+1}u_1u_i}}(e) + f_{\mathcal{B}_{u_iu_1u_{i-1}}}(e) \leq 1 + 3$. (5) If $e = (u_1, v_{\lfloor\frac{m}{2}\rfloor})(u_i, v_{\lfloor\frac{m}{2}\rfloor+2})$ or $(u_1, v_{\lfloor\frac{m}{2}\rfloor+2})(u_i, v_{\lfloor\frac{m}{2}\rfloor})$ for $i \geq 3$, then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_{i+1}u_1u_i}}(e) + f_{\mathcal{B}_{u_iu_1u_{i-1}}}(e) \leq 1 + 3$. (6) If $e \in u_1u_i\rho P_m - u_1u_2\rho P_m$ for $i \geq 3$ and not as in (3), then $f_{\mathcal{B}(S_n\rho P_m)} = f_{u_{i+1}u_1u_i} + f_{u_iu_1u_{i-1}} \leq 2 + 2$.

**Case 2.** $m$ is even. Then by Remarks 3.1.7, 3.1.13, and as in Case 1, we have the following: (1) If $e = (u_1, v_j)(u_2, v_{j+1})$ or $(u_1, v_{j+1})(u_2, v_j)$ or $(u_1, v_j)(u_1, v_{j+1})$ such that $j = \lfloor\frac{m}{2}\rfloor$, then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_1u_2}} + f_{\mathcal{B}_{u_3u_1u_2}} \leq 3 + 1$. (2) If $e \in u_1u_2\rho P_m$ which is not as in (1), then $f_{\mathcal{B}(S_n\rho P_m)} = f_{\mathcal{B}_{u_1u_2}} + f_{\mathcal{B}_{u_3u_1u_2}} \leq 2 + 2 = 4$. (3) If $e = (u_i, v_j)(u_i, v_{j+1})$ or $(u_1, v_j)(u_i, v_{j+1})$ or $(u_1, v_{j+1})(u_i, v_j)$ for $i \geq 3$ such that $j \in \{\lfloor\frac{m}{2}\rfloor, \lfloor\frac{m}{2}\rfloor+1\}$, then $f_{\mathcal{B}(S_n\rho P_m)}(e) = f_{\mathcal{B}_{u_{i+1}u_1u_i}}(e) + f_{\mathcal{B}_{u_iu_1u_{i-1}}}(e) \leq 1+3$. (4) If $e \in u_1u_i\rho P_m - u_1u_2\rho P_m$ for $i \geq 3$ and not as in (3), then $f_{\mathcal{B}(S_n\rho P_m)} = f_{u_{i+1}u_1u_i} + f_{u_iu_1u_{i-1}} \leq 2 + 2$. □

**Lemma 3.1.18** ([17]). If $m \geq 3$, then $b(ab\rho P_m) \geq 3$.

The following result follows immediately from Lemma 3.1.18, MacLan's Theorem and Kuratowiski's Theorem and Theorem 3.1.17.

**Corollary 3.1.19.** $3 \leq b(S_n\rho P_m) \leq 4$, for all $n \geq 2$, $m \geq 3$.

## 3.2 The basis number of $S_n\rho S_m$

Consider $S_m$ to be a star with vertex set $\{v_1, v_2, \ldots, v_m\}$ and $d_{S_m}(v_1) = m-1$. Note that the automorphism group of $S_m$ is isomorphic to the symmetric group on the set $\{v_2, v_3, \ldots, v_m\}$. Therefore, for any $\gamma \in$Aut$(G), \gamma(v_1) = v_1$. Moreover, for any two vertices $v_i, v_j$ such that $2 \leq i, j \leq m$ there is an automorphism $\alpha$ such that $\alpha(v_i) = v_j$. Hence, the graph $ab\rho S_m$ is decomposable into $(a\square S_m) \cup (b\square S_m) \cup \{(a, v_1)(b, v_1)\} \cup ab[N_{m-1}]$ where $N_{m-1}$ is the null graph with vertex set $\{v_2, v_3, \ldots, v_m\}$ and $ab[N_{m-1}]$ is the lexicographic product of $ab$ and $N_{m-1}$. Now, we use the following sets of cycles which where introduced by Jaradat in [30].

$$\mathcal{H}_{ab} = \{(a, v_j)(b, v_i)(a, v_{j+1})(b, v_{i+1})(a, v_j) \mid 2 \leq i, j \leq m - 1\},$$

$$\mathcal{G}_{ab} = \left\{ \mathcal{G}_{ab}^{(j)} = (a, v_1)(a, v_j)(b, v_2)(a, v_{j+1})(a, v_1) \mid 2 \le j \le m-1 \right\},$$

$$\mathcal{S}_{ab} = \{(a, v_1)(a, v_2)(b, v_2)(b, v_1)(a, v_1)\}.$$

Note that $\mathcal{H}_{ab}$ is the Schemeichel's 4-fold basis of $\mathcal{C}(ab[N_{m-1}])$ (see Theorem 2.4 in [22]). Moreover, (1) if $e = (a, v_2)(b, v_m)$ or $e = (a, v_m)(b, v_2)$ or $e = (a, v_2)(b, v_2)$ or $e = (a, v_m)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) = 1$. (2) If $e = (a, v_2)(b, v_l)$ or $(a, v_j)(b, v_2)$ or $(a, v_m)(b, v_l)$ or $(a, v_j)(b, v_m)$, then $f_{\mathcal{H}_{ab}}(e) \le 2$. (3) If $e \in E(ab[N_{m-1}])$ and is not of the above form, then $f_{\mathcal{H}_{ab}}(e) \le 4$.

The following result of Jaradat [17], and Jaradat et al. [16] will be used in the coming results of this section.

**Lemma 3.2.1 (Jaradat).** $\mathcal{L}_{ab} = \mathcal{H}_{ab} \cup \mathcal{G}_{ab} \cup \mathcal{G}_{ba} \cup \mathcal{S}_{ab}$ is linearly independent subset of cycles of $\mathcal{C}(ab\rho S_m)$.

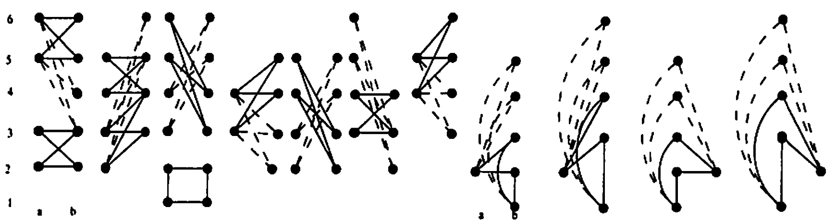

Figure 5: Cycles of $\mathcal{L}_{ab}$ for $m = 6$

**Proposition 3.2.2 (Jaradat et al.)** *Let $A$ and $B$ be two linearly independent sets of cycles such that $E(A) \cap E(B)$ is an edge set of a forest. Then $A \cup B$ is linearly independent.*

Let $cab$ be any star of order 3, then we define the following sets of cycles:

$$\mathcal{W}_{cab} = \{(c, v_1)(c, v_2)(a, v_2)(b, v_m)(b, v_1)(a, v_1)(c, v_1)\}$$

and

$$\mathcal{E}_{cab} = \left\{ \mathcal{E}_{cab}^{(j)} = (c, v_2)(a, v_j)(b, v_m)(a, v_{j+1})(c, v_2) \mid 2 \le j \le m-1 \right\}.$$

**Lemma 3.2.3.** $\mathcal{Y}_{cab} = \mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca} \cup \mathcal{W}_{cab}$ is linearly independent.
**Proof.** We use mathematical induction on $m$ to show that $\mathcal{E}_{cab}$ is linearly independent. If $m = 3$, then $\mathcal{E}_{cab}$ consists only of one cycle $\mathcal{E}_{cab}^{(1)}$. Thus $\mathcal{E}_{cab}$

is linearly independent. Note that $\mathcal{E}_{cab} = (\cup_{j=2}^{m-2}\mathcal{E}_{cab}^{(j)}) \cup \mathcal{E}_{cab}^{(m-1)}$. Assume that $m$ is grater than 3 and it is true for less than $m$. Since $\mathcal{E}_{cab}^{(m-1)}$ contains the edge $(c, v_2)(a, v_m)$ which is not in any cycle of $\cup_{j=2}^{m-2}\mathcal{E}_{cab}^{(j)}$, $\mathcal{E}_{cab}$ is linearly independent. By Lemma 3.2.1, we have $\mathcal{H}_{ca}$ and $\mathcal{G}_{ca}$ are linearly independent. Note that $E(\mathcal{H}_{ca}) \cap E(\mathcal{E}_{cab}) = \{(c, v_2)(a, v_j) \mid 2 \le j \le m\}$ which is an edge set of a star. Then, by Proposition 3.2.2, $\mathcal{E}_{cab} \cup \mathcal{H}_{ca}$ is linearly independent. Now, any linear combination of cycles of $\mathcal{G}_{ca}$ contains an edge of the form $(c, v_1)(c, v_j), j \ge 2$ which is not in any cycle of $\mathcal{E}_{cab} \cup \mathcal{H}_{ca}$. Thus $\mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca}$ is linearly independent. Finally, $\mathcal{W}_{cab}$ contains the edge $(c, v_1)(a, v_1)$ which is not in any cycle of $\mathcal{E}_{cab} \cup \mathcal{H}_{ca} \cup \mathcal{G}_{ca}$. Thus $\mathcal{Y}_{cab}$ is linearly independent. □

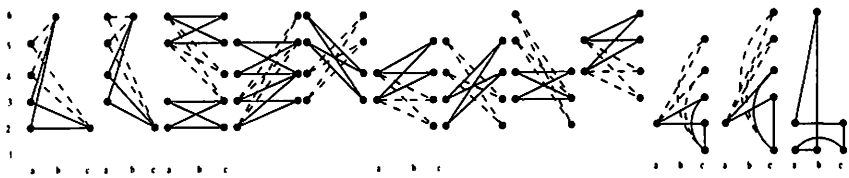

Figure 6: Cycles of $\mathcal{Y}_{cab}$ for $m = 6$

**Remark 3.2.4.** Let $e \in E(abc\rho S_m)$, then by the aid of Figures 5 and 6, we have: (1) If $e = (a, v_1)(b, v_1)$, then $f_{\mathcal{L}_{ab}}(e) = 1$ and $f_{\mathcal{Y}_{cab}}(e) = 1$. (2) If $e = (a, v_2)(b, v_2)$, then $f_{\mathcal{L}_{ab}}(e) = 4$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (3) If $e = (a, v_1)(a, v_2)$, then $f_{\mathcal{L}_{ab}}(e) = 2$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (4) If $e = (a, v_1)(a, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) \le 2$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (5) If $e = (b, v_1)(b, v_2)$, then $f_{\mathcal{L}_{ab}}(e) = 2$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (6) If $e = (b, v_1)(b, v_j)$, $2 \le j < m$, then $f_{\mathcal{L}_{ab}}(e) \le 2$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (7) If $e = (b, v_1)(b, v_m)$, then $f_{\mathcal{L}_{ab}}(e) = 1$ and $f_{\mathcal{Y}_{cab}}(e) \le 1$. (8) If $e = (a, v_2)(b, v_j)$, $2 < j < m$, then $f_{\mathcal{L}_{ab}}(e) \le 4$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (9) If $e = (a, v_2)(b, v_m)$, then $f_{\mathcal{L}_{ab}}(e) \le 2$ and $f_{\mathcal{Y}_{cab}}(e) \le 2$. (10) If $e = (a, v_j)(b, v_2)$, $2 < j \le m$, then $f_{\mathcal{L}_{ab}}(e) \le 4$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (11) If $e = (a, v_j)(b, v_m)$, $2 < j \le m$, then $f_{\mathcal{L}_{ab}}(e) \le 2$ and $f_{\mathcal{Y}_{cab}}(e) \le 2$. (12) If $e = (a, v_j)(b, v_k)$, $2 \le j, k \le m$, and not as in (1)-(11), then $f_{\mathcal{L}_{ab}}(e) \le 4$ and $f_{\mathcal{Y}_{cab}}(e) = 0$. (13) If $e = (a, v_1)(c, v_1)$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) = 1$. (14) If $e = (c, v_1)(c, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 2$. (15) If $e = (a, v_2)(c, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 4$. (16) If $e = (c, v_2)(a, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 4$. (17) If $e = (a, v_m)(c, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 2$. (18) If $e = (c, v_m)(a, v_j)$, $2 \le j \le m$, then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 2$. (19) If $e = (a, v_j)(c, v_k)$, $2 \le j, k \le m$, and not as in (13)-(18), then $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 4$.

Throughout the rest of this work, we consider

$$\mathcal{F}_{ab} = \mathcal{L}_{ab} \text{ and } \mathcal{F}_{cab} = \mathcal{Y}_{cab}.$$

Let $V(S_n) = \{u_1, u_2, \ldots, u_n\}$ with $d_{S_n}(u_1) = n - 1$. Then the graph $S_n \rho S_m$ is decomposable into

$$\cup_{i=1}^{n} u_i \Box S_m \cup (\cup_{i=2}^{n}(\{(u_1, v_1)(u_i, v_1)\} \cup u_1 u_i[N_{m-1}])$$

where $N_{m-1}$ is the graph defined as above. Hence, $|E(S_n \rho S_m)| = n(m-1) + (n-1)(1 + (m-1)^2)$. Therefore,

$$\dim C(S_n \rho P_m) = nm^2 - m^2 - 2nm + 2m + n - 1. \qquad (5)$$

**Theorem 3.2.5.** For any two stars $S_n$ and $S_m$ of order $n, m \geq 2$, we have that $b(S_n \rho S_m) \leq 4$. Moreover the equality holds if $n \geq 4$ and $m \geq 6$ .
**Proof.** Define $\mathcal{B}(S_n \rho S_m) = (\cup_{i=2}^{n-1} \mathcal{F}_{u_{i+1} u_1 u_i}) \cup \mathcal{F}_{u_1 u_2}$. We now show that $\mathcal{B}(S_n \rho S_m)$ is linearly independent by using the mathematical induction on $n$. If $n = 2$, then $\mathcal{B}(S_n \rho S_m) = \mathcal{F}_{u_1 u_2}$ and it is linearly independent by Lemma 3.2.1. If $n = 3$, then $\mathcal{B}(S_n \rho S_m) = \mathcal{F}_{u_1 u_2} \cup \mathcal{F}_{u_3 u_1 u_2}$. Note that

$$\begin{aligned} E(\mathcal{F}_{u_1 u_2}) \cap E(\mathcal{F}_{u_3 u_1 u_2}) &= \{(u_1, v_i)(u_2, v_m) \mid 2 \leq i \leq m\} \cup \\ &\quad \{(u_2, v_m)(u_2, v_1), (u_1, v_1)(u_2, v_1)\} \end{aligned}$$

which is an edge set of tree. Hence, $\mathcal{B}(S_n \rho S_m)$ is linearly independent by Proposition 3.2.2. Assume that $n \geq 4$ and it is true for less than $n-1$. Note that $\mathcal{B}(S_n \rho S_m) = (\mathcal{F}_{u_1 u_2} \cup (\cup_{i=2}^{n-2} \mathcal{F}_{u_{i+1} u_1 u_i})) \cup \mathcal{F}_{u_n u_1 u_{n-1}}$. By induction steps we have $\mathcal{F}_{u_1 u_2} \cup (\cup_{i=2}^{n-2} \mathcal{F}_{u_{i+1} u_1 u_i})$ is linearly independent. Since

$$\begin{aligned} & E(\mathcal{F}_{u_1 u_2} \cup (\cup_{i=2}^{n-2} \mathcal{F}_{u_{i+1} u_1 u_i})) \cap E(\mathcal{F}_{u_n u_1 u_{n-1}}) \\ = \; & \{(u_{n-1}, v_m)(u_1, v_i) \mid 2 \leq i \leq m\} \cup \{(u_{n-1}, v_1) \\ & (u_{n-1}, v_m), (u_{n-1}, v_1)(u_1, v_1)\} \end{aligned}$$

which is an edges set of a tree, as a result, by Proposition 3.2.2, $\mathcal{B}(S_n \rho S_m)$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{F}_{u_1 u_2}| &= |\mathcal{L}_{ab}| \\ &= (m-2)^2 + 2(m-2) + 1, \end{aligned}$$

and

$$\begin{aligned} |\mathcal{F}_{cab}| &= |\mathcal{Y}_{cab}| \\ &= (m-2)+(m-2)^2+(m-2)+1 \\ &= (m-2)^2+2(m-2)+1. \end{aligned}$$

Thus,

$$\begin{aligned} |\mathcal{B}(S_n\rho S_m)| &= |\mathcal{F}_{u_1u_2}| + \sum_{i=2}^{n-1} |\mathcal{F}_{u_{i+1}u_1u_i}| \\ &= \sum_{i=1}^{n-1}((m-2)^2+2(m-2)+1) \\ &= (n-1)((m-2)^2+2(m-2)+1) \\ &= nm^2-m^2-2mn+2m+n-1 \\ &= \dim C(S_n\rho S_m). \end{aligned}$$

where the last equality follows from (5). Therefore, $\mathcal{B}(S_n\rho S_m)$ form a basis for $C(S_n\rho S_m)$. Now, we prove that $\mathcal{B}(S_n\rho S_m)$ is a 4-fold basis. Note that

$$\begin{aligned} E(\mathcal{F}_{u_1u_2}) \cap E(\mathcal{F}_{u_{i+1}u_1u_i}) &= \emptyset \text{ if } i>3 \text{ and} \\ E(\mathcal{F}_{u_{i+1}u_1u_i}) \cap E(\mathcal{F}_{u_{k+1}u_1u_k}) &= \emptyset \text{ whenever } |k-i|>1. \end{aligned}$$

Thus, (i) if $e \in E(u_1u_2\rho S_m)$, then $f_{\mathcal{B}(S_n\rho S_m)}(e) = f_{\mathcal{F}_{u_1u_2}}(e) + f_{\mathcal{F}_{u_3u_1u_2}}(e)$, which is by (1)-(12) of Remark 3.2.4, less than or equal to 4. (ii) If $e = (u_1,v_2)(u_i,v_j)$ or $(u_i,v_2)(u_1,v_j)$, $2 \le j \le m$ and $i \ge 3$ or $(u_1,v_j)(u_i,v_k)$, $2 \le j \le m$, $2 < k < m$, then $f_{\mathcal{B}(S_n\rho S_m)}(e) = f_{\mathcal{F}_{u_ju_1u_{j-1}}}(e) \le 4$, then by (16), (17) and (19) of Remark 3.2.4 $f_{\mathcal{B}(S_n\rho S_m)}(e) = f_{\mathcal{F}_{u_ju_1u_{j-1}}}(e) \le 4$. (iii) If $e = (u_1,v_j)(u_i,v_k)$, $2 \le j \le m$ and $2 < j < m$, then by (19) of Remark 3.2.4 $f_{\mathcal{L}_{ab}}(e) = 0$ and $f_{\mathcal{Y}_{cab}}(e) \le 4$. (iv) If $e \in E(u_1u_j\rho S_m) - E(u_1u_2\rho S_m)$ and not as in (2), then $f_{\mathcal{B}(S_n\rho S_m)}(e) = f_{\mathcal{F}_{u_ju_1u_{j-1}}}(e) + f_{\mathcal{F}_{u_{j+1}u_1u_j}}(e) \le 2+2=4$ by (12)-(17) of Remark 3.2.1. We next show that $b(S_n\rho S_m) = 4$ for all $n \ge 4$ and $m \ge 6$. Suppose that $\mathcal{B}$ is a 3-fold basis of $C(S_n\rho S_m)$, for $n \ge 4$ and $m \ge 6$. Since the girth of $S_n\rho S_m$ is 4, as a result

$$4\dim C(S_n\rho S_m) \le 3 \mid E(S_n\rho S_m) \mid$$

and so

$$4(nm^2-2nm-m^2+2m+n-1) \le 3(nm^2-nm-m^2+2m+n-2),$$

which implies that,

$$(m^2+1)(n-1)-m(5n-2)+3 \le 0.$$

Thus,

$$(m^2+1)(n-1) \leq 5m(n-1) + 3(m-1),$$

which implies that

$$m \leq 5 + 3(m-1)/m(n-1) - 1/m.$$

But for $n \geq 4$, we have

$$3(m-1)/m(n-1) - 1/m < 1.$$

Thus, $m \leq 5$. This is a contradiction. Thus $b(S_n\rho S_m) \geq 4$, for all $n \geq 4$ and $m \geq 6$. Therefore, $b(S_n\rho S_m) = 4$, for all $n \geq 4$, $m \geq 6$. □

**Theorem 3.2.6.** $\mathcal{B}(S_n\rho S_m)$ is a required basis of $S_n\rho S_m$ for each $n \geq 4$ and $m \geq 6$.

## 3.3 The basis number of $S_n\rho W_m$

Now, consider $W_m$ to be the wheel graph with vertex set $\{v_1, v_2, \ldots, v_m\}$ and $d_{W_m}(v_1) = m-1$. Note that for $m \geq 5$ and for each $2 \leq i, j \leq m$, there exist $\alpha \in \mathrm{Aut}(W_m)$ such that $\alpha(v_i) = v_j$. Let $a$ be a vertex. Then we recall the following sets of cycles of Jaradat [17]:

$$\mathcal{P}_a = \left\{\mathcal{P}_a^{(j)} = (a,v_1)(a,v_j)(a,v_{j+1})(a,v_1) \mid j = 2,3,\ldots,m-1\right\},$$

$$\mathcal{I}_a = \{(a,v_2)(a,v_3)\ldots(a,v_m)(a,v_2)\}.$$

**Lemma 3.3.1.** $(\cup_{i=1}^n \mathcal{P}_{u_i}) \cup (\cup_{i=1}^n \mathcal{I}_{u_i}) \cup \mathcal{B}(S_n\rho S_m)$ is linearly independent.
**Proof.** By Theorem 3.2.1, $\mathcal{B}(S_n\rho S_m)$ is linearly independent. It is easy to verify that $(\cup_{i=1}^n \mathcal{I}_{u_i})$ is a set of union of edge disjoint cycles, thus $(\cup_{i=1}^n \mathcal{I}_{u_i})$ is linearly independent. Note that, for each $i = 1,2,\ldots,n$, $\mathcal{P}_{u_i}^{(j)}$ contains the edge $(u_i,v_j)(u_i,v_{j+1})$ which does not appear in any other cycle of $\mathcal{P}_{u_i}$. Thus, $\mathcal{P}_{u_i}$ is linearly independent for each $i$. Since $E(\mathcal{P}_{u_i}) \cap E(\mathcal{P}_{u_j}) = \emptyset$ whenever $i \neq j$, we have that $\cup_{i=1}^n \mathcal{P}_{u_i}$ is linearly independent. Since $E(\cup_{i=1}^n \mathcal{P}_{u_i}) - (\cup_{i=1}^n E(u_i \Box P_{m-1})$ is a forest, any linear combination of cycles of $(\cup_{i=1}^n \mathcal{P}_{u_i})$ contains at least one edge of $(\cup_{i=1}^n E(u_i \Box P_{m-1})$, which is not in any cycle of $\mathcal{B}(S_n\rho S_m)$ where $P_{m-1} = v_2v_3\ldots v_m$. Thus $\mathcal{B}(S_n\rho S_m) \cup (\cup_{i=1}^n \mathcal{P}_{u_i})$ is linearly independent. Similarly, any linear combination of cycles of $(\cup_{i=1}^n \mathcal{I}_{u_i})$ contains an edge of the form $(u_i,v_2)(u_i,v_m)$, $1 \leq i \leq n$ which is not in any cycle of $\mathcal{B}(S_n\rho S_m) \cup (\cup_{i=1}^n \mathcal{P}_{u_i})$. Thus $\mathcal{B}(S_n\rho S_m) \cup (\cup_{i=1}^n \mathcal{P}_{u_i}) \cup (\cup_{i=1}^n \mathcal{I}_{u_i})$ is linearly independent. □

Note that $S_n\rho W_m$ is decomposable into $S_n\rho S_m \cup (\cup_{i=1}^{n}(a_i\Box C))$ where $C = v_2v_3\ldots v_mv_2$. Thus, $|E(S_n\rho W_m)| = |E(S_n\rho S_m)| + (m-1)n$. Hence,

$$\dim \mathcal{C}(P_n\rho W_m) = (n-1)m^2 + 2m - mn - 1. \tag{6}$$

**Theorem 3.3.2.** For any star $S_n$ with $n \geq 2$ and wheel $W_m$ with $m \geq 5$, we have that $b(S_n\rho W_m) \leq 4$. Moreover, the equality holds if $n \geq 2$ and $m \geq 12$.

**Proof.** Define $\mathcal{B}(S_n\rho W_m) = \mathcal{B}(S_n\rho S_m)\cup(\cup_{i=1}^{n}\mathcal{P}_{u_i})\cup(\cup_{i=1}^{n}\mathcal{I}_{u_i})$. By Lemma 3.3.1, $\mathcal{B}(S_n\rho W_m)$ is linearly independent. Now,

$$|\mathcal{P}_{u_i}| = (m-2),$$

and

$$|\mathcal{I}_{u_i}| = 1.$$

Hence,

$$\begin{aligned} |\mathcal{B}(S_n\rho W_m)| &= |\mathcal{B}(S_n\rho S_m)| + \sum_{i=1}^{n}\mathcal{P}_{u_i} + \sum_{i=1}^{n}\mathcal{I}_{u_i} \\ &= nm^2 - m^2 - 2mn + 2m + n - 1 + n(m-2) + n \\ &= m^2(n-1) - nm + 2m - 1 \\ &= \dim \mathcal{C}(S_n\rho W_m). \end{aligned}$$

where the last equality follows from (6). Thus, $\mathcal{B}(S_n\rho W_m)$ form a basis for $\mathcal{C}(S_n\rho W_m)$. Now, we prove that $\mathcal{B}(S_n\rho W_m)$ is a 4-fold basis for all $n \geq 2$ and $m \geq 5$. By Remark 3.2.4 and Theorem 3.2.5 we have the following: (1) If $e = (u_i,v_1)(u_i,v_j)$ such that $2 \leq j \leq m$, then $f_{\mathcal{B}(S_n\rho W_m)}(e) = f_{\mathcal{B}(S_n\rho S_m)}(e) + f_{\mathcal{P}_{u_i}}(e) \leq 2+2$. (2) If $e = (u_i,v_j)(u_i,v_{j+1})$ such that $1 \leq j \leq m-1$, then $f_{\mathcal{B}(S_n\rho W_m)}(e) = f_{\mathcal{I}_{u_i}}(e) + f_{\mathcal{P}_{u_i}}(e) \leq 1+1$. (3) If $e = (u_i,v_2)(u_i,v_m)$, then $f_{\mathcal{B}(S_n\rho W_m)}(e) = f_{\mathcal{I}_{u_i}}(e) = 1$. (4) If $e \in E(S_n\rho W_m)$ and is not as in (1)-(3), then $f_{\mathcal{B}(S_n\rho W_m)}(e) = f_{\mathcal{B}(S_n\rho S_m)}(e) \leq 4$. To show that $b(S_n\rho W_m) \geq 4$, for any $n \geq 2$ and $m \geq 12$, we have to exclude any possibility for the cycle space $\mathcal{C}(S_n\rho W_m)$ to have a 3-fold basis for any $n \geq 2$ and $m \geq 12$. To that we use the same argument as in Theorem 3.18 of [17]. For the completeness we give the proof. Suppose that $\mathcal{B}$ is a 3-fold basis of cycle space $\mathcal{C}(S_n\rho W_m)$ for any $n \geq 2$ and $m \geq 12$. Since the girth of $S_n\rho S_m$ is 4, then we have the following three cases:

**Case1:** Suppose that $\mathcal{B}$ consists only of 3-cycles. Then $|\mathcal{B}| \leq 3(m-1)n$ because any 3-cycle must contain an edge of $E(a_i\Box(v_2v_3\ldots v_mv_2))$, for

$i = 1, 2, \ldots, n$ and each edge is of fold at most 3. This is equivalent to the inequality $m^2(n-1) - mn + 2m - 1 \leq 3(m-1)n$ which implies that $m^2(n-1) - 4m(n-1) - 2m + 3n - 1 \leq 0$ and so $m \leq 4 + 3/(n-1) - 3n/m(n-1) + 1/m(n-1)$, which implies that $m \leq 4 + 2 - 2/m(n-1)$. Thus, $m < 6$. This is a contradiction.

**Case 2:** Suppose that $\mathcal{B}$ consists only of cycles of length grater than or equal 4. Then $4 \mid \mathcal{B} \mid \leq 3 \mid E(S_n \rho W_m) \mid$ because the length of each cycle of $\mathcal{B}$ is grater than or equal to 4 and each edge is of fold at most 3. Thus $4(m^2(n-1) - nm + 2m - 1) \leq 3(m^2(n-1) + 2m - 2)$ which is equivalent to $m^2(n-1) - 4m(n-1) - 2m + 2 \leq 0$ and so $m \leq 4 + 2 - 2/m(n-1) < 6$. This is a contradiction.

**Case 3:** Suppose that $\mathcal{B}$ consists of $s$ 3-cycles and $t$ cycles of length grater than or equal to 4. Then $t \leq \lfloor (3(m^2(n-1) + 2m - 2) - 3s)/4 \rfloor$ because the length of each cycle of $s$ is 3 and each cycle of $t$ is at least 4, and the fold of each edge at most 3. Hence $\mid \mathcal{B} \mid = s + t \leq s + \lfloor (3(m^2(n-1) + 2m - 2) - 3s)/4 \rfloor$ which implies that $4(m^2(n-1) - mn + 2m - 1) \leq s + 3(m^2(n-1) + 2m - 2)$. Thus $4(m^2(n-1) - nm + 2m - 1) \leq 3(m-1)n + 3(m^2(n-1) + 2m - 2)$. By simplifying the inequality we have that $m^2(n-1) - 7m(n-1) - 5m + 3n + 2 \leq 0$. Hence, $m \leq 7 + 5/(n-1) - 3n/m(n-1) < 12$. This is a contradiction.
□

**Theorem 3.3.2.** For each $n \geq 2, m \geq 12$. $\mathcal{B}(S_n \rho W_m)$ is a required basis for $\mathcal{C}(S_n \rho W_m)$.

# 4 The minimum cycle bases of wreath product of graphs.

In this section, we construct minimum cycle bases for the wreath products of a star by a path, two stars and a star by a path. Also, we give their total lengths and the length of longest cycles

## 4.1 The minimum cycle basis of $S_n \rho P_m$

**Lemma 4.1.1.** Let $m$ be an odd integer. Then $\mathcal{A}^*_{ab} = \mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba}$ is linearly independent.

**Proof.** By using the proof of Lemma 3.1.4, we have that $\mathcal{K}_{ab}$ and $\mathcal{R}_{ab} \cup \mathcal{R}_{ba}$ are linearly independent sets. Clearly, any linear combination of cycles of $\mathcal{K}_{ab}$ contains an edge of the form $(a, v_j)(b, v_j)$ , $1 \leq j \leq m$ which is not in any cycle of $\mathcal{R}_{ab} \cup \mathcal{R}_{ba}$. Thus, $\mathcal{A}^*_{ab}$ is linearly independent. □

**Lemma 4.1.2.** $\mathcal{A}^* = \cup_{i=2}^{n} \mathcal{A}^*_{u_1 u_i}$ is linearly independent .

**Proof.** The proof of this lemma follows immediately from using the mathematical induction, Proposition 3.2.2 and Lemmas 4.1.1 and from noting that $E(\cup_{i=2}^{n-1}\mathcal{A}^*_{u_1u_i}) \cap E(\mathcal{A}^*_{u_1u_n}) = E(u_1 \square P_m)$. $\square$

Now, we define the following cycles:

$$\mathcal{Z}^{(5)}_{ab} = (a, v_{\lfloor \frac{m}{2} \rfloor+1})(b, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor+1})(a, v_{\lfloor \frac{m}{2} \rfloor+1}),$$

and

$$\mathcal{Z}^{(6)}_{ab} = (a, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor+1})(a, v_{\lfloor \frac{m}{2} \rfloor}).$$

**Lemma 4.1.3.** Let $m$ be an even integer. Then $\mathcal{T}^*_{ab} = \mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}^{(3)}_{ab}, \mathcal{Z}^{(4)}_{ab}, \mathcal{Z}^{(5)}_{ab}\right\} - \left\{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ab}, \mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ba}, \mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{ab}\right\}$ is linearly independent.
**Proof.** By Lemma 3.1.6, $\mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}^{(3)}_{ab}, \mathcal{Z}^{(4)}_{ab}\right\} - \left\{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ab}, \mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{ab}\right\}$ is linearly independent. Thus $\mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ba} \cup \left\{\mathcal{Z}^{(3)}_{ab}, \mathcal{Z}^{(4)}_{ab}\right\} - \left\{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ab}, \mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ba}, \mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{ab}\right\}$ is linearly independent. Now the cycle $\mathcal{Z}^{(5)}_{ab}$ contains the edge $(b, v_{\lfloor \frac{m}{2} \rfloor})(b, v_{\lfloor \frac{m}{2} \rfloor+1})$ which is not in any cycle of $\mathcal{K}_{ab} \cup \mathcal{R}_{ab} \cup \mathcal{R}_{ab} \cup \left\{\mathcal{Z}^{(3)}_{ab}, \mathcal{Z}^{(4)}_{ab}\right\} - \left\{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ab}, \mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ba}, \mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{ab}\right\}$. Thus $\mathcal{T}^*_{ab}$ is linearly independent. $\square$

**Lemma 4.1.4.** $\mathcal{T}^* = \cup_{i=2}^{n}\mathcal{T}^*_{u_1u_i}$ is linearly independent.
**Proof.** Following, word by word, the same arguments as in the proof of Lemma 4.1.2, we have the result. $\square$

**Theorem 4.1.5.** $\mathcal{B}^*(S_n \rho P_m) = \begin{cases} \mathcal{A}^*, & \text{if } m \text{ is odd,} \\ \mathcal{T}^*, & \text{if } m \text{ is even.} \end{cases}$ is a minimal cycle basis of $S_n \rho P_m$.
**Proof.** By Lemma 4.1.2 and 4.1.4, $\mathcal{B}^*(S_n \rho P_m)$ is linearly independent. Now,

$$\begin{aligned} |\mathcal{A}^*_{u_1u_i}| &= |\mathcal{A}^*_{ab}| \\ &= |\mathcal{K}_{ab}| + |\mathcal{R}_{ab}| + |\mathcal{R}_{ba}| \\ &= (m-1) + \lfloor \frac{m}{2} \rfloor + \lfloor \frac{m}{2} \rfloor \\ &= (m-1) + 2\lfloor \frac{m}{2} \rfloor \end{aligned}$$

if $m$ is odd, and

$$\begin{aligned} |\mathcal{T}^*_{u_1u_i}| &= |\mathcal{T}^*_{ab}| \\ &= |\mathcal{K}_{ab} - \{\mathcal{K}^{\lfloor \frac{m}{2} \rfloor}_{ab}\}| + |\mathcal{R}_{ab} - \{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ab}\}| + |\mathcal{R}_{ba} - \{\mathcal{R}^{\lfloor \frac{m}{2} \rfloor}_{ba}\}| \end{aligned}$$

$$
\begin{aligned}
&+|\{\mathcal{Z}_{ab}^{(3)}, \mathcal{Z}_{ab}^{(4)}, \mathcal{Z}_{ab}^{(5)}\}| \\
= \;& (m-2) + (\lfloor \frac{m}{2} \rfloor - 1) + (\lfloor \frac{m}{2} \rfloor - 1) + 3 \\
= \;& (m-1) + 2\lfloor \frac{m}{2} \rfloor
\end{aligned}
$$

if $m$ is even. Thus,

$$
\begin{aligned}
|\mathcal{A}^*| &= \sum_{i=2}^{n} |\mathcal{A}^*_{u_1 u_i}| \\
&= (n-1)((m-1) + 2\lfloor \frac{m}{2} \rfloor), \\
&= \dim \mathcal{C}(S_n \rho P_m)
\end{aligned}
$$

if $m$ is odd, and

$$
\begin{aligned}
|\mathcal{T}^*| &= \sum_{i=2}^{n} |\mathcal{T}^*_{u_1 u_i}| \\
&= (n-1)((m-1) + 2\lfloor \frac{m}{2} \rfloor) \\
&= \dim \mathcal{C}(S_n \rho P_m)
\end{aligned}
$$

if $m$ is even. Thus $\mathcal{B}^*(S_n \rho P_m)$ is a cycle basis of $\mathcal{C}(S_n \rho P_m)$. Recall that a minimal cycle basis is obtained by a greedy algorithm, that is an algorithm that selects independent cycles starting with the shortest ones from the set of all cycles. We consider two cases:

**Case 1:** $m$ is odd. Then the girth of $S_n \rho P_m$ is 4. Since each cycle of $\mathcal{B}^*(S_n \rho P_m)$ is of length 4, as a result $\mathcal{B}^*(S_n \rho P_m)$ is a minimum cycle basis.

**Case 2:** $m$ is even. Note that the only 3-cycles of $S_n \rho P_m$ are $\cup_{i=2}^{n} \Delta_{u_1 u_i}$ where $\Delta_{u_1 u_i} = \{\mathcal{Z}^{(3)}_{u_1 u_i}, \mathcal{Z}^{(4)}_{u_1 u_i}, \mathcal{Z}^{(5)}_{u_1 u_i}, \mathcal{Z}^{(6)}_{u_1 u_i}\}$ and only three cycles of the four cycles of $\Delta_{u_1 u_i}$ are linearly independent for each $i = 2, 3, \ldots n$. Thus $(\cup_{i=2}^{n} \mathcal{Z}^{(3)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(4)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(5)}_{u_1 u_i})$ is a set consisting of the largest number of 3-cycles linearly independent of $\mathcal{C}(S_n \rho P_m)$. Since $(\cup_{i=2}^{n} \mathcal{Z}^{(3)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(4)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(5)}_{u_1 u_i}) \subseteq \mathcal{B}^*(S_n \rho P_m)$ and $\mathcal{B}^*(S_n \rho P_m) - (\cup_{i=2}^{n} \mathcal{Z}^{(3)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(4)}_{u_1 u_i}) \cup (\cup_{i=2}^{n} \mathcal{Z}^{(5)}_{u_1 u_i})$ are 4-cycles, $\mathcal{B}^*(S_n \rho P_m)$ is a minimum cycle basis. □

**Corollary 4.1.6.** $l(S_n \rho P_m) = \begin{cases} 8mn - 8m - 8n + 8, & \text{if } m \text{ is odd} \\ 8mn - 8m - 7n + 7, & \text{if } m \text{ is even,} \end{cases}$ and $\lambda(S_n \ \rho P_m) = 4$.

## 4.2 The minimum cycle basis of $S_n \rho S_m$

**Lemma 4.2.1.** $\mathcal{B}^*(S_n\rho S_m) = (\mathcal{B}(S_n\rho S_m) - (\cup_{i=2}^{n-1} \mathcal{W}_{u_{i+1}u_1u_i})) \cup (\cup_{i=3}^{n} \mathcal{S}_{u_1u_i})$ is linearly independent.

**Proof.** By Theorem 3.2.5, $\mathcal{B}(S_n\rho S_m)$ is linearly independent. Thus $\mathcal{B}(S_n\rho S_m) - (\cup_{i=2}^{n-1} \mathcal{W}_{u_{i+1}u_1u_i})$ is linearly independent. Now, $\cup_{i=3}^{n} \mathcal{S}_{u_1u_i}$ is a cycle basis of the planar graph $S_n\rho v_1v_2$ which obtained by pasting all the cycles of $\cup_{i=3}^{n} \mathcal{S}_{u_1u_i}$ at the common edge of the successive cycles. Thus, $\cup_{i=3}^{n} \mathcal{S}_{u_1u_i}$ is linearly independent. Note that

$$\begin{aligned} & E(\mathcal{B}(S_n\rho S_m) - (\cup_{i=2}^{n-1} \mathcal{W}_{u_{i+1}u_1u_i})) \cap E(\cup_{i=3}^{n} \mathcal{S}_{u_1u_i}) \\ = & \{(u_1,v_1)(u_1,v_2)\} \cup \{(u_1,v_2)(u_i,v_2),(u_i,v_1)(u_i,v_2) \mid 3 \le i \le n\} \end{aligned}$$

which is an edge set of a forest. Then, by Proposition 3.2.2, $\mathcal{B}^*(S_n\rho S_m)$ is linearly independent. □

**Theorem 4.2.2.** $\mathcal{B}^*(S_n\rho S_m)$ is a minimal cycle basis of $S_n\rho S_m$.

**Proof.** By Lemma 4.2.1 and since

$$\begin{aligned} |\mathcal{B}^*(S_n\rho S_m)| &= |\mathcal{B}(S_n\rho S_m)| - \sum_{i=2}^{n-1} |\mathcal{W}_{u_{i+1}u_1u_i}| + \sum_{i=3}^{n} |\mathcal{S}_{u_1u_i}| \\ &= (nm^2 - m^2 - 2nm + 2m + n - 1) - (n-2) + (n-2) \\ &= (nm^2 - m^2 - 2nm + 2m + n - 1) \\ &= \dim \mathcal{C}(S_n\rho S_m), \end{aligned}$$

$\mathcal{B}^*(S_n\rho S_m)$ is a cycle basis for $\mathcal{C}(S_n\rho S_m)$. Since the girth of $S_n\rho S_m$ is 4, and each cycle of $\mathcal{B}^*(S_n\rho S_m)$ is of length 4, as a result $\mathcal{B}^*(S_n\rho S_m)$ is a minimum cycle basis. □

**Corollary 4.2.3.** $l(S_n\rho S_m) = 4(nm^2 - m^2 - 2nm + 2m + n - 1)$ and $\lambda(S_n\rho S_m) = 4$.

## 4.3 The minimum cycle basis of $S_n \rho W_m$

In the following result $B_{u_i\square W_m}$ denotes to the cycle basis of the wheel $u_i\square W_m$ consisting of 3-cycles.

**Lemma 4.3.1 (Jaradat).** $(\cup_{k=2}^{m} \mathcal{V}_{ab}^{(k)}) \cup (\mathcal{V}_{ba}^{(l)})$ is linearly independent for any $2 \le l \le m$.

**Lemma 4.3.2.** $\mathcal{B}^*(S_n\rho W_m) = (\cup_{i=2}^{n} \cup_{j=2}^{m} \mathcal{V}_{u_1u_i}^{(j)}) \cup (\cup_{i=2}^{n} \mathcal{V}_{u_iu_1}^{(m)}) \cup (\cup_{i=1}^{n} \mathcal{B}_{u_i\square W_m}) \cup (\cup_{i=2}^{n} \mathcal{S}_{u_1u_i})$ is linearly independent.

**Proof.** By Lemma 4.3.1, $(\cup_{j=2}^{m}\mathcal{V}_{u_1u_i}^{(j)}) \cup \mathcal{V}_{u_iu_1}^{(m)}$ is linearly independent for each $i$. Note that

$$\begin{aligned} & E((\cup_{j=2}^{m}\mathcal{V}_{u_1u_{k+1}}^{(j)}) \cup \mathcal{V}_{u_{k+1}u_1}^{(m)}) \cap E(\cup_{i=2}^{k-1}((\cup_{j=2}^{m}\mathcal{V}_{u_1u_{i+1}}^{(j)}) \cup \mathcal{V}_{u_{i+1}u_1}^{(m)})) \\ = \; & E(u_1\Box v_2v_3\dots v_m) \end{aligned}$$

which is an edge set of a path for each $1 \leq k \leq n-1$. Thus, $(\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}_{u_1u_i}^{(j)}) \cup (\cup_{i=2}^{n}\mathcal{V}_{u_iu_1}^{(m)})$ is linearly independent. Now, for each $i = 1, 2, \dots, n$, $\mathcal{B}_{u_i\Box W_m}$ is cycle basis of $a_i\Box W_m$. Since $E(\mathcal{B}_{u_i\Box W_m}) \cap E(\mathcal{B}_{u_j\Box W_m}) = \emptyset$ whenever $i \neq j$, $\cup_{i=1}^{n}\mathcal{B}_{u_i\Box W_m}$ is linearly independent. Since, $\cup_{i=1}^{n}(u_i\Box P_{m-1})$ is a forest of $\cup_{i=1}^{n}\mathcal{B}_{u_i\Box W_m}$ where $P_{m-1} = v_2v_3\dots v_m$, as a result any linear combination of $\cup_{i=1}^{n}\mathcal{B}_{u_i\Box W_m}$ must contain at least one edge of $\cup_{i=1}^{n}E(u_i\Box (W_m - P_{m-1}))$ which is not in any cycle of $(\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}_{u_1u_i}^{(j)}) \cup (\cup_{i=2}^{n}\mathcal{V}_{u_iu_1}^{(m)})$, as a result $(\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}_{u_1u_i}^{(j)}) \cup (\cup_{i=2}^{n}\mathcal{V}_{u_iu_1}^{(m)}) \cup (\cup_{i=1}^{n}\mathcal{B}_{u_i\Box W_m})$ is linearly independent. By Lemma 4.2.1, $\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}$ is linearly independent. Note that, any linear combination of cycles of $\cup_{i=2}^{n}\mathcal{S}_{u_1u_i}$ contains an edge of $E(S_n\Box v_1)$ which is not in any cycle of $(\cup_{i=2}^{n}\cup_{j=2}^{m}\mathcal{V}_{u_1u_i}^{(j)}) \cup (\cup_{i=2}^{n}\mathcal{V}_{u_iu_1}^{(m)})$. Therefore, $\mathcal{B}^*(S_n\rho W_m)$ is linearly independent. □

**Theorem 4.3.3.** $\mathcal{B}^*(S_n\rho W_m)$ is a minimal cycle basis of $S_n\rho W_m$.

**Proof.** Note that

$$|\mathcal{B}_{u_i\Box W_m}| = \dim \mathcal{C}(W_m) = m - 1,$$

and

$$|\mathcal{V}_{ab}^{(k)}| = (m-2).$$

Thus,

$$\begin{aligned} |\mathcal{B}^*(S_n\rho W_m)| &= \sum_{i=2}^{n}\sum_{j=2}^{m}|\mathcal{V}_{u_1u_i}^{(j)}| + \sum_{i=2}^{n}|\mathcal{V}_{u_iu_1}^{(m)}| + \sum_{i=1}^{n}|\mathcal{B}_{u_i\Box W_m}| \\ &\quad + \sum_{i=2}^{n}|\mathcal{S}_{u_1u_i}| \\ &= \sum_{i=2}^{n}\sum_{j=2}^{m}(m-2) + \sum_{i=2}^{n}(m-2) + \sum_{i=1}^{n}(m-1) + \sum_{i=2}^{n}1 \\ &= (n-1)(m-1)(m-2) + (n-1)(m-2) + n(m-1) \\ &\quad + (n-1) \\ &= m^2(n-1) - mn + 2m - 1 \\ &= \dim \mathcal{C}(S_n\rho W_m). \end{aligned}$$

Hence, by Lemma 4.3.2, $\mathcal{B}^*(S_n\rho W_m)$ form a cycle basis for $S_n\rho W_m$. We now show that $\mathcal{B}^*(S_n\rho W_m)$ is minimal cycle basis. Since the girth of $S_n\rho W_m$ is 3 and each cycle of $\mathcal{B}^*(S_n\rho W_m) - (\cup_{i=2}^{n} S_{u_1u_i})$ is of length three, also, since any cycle contains an edge of $S_n\Box v_1$ is of length at least 4 and since the length of any cycle of $\cup_{i=2}^{n} S_{u_1u_i}$ is 4, as a result, by the greedy algorithm, $\mathcal{B}^*(S_n\rho W_m)$ is a minimal cycle bases of $S_n\rho W_m$. □

**Corollary 4.3.4.** $l(S_n\rho W_m) = 3(m^2(n-1) - nm + 2m - n) + 4(n-1)$ and $\lambda(S_n\rho W_m) = 4$.

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