CONTRIBUTIONS TO STRENGTH SIX BALANCED ARRAYS USING HÖLDER AND MINKOWSKI INEQUALITIES

D.V. CHOPRA, RICHARD M. LOW, AND R. DIOS

ABSTRACT. In this paper, we obtain some new results, using inequalities such as Hölder and Minkowski, etc., on the existence of balanced arrays (B-arrays) with two levels and of strength six. We then discuss the use of these results to obtain the maximum number of constraints for B-arrays with given values of the parameter vector $\underline{\mu}'$. We also include some illustrative examples.

1. INTRODUCTION AND PRELIMINARIES

First of all, for the sake of completeness, we present some basic concepts and definitions concerning balanced arrays (B-arrays). The symbols $\lambda(\underline{\alpha}), P(\underline{\alpha})$, and $w(\underline{\alpha})$ denote, respectively, the frequency of the column vector $\underline{\alpha}$, the column vector obtained by permuting the elements of $\underline{\alpha}$, and the weight of the column vector $\underline{\alpha}$ (the weight of a vector $\underline{\alpha}$ is the number of non-zero elements in it).

Definition. An array T of size $(m \times N)$ with two levels or elements (say, 0 and 1) is called a *B-array of strength* t $(\leq m)$ if in every $(t \times N)$ submatrix T^* of T (the number of such submatrices is $\binom{m}{t}$), the following condition is satisfied: $\lambda(\underline{\alpha}) = \lambda(P(\underline{\alpha})) = \mu_i$ (say).

Remarks: The vector $\underline{\mu}' = (\mu_0, \mu_1, \ldots, \mu_t)$ is called the *index set* of the B-array T and N is known once we are given $\underline{\mu}'$, ie. $N = \sum_{i=0}^{t} \binom{t}{i}\mu_i$. For $t = 6$, we have $N = \sum_{i=0}^{6} \binom{6}{i}\mu_i$.

Definition. A B-array T is reduced to an *orthogonal array* if $\mu_i = \mu$, for each i.

In the case of an orthogonal array, $N = 2^t\mu$, which for $t = 6$, is reduced to $N = 2^6\mu$. Thus, O-arrays form a subset of B-arrays.

Date: July 17, 2009. v.0.1b.

2000 *Mathematics Subject Classification.* 05B15.

Key words and phrases. balanced arrays, strength six, Hölder and Minkowski inequalities.

O-arrays, a special case of B-arrays, have been extensively used in cryptography, computer technology, information theory, coding theory, and in the famous Taguchi techniques relating to quality control in industry. R.C. Bose [2] applied O-arrays to point out the connections between the problems of information theory and experimental designs. Factorial designs form a very important and integral part of statistical design of experiments, and these designs have found great use in almost all areas of scientific investigations such as medicine, industry, technology, agriculture, social sciences, etc. These combinatorial arrays have been greatly used to construct factorial designs. O-arrays were first introduced into statistics, under the name of *hypercubes*, by C.R. Rao [14]. In order that factorial designs be made available to the researcher for all N, the concept of B-arrays (under the name of *partially balanced arrays*) was introduced into statistics by I.M. Chakravarti [3] on the suggestion of C.R. Rao. In a way, B-arrays are a generalization of O-arrays. B-arrays are also related to other combinatorial structures. For example, balanced incomplete block designs (BIBDS) are related, in some fashion, to B-arrays of strength two. Houghton, et. al. [9] used this connection to show the non-existence of the famous BIB design $(46, 6, 1)$. Sinha, et. al. [19] have pointed out the relationship of strength two B-arrays with rectangular designs, group divisible designs, and nested balanced incomplete block designs. These combinatorial arrays with different strengths are appropriate for different experimental objectives. For example, if the objective of the researcher is to estimate all the effects up to and including three-factor interactions (under the assumption that all higher-order interactions are negligible), then we need to construct a B-array (or an O-array) with strength six. To gain further insight into the importance and usefulness of B-arrays to combinatorics and experimental designs, the interested reader may consult the list of references (which is not exhaustive, by any means) at the end of this paper, and also additional references mentioned therein.

The problem of the existence of B-arrays for a given set of arbitrary parameters $\underline{\mu}'$ and m ($\geq t+1 = 7$) is clearly a very complex and challenging problem. Another problem of great interest, addressed within the literature by numerous researchers, is to obtain the maximum number of constraints m, for a given $\underline{\mu}'$. Such problems for O-arrays have been addressed, among others, by Bose and Bush [1], C.R. Rao [14, 15], Seiden and Zemach [18], and for B-arrays by Chopra/Bsharat, Dios and Low [4, 5, 6, 7], Rafter and Seiden [13], Saha et. al. [17], Yamamoto et. al. [21], etc.

In this paper, we obtain some inequalities (using the Hölder and Minkowski inequalities) involving the parameters m and $\underline{\mu}'$ of a B-array with strength six. In order for such a B-array to exist, it is necessary that each of these inequalities must be satisfied. We also discuss the use of these results to obtain, for a given $\underline{\mu}'$, the maximum value of m. In addition, some illustrative examples using some specific values of $\underline{\mu}'$ are provided.

2. Main Results with Discussion

The following results are easy to establish.

Lemma 1. *A B-array T with $t = 6$, $m = 6$, and arbitrary index set $\underline{\mu}'$ always exists.*

Lemma 2. *If an array T with $t = 6$ does not exist for some $m = k$ (say, $k \geq 7$), then it does not exist for any $m \geq k + 1$.*

Lemma 3. *A B-array T of strength six (with index set $\underline{\mu}'$) is also of strength k, where $0 \leq k \leq 6$.*

Remark: It is not difficult to see that, considered as an array of strength k, its elements are linear combinations of the elements $\mu_0, \mu_1, \mu_2, \ldots, \mu_6$. Let $A(j,k)$ be the jth element $(0 \leq j \leq k)$ of the parameter vector of T when considered as an array of strength k, where $A(j,k)$ in terms of the μ_is is given by

$$A(j,k) = \sum_{i=0}^{6-k} \binom{6-k}{i} \mu_{i+j}, \quad \text{where } j = 0, 1, \ldots, k, \text{ and } k \leq 6. \tag{2.1}$$

From (2.1), it is obvious that $A(t,t) = A(6,6) = \mu_6$, $A(j,6) = \mu_j$, and $A(j,0) = N = A(0,0)$.

The next lemma expresses the moments of the weights of the columns of T about 0 as a polynomial function in terms of its parameters m, $\mu_0, \mu_1, \ldots, \mu_6$.

Lemma 4. *Consider a strength six B-array T with parameters m and $\underline{\mu}'$. Let x_j $(j = 0, 1, \ldots, m)$ denote the frequency of the columns of weight j in T. Then, the following must hold:*

$$L_0 = \sum_{j=0}^{m} x_j = N, \tag{2.2}$$

$$L_k = \sum_{j=0}^{m} j^k x_j = \sum_{r=1}^{k} a_r m_r A(r,r), \quad (1 \leq k \leq 6),$$

where $m_r = m(m-1)(m-2)\cdots(m-r+1)$, and a_r are known integers which appear while deriving $\sum jx_j$, $\sum j^2 x_j, \ldots, \sum j^k x_j$.

Remark: For computational ease and convenience, we provide next the values of a_r for $k = 1, 2, \ldots, 6$: (1), (1,1), (1,3,1), (1,7,6,1), (1,15,25,20,1), and (1,31,90,65,15,1). For example, the last set of six integers refers to

$k = 6$ with values of $(a_1, a_2, \ldots, a_6)$, etc.

Next, we state (for future use), the inequalities due to Minkowski and Hölder.

Minkowski's Inequality: For $x_i, y_i \geq 0$ and $p > 1$, we have

$$\left[\sum_{i=1}^{n}(x_i + y_i)^p\right]^{\frac{1}{p}} \leq \left[\sum_{i=1}^{n} x_i^p\right]^{\frac{1}{p}} + \left[\sum_{i=1}^{n} y_i^p\right]^{\frac{1}{p}}. \tag{2.3}$$

Hölder's Inequality: For $x_i, y_i \geq 0$, $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, we have

$$\sum_{i=1}^{n} x_i^{\frac{1}{p}} y_i^{\frac{1}{q}} \leq \left[\sum_{i=1}^{n} x_i\right]^{\frac{1}{p}} \left[\sum_{i=1}^{n} y_i\right]^{\frac{1}{q}}. \tag{2.4}$$

Remark: In what follows, we use the symbols a and b to denote $\overline{j}$ and $\overline{j^2}$ respectively (ie. $a = \overline{j} = \frac{\sum j x_i}{N} = \frac{L_1}{N}$, and $b = \overline{j^2} = \frac{\sum j^2 x_i}{N} = \frac{L_2}{N}$).

Theorem 1. *For a B-array T of strength six with parameters m and $\underline{\mu}'$ to exist, the following conditions must be satisfied:*

$$N^5 L_6 + 15N^3 L_4 L_1^2 + 15N L_2 L_1^4 \geq 6N^4 L_1 L_5 + 20N^2 L_3 L_1^3 + 5L_1^6. \tag{2.5}$$

$$N^3 L_4 + 6N L_2 L_1^2 \geq 4N^2 L_1 L_3 + 3L_1^4. \tag{2.6}$$

$$N L_2 \geq L_1^2. \tag{2.7}$$

Proof. (Outline). Clearly, $\sum (j - a)^k \geq 0$, where $a = \frac{L_1}{N}$ and k is an even integer satisfying $0 \leq k \leq 6$. To obtain (2.5), (2.6) and (2.7), we pick $k = 6, 4$, and 2, respectively. Expanding the left-hand side of the inequality and simplifying yield the desired results. □

Theorem 2. *Let T be a balanced array with parameters m, $t = 6$, and $\underline{\mu}'$. Then for T to exist, the following must hold:*

$$L_4 + L_5 \leq \sqrt[3]{L_3 L_6}\left(\sqrt[3]{L_3} + \sqrt[3]{L_6}\right). \tag{2.8}$$

$$L_2 + L_4 \leq \sqrt[3]{L_0 L_6}\left(\sqrt[3]{L_0} + \sqrt[3]{L_6}\right). \tag{2.9}$$

$$L_3 + L_4 \leq \sqrt[3]{L_2 L_5}\left(\sqrt[3]{L_2} + \sqrt[3]{L_5}\right). \tag{2.10}$$

$$L_2 + 2L_3 + 2L_4 + L_5 \leq \left(\sqrt[5]{L_1}\right)^4 \sqrt[5]{L_6} + 2\left(\sqrt[5]{L_1}\right)^3\left(\sqrt[5]{L_6}\right)^2 + 2\left(\sqrt[5]{L_1}\right)^2\left(\sqrt[5]{L_6}\right)^3 + \sqrt[5]{L_1}\left(\sqrt[5]{L_6}\right)^4. \tag{2.11}$$

Proof. (Outline). Here, we use Minkowski's Inequality with $p = 3$ for the first three results, and $p = 5$ for the last one. We also replace x_i by $jx_j^{\frac{1}{3}}$ and y_i by $j^2 x_j^{\frac{1}{3}}$ for (2.8); x_i by $x_j^{\frac{1}{3}}$ and y_i by $j^2 x_j^{\frac{1}{3}}$ for (2.9); and x_i by $j^{\frac{2}{3}} x_j^{\frac{1}{3}}$ and y_i by $j^{\frac{5}{3}} x_j^{\frac{1}{3}}$ for (2.10). After some simplification, we obtain the first three results. For the last one (2.11), we set $p = 5$ and replace x_i by $j^{\frac{1}{5}} x_j^{\frac{1}{5}}$ and y_i by $j^{\frac{6}{5}} x_j^{\frac{1}{5}}$ and simplify. For example, for $p = 3$, we have

$$\left[\sum (x_i^3 + 3x_i^2 y_i + 3x_i y_i^2 + y_i^3)\right]^{\frac{1}{3}} \leq \left(\sum x_i^3\right)^{\frac{1}{3}} + \left(\sum y_i^3\right)^{\frac{1}{3}}.$$

Raising both sides to the power 3, we have

$$\sum (x_i^2 y_i + x_i y_i^2) \leq \left(\sum x_i^3\right)^{\frac{2}{3}} \left(\sum y_i^3\right)^{\frac{1}{3}} + \left(\sum x_i^3\right)^{\frac{1}{3}} \left(\sum y_i^3\right)^{\frac{2}{3}}.$$

The results follow by replacing x_i and y_i by the above symbols. □

Theorem 3. *Consider a B-array T with parameters m, $t = 6$, and $\underline{\mu}'$. For T to exist, the following conditions must hold:*

$$L_5^6 \leq L_0 L_6^5. \tag{2.12}$$

$$L_3^3 \leq L_1 L_4^2. \tag{2.13}$$

$$L_3^4 \leq L_2^3 L_6. \tag{2.14}$$

$$L_4^5 \leq L_0 L_5^4. \tag{2.15}$$

$$L_4^3 \leq L_2 L_5^2. \tag{2.16}$$

$$L_5^4 \leq L_2 L_6^3. \tag{2.17}$$

$$L_4^4 \leq L_1 L_5^3. \tag{2.18}$$

$$L_2^4 \leq L_1^3 L_5. \tag{2.19}$$

Proof. (Outline). Here, we use Hölder's Inequality (2.4). We briefly describe the derivation of (2.12). In (2.4), take $p = 6$, $q = \frac{6}{5}$. This yields

$$\left[\sum \sqrt[6]{x_i y_i^5}\right]^6 \leq \left(\sum x_i\right)\left(\sum y_i\right)^5.$$

Replacing x_i by x_j and y_i by $j^5 x_j$ implies $L_5^6 \leq L_0 L_6^5$, which is (2.12). To obtain (2.13), take $p = 3$ and replace x_i by jx_j and y_i by $j^4 x_j$. To obtain (2.16), take $p = 3$ and replace x_i by $j^2 x_j$ and y_i by $j^5 x_j$. We select $p = 4$ to derive (2.14), (2.17), (2.18), and (2.19). In these cases, the (x_i, y_i) substitutions are $(j^6 x_j, j^2 x_j)$, $(j^2 x_j, j^6 x_j)$, $(jx_j, j^5 x_j)$, $(j^5 x_j, jx_j)$, respectively. Finally, to derive (2.15), choose $p = 5$ and $(x_i, y_i) = (x_j, j^5 x_j)$. □

3. Discussions and Illustrations

A computer program was prepared and a great variety of B-arrays with given $\underline{\mu}'$ and values of m (starting with $m = 6$) were run on the inequalities presented here. If the first contradiction of an inequality occurred at $m = m^* + 1$ (say), then $\max(m) = m^*$ was noted. Thus, the least of all such m^* was the number of constraints for which the B-array T with the given $\underline{\mu}'$ could possibly exist.

It was observed that there was no one single inequality which was the best for each array tested. An inequality could be best for one $\underline{\mu}'$ but did not do well for another $\underline{\mu}'$. For example, taking $\underline{\mu}' = (9, 8, 8, 8, 6, 7, 8)$, we obtain (using (2.5)) $m \leq 12$ which is the best, while all other m values are at least 500. Now, taking $\underline{\mu}' = (8, 7, 7, 5, 6, 6, 8)$, we found that $m \leq 36$ using (2.6) while the other inequalities gave us m at least 500. Here in this case, (2.6) was the best inequality. Finally, we observe that (2.10) and (2.16) are the best ones, each giving us $m \leq 10$, for $\underline{\mu}' = (1, 2, 1, 1, 4, 3, 2)$ while (2.5) gives us $m > 500$ and (2.6) gives us $m \leq 19$. Next, we compare these results with some of the earlier known results within the literature. From Chopra and Bsharat [4], we have $\max(m) = 11$ for the array $(1, 1, 2, 1, 4, 1, 1)$, using (2.4); while here we obtain $\max(m) = 9$, using (2.10) and (2.16) which is an improvement. From Dios and Chopra [7], we have $\max(m) = 21$ for the array $(4, 4, 3, 2, 3, 4, 4)$, using (2.2) given in [7]; while here we obtain $\max(m) = 18$, using (2.6) of this paper which is an improvement of the earlier result. Thus, the results presented here show improvements in some cases over the earlier published results.

References

[1] R.C. Bose and K.A. Bush, Orthogonal arrays of strength two and three, *Ann. Math. Statist.* 23 (1952), 508-524.

[2] R.C. Bose, On some connections between the design of experiments and information theory, *Bull. Internat. Statist. Inst.* 38 (1961), 257-271.

[3] I.M. Chakravarti, Fractional replication in asymmetrical factorial designs and partially balanced arrays, *Sankhya* 17 (1956), 143-164.

[4] D.V. Chopra and M. Bsharat, Some results on bi-level balanced arrays, *Congr. Numer.* 181 (2006), 89-95.

[5] D.V. Chopra, R.M. Low and R. Dios, On the maximum number of constraints for some balanced arrays, *Congr. Numer.* 190 (2008), 5-11.

[6] D.V. Chopra, R.M. Low and R. Dios, Contributions to orthogonal arrays of strength six, *Congr. Numer.* 186 (2007), 193-198.

[7] R. Dios and D.V. Chopra, Further investigations on balanced arrays of strength six, *J. Combin. Math. Combin. Comput.* 48 (2004), 89-94.

[8] A.S. Hedayat, N.J.A. Sloane and J. Stufken, *Orthogonal Arrays: Theory and Applications* (1999), Springer-Verlag (N.Y.).

[9] S.K. Houghton, I. Thiel, J. Jansen and C.W. Lam, There is no $(46, 6, 1)$ block design, *J. Combin. Designs* 9 (2001), 60-71.

[10] J.P.C Kleijnen and O. Pala, Maximizing the simulation output: a competition, *Simulation.* 7 (1999), 168-173.

[11] J.Q. Longyear, Arrays of strength t on two symbols, *J. Statist. Plann. Inf.* 10 (1984), 227-239.

[12] D.S. Mitrinovic, *Analytic Inequalities* (1970), Springer-Verlag (N.Y.).

[13] J.A. Rafter and E. Seidon, Contributions to the theory and construction of balanced arrays, *Ann. Statist.* 2 (1974), 1256-1273.

[14] C.R. Rao, Hypercubes of strength t leading to confounded designs in factorial experiments, *Bull. Calcutta Math. Soc.* 38 (1946), 67-78.

[15] C.R. Rao, Factorial experiments derivable from combinatorial arrangements of arrays, *J. Roy. Statist. Soc. Suppl.* 9 (1947), 128-139.

[16] C.R. Rao, Some combinatorial problems of arrays and applications to design of experiments, *A Survey of Combinatorial Theory* (edited by J.N. Srivastava, et. al.) (1973), North-Holland Publishing Co., 349-359.

[17] G.M. Saha, R. Mukerjee and S. Kageyama, Bounds on the number of constraints for balanced arrays of strength t, *J. Statist. Plann. Inf.* 18 (1988), 255-265.

[18] E. Seiden and R. Zemach, On orthogonal arrays, *Ann. Math. Statist.* 27 (1966), 1355-1370.

[19] K. Sinha, G.M. Saha and S. Kageyama, Balanced arrays of strength two from block designs, *Combin. Designs.* 10 (2002), 303-312.

[20] W.D. Wallis, *Combinatorial Designs*, Second Edition. (2007), Marcel Dekker Inc., New York.

[21] S. Yamamoto, M. Kuwada and R. Yuan, On the maximum number of constraints for s-symbol balanced arrays of strength t, *Commun. Statist. Theory Meth.* 14 (1985), 2447-2456.

Department of Mathematics and Statistics, Wichita State University, Wichita, KS 67260-0033, USA.
E-mail address: dharam.chopra@wichita.edu

Department of Mathematics, San Jose State University, San Jose, CA 95192, USA.
E-mail address: low@math.sjsu.edu

Department of Mathematical Sciences, New Jersey Institute of Technology, Newark, NJ 07102-1982, USA.
E-mail address: dios@adm.njit.edu