A Uniquely 3-List Colorable, Planar and K_4-Free Graph

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Abstract

Let G be a graph with v vertices. If there exists a collection of list of colors $S_1, S_1, \ldots, S_v$ on its vertices, each of size k, such that there exists a unique proper coloring for G from this list of colors, then G is called a *uniquely k-list colorable graph.* In this note we present a uniquely 3-list colorable, planar and K_4-free Graph. It is a counterexample for a conjecture by Ch. Eslahchi, M. Ghebleh and H. Hajiabolhassan [3].

1 Introduction

We consider simple graphs which are finite, and have no loops or multiple edges. For the definition of basic concepts not given here, one may refer to a text book in graph theory; for example [5].

Let G be a graph with the vertex set $\{1, 2, \ldots, v\}$, and let $S_1, S_2, \ldots, S_v$ be a collection of list of colors on its vertices. If there exists a proper coloring c for G such that $c(v) \in S_v$ for all $v \in V(G)$, then G is called to have a *list coloring.* By a proper coloring we mean for adjacent vertices u and v, we have $c(u) \neq c(v)$. For a survey on list coloring we refer reader to [1]. A graph G is called *uniquely k-list colorable,* or UkLC for short, if there exists a collection of list of colors $S_1, S_1, \ldots, S_k$, on its vertices, each of size k such that there exists a unique proper coloring for G from this collection of list of colors. This concept was introduced by Mahdian and Mahmoodian [4] and independently by Dinitz and Martin [2]. Mahdian and Mahmoodian [4] characterized uniquely 2-list colorable graphs as follows.

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Theorem A [4] *A graph G is not* U2LC *if and only if each of its blocks is either a cycle, a complete graph, or a complete bipartite graph.*

Ch. Eslahchi, M. Ghebleh and H. Hajiabolhassan [3] conjectured that every uniquely 3-list colorable planar graph has K_4 as a subgraph. We present a graph which is uniquely 3-list colorable, planar and K_4-free. So we prove that mentioned conjecture is not true in general. We construct this graph in three levels. The basic construction is due to level one and the other constructions are derived from the level one construction.

2 A counterexample

First we construct a graph G_1 with 17 vertices; then we add the two other graphs with 15 and 12 vertices to G to obtain the desired graph. We label vertices of G_1 by $v_1, v_2, \ldots, v_{17}$. Let the list of colors for each vertex v be $L(v) = \{i, j, k\}$ such that $i, j, k \in \{1, 2, 3, 4\}$ and $i \neq j \neq k$. Now, we construct G_1 as follows:

$$V(G_1) = \{v_1, v_2, \ldots, v_{17}\},$$

and the list of colors:
$L(v_1) = \{1,2,3\}, L(v_2) = \{2,3,4\}, L(v_3) = \{1,3,4\}, L(v_4) = \{2,3,4\},$
$L(v_5) = \{1,2,4\}, L(v_6) = \{1,2,3\}, L(v_7) = \{1,2,4\}, L(v_8) = \{1,2,4\},$
$L(v_9) = \{1,2,3\}, L(v_{10}) = \{1,2,3\}, L(v_{11}) = \{1,3,4\}, L(v_{12}) = \{1,2,3\},$
$L(v_{13}) = \{2,3,4\}, L(v_{14}) = \{2,3,4\}, L(v_{15}) = \{1,2,3\}, L(v_{16}) = \{1,2,3\},$
$L(v_{17}) = \{1,3,4\}.$
For more information about edges of G_1 and construction of G_1, see diagram of G_1 in Figure 1.

Lemma 1 *The graph G_1 is planar and K_4-free.*

Proof. We can easily infer from the Figure 1, the graph G_1 is a planar and K_4-free graph. ■

Definition 1 *A* **partial list coloring** *of G is an assignment of colors to some vertices of G from its list such that no two adjacent vertices have the same color.*

Note that in a partial list coloring of G some vertices may not be necessarily colored.

Definition 2 *Let G be a graph and $V(G)$ be a partial ordered set with a binary relation $\preceq$. A partial list coloring of G is a* **strict partial list**

coloring *with respect to* $\preceq$ *if for each colored* $v \in V(G)$*, all* $u \preceq v$ *are colored too.*

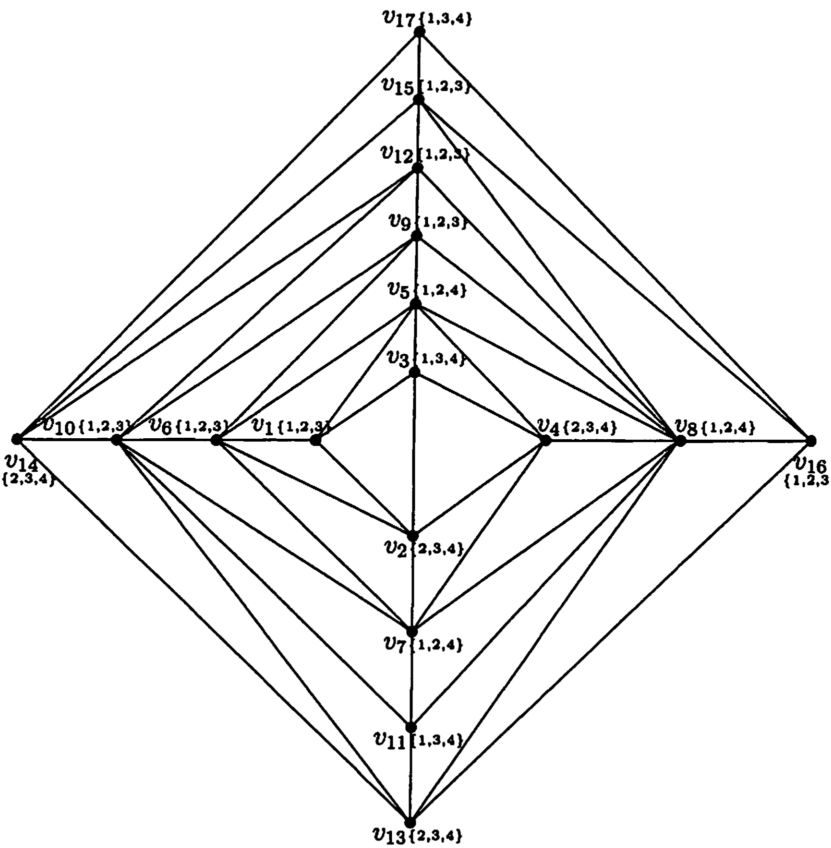

Figure 1: G_1 is planar and K_4-free

We define a natural partial order relation, $\preceq$, on $V(G_1)$ i.e. we say the $v_i \preceq v_j$ if and only if, $i \leq j$. All maximal partial strict list colorings of G_1 with respect to $\preceq$ are appeared in Appendix and we show each of them by its label C_i, where $1 \leq i \leq 41$. These maximal strict partial list colorings play an important role in our discussion.

Lemma 2 *The graph* G_1 *can be colored by its list of colors in two different ways only.*

Proof. We see in the table in Appendix that there are only two ways for coloring G_1 by its list of colors and they are C_{17} and C_{19}. ■

In the second step, we add two other graphs to G_1 in such a way that the resulting graph has the desired properties. Let $G_2 = G_1[\{v_1, v_2, \ldots, v_{15}\}]$ and $G_3 = G_1[\{v_1, v_2, \ldots, v_{12}\}]$ be the two induced subgraphs of G_1. We denote the vertices of G_2 and G_3 properly by $v_1', v_2', \ldots, v_{15}'$ and $v_1'', v_2'', \ldots, v_{12}''$, respectively. Obviously G_2 and G_3 are both palnar and K_4-free. Figure 1 shows that $v_7v_{10}v_{11}$ is a face of G_1, and we know that G_2 is induced subgraph of G_1; so $v_7'v_{10}'v_{11}'$ is a face of G_2. Therefore we can embed G_2 in face $v_7v_{10}v_{11}$ of G_1, such that $v_7'v_{10}'v_{11}'$ be the outer face of G_2. Now, we join G_1 to G_2 by the edges $\{v_{10}, v_{10}'\}$ and $\{v_{11}, v_{11}'\}$. Similarly, we embed G_3 in the face $v_2'v_3'v_4'$ of G_2, such that $v_2''v_3''v_4''$ be the outer face and we join G_2 to G_3 by the edges $\{v_2', v_2''\}$ and $\{v_3', v_3''\}$. The resulting graph H has $17 + 15 + 12 = 44$ vertices.

Theorem 1 *The graph H is planar, K_4-free and uniquely 3-list colorable.*

Proof. From construction of H, we note that H is planar and K_4-free; so it remains to prove that H is uniquely 3-list colorable. It can be easily seen that every coloring appeared in Appendix is also a coloring for G_2 and G_3 if it is restricted to the vertices of G_2 and G_3 respectively. Suppose the list of colors for vertices of H be those induced by G_1. By Theorem 1 we see that the only two possible ways for coloring G_1, are C_{17} and C_{19}. Since the color of v_{10} in C_{17} and C_{19} is 1, and there is an edge between v_{10} and v_{10}', for coloring the vertices of G_2 in H we must choose a coloring of $v_1', v_2', \ldots, v_{15}'$, such that color of v_{10}' is not 1. Therefore, by the table in Appendix, we find that C_{16} and C_{28} are the only two ways for coloring G_2. For coloring G_3 in H, we must choose a coloring which colors $v_1'', v_2'', \ldots, v_{12}''$ such that color of v_3'' is not 1 because there is an edge between v_3' and v_3'' and also color of v_3' in C_{16} and C_{28} is 1. Therefore, by the table in Appendix, we find that C_2 is the only way for coloring G_3. So, G_3 must be colored by C_2. Since the color of v_2'' in C_2 is 2 and $\{v_2', v_2''\}$ is an edge in H and also color of v_2' in C_{28} is 2 and in C_{16} is 4, we infer that G_2 must be colored with C_{16}. Now, we know that G_2 is colored with C_{16}. For coloring G_1, we see that v_{11} and v_{11}' are adjacent, so color of v_{11}' must differ from color of v_{11}. Color of v_{11}' is 4 in C_{16}, while color of v_{11} in C_{19} is 4 and color of v_{11} in C_{17} is 3; so we find that G_1 must be colored with C_{17}. All of these statements show that we must color G_1, G_2 and G_3 with C_{17}, C_{16} and C_2 respectively. So we proved that there exists a unique way for coloring H by its list of colors. ■

3 Appendix

Here we list all possible maximal strict partial list colorings of G_1 with respect to $\preceq$, that is introduced earlier. We list all vertices in a row and write color of each vertex under it. Correctness of this table can be checked by hand.

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}	v_{11}	v_{12}	v_{13}	v_{14}	v_{15}	v_{16}	v_{17}
C_1	1	2	3	4	2	3	1										
C_2	1	2	4	3	2	3	1	4	1	2	3	3					
C_3	1	2	4	3	2	3	4	1									
C_4	1	3	4	2													
C_5	1	4	3	2	4	2	1										
C_6	1	4	3	2	4	3	1										
C_7	2	3	1	2	4	1	4	1	2	3							
C_8	2	3	1	2	4	1	4	1	3	2	3						
C_9	2	3	1	4													
C_{10}	2	3	4	2	1												
C_{11}	2	4	1	2	4	1											
C_{12}	2	4	1	2	4	3	1										
C_{13}	2	4	1	3	4	1	2	1	2	3	4						
C_{14}	2	4	1	3	4	1	2	1	3								
C_{15}	2	4	1	3	4	3	1	2	1	2	3	3	4				
C_{16}	2	4	1	3	4	3	1	2	1	2	4	3	3	4	1		
C_{17}	2	4	1	3	4	3	2	1	2	1	3	3	2	4	2	3	1
C_{18}	2	4	1	3	4	3	2	1	2	1	3	3	4	2			
C_{19}	2	4	1	3	4	3	2	1	2	1	4	3	2	4	2	3	1
C_{20}	2	4	1	3	4	3	2	1	2	1	4	3	3	2			
C_{21}	2	4	1	3	4	3	2	1	2	1	4	3	3	4	2		
C_{22}	2	4	3	2	1	3	1	4	2								
C_{23}	2	4	3	2	4	1											
C_{24}	2	4	3	2	4	3	1										
C_{25}	3	2	1	3	2	1	4	1	3	2	3						
C_{26}	3	2	1	3	4	1	4	1	2	3							
C_{27}	3	2	1	3	4	1	4	1	3	2	3						
C_{28}	3	2	1	3	4	1	4	2	3	2	1	1	3	4	3	1	
C_{29}	3	2	1	3	4	1	4	2	3	2	1	1	4	3			
C_{30}	3	2	1	3	4	1	4	2	3	2	3	1	4	3			
C_{31}	3	2	1	4	2	1											
C_{32}	3	2	4	3	1												
C_{33}	3	2	4	3	2	1	4	1	3	2	3						
C_{34}	3	4	1	2	4	1											
C_{35}	3	4	1	2	4	2	1										
C_{36}	3	4	1	3	2	1	2	1	3								
C_{37}	3	4	1	3	2	1	2	4	3								
C_{38}	3	4	1	3	4	1	2	1	2	3	4						
C_{39}	3	4	1	3	4	1	2	1	3								
C_{40}	3	4	1	3	4	2	1	2	1	3	4						
C_{41}	3	4	1	3	4	2	1	2	3								

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References

[1] N. ALON, *Restricted colorings of graphs*, in "Surveys in Combinatorics", Proc. 14th British Combinatorial Conference, London Mathematical Society Lecture Notes Series 187, 1993, pp. 1–33.

[2] J. H. DINITZ AND W. J. MARTIN, *The stipulation polynomial of a uniquely list-colorable graph*, Austral. J. Combin., **11** (1995), pp. 105–115.

[3] C. ESLAHCHI, M. GHEBLEH, AND H. HAJIABOLHASSAN, *Some concepts in list coloring*, Journal of Combinatorial Mathematics and Combinatorial Computing, **41** (2002), pp. 151–160.

[4] M. MAHDIAN AND E. S. MAHMOODIAN, *A characterization of uniquely 2-list colorable graphs*, Ars Combin., **51** (1999), pp. 295–305.

[5] D. B. WEST, *Introduction to Graph Theory*, Prentice-Hall, Inc, USA, 2001 2nd ed.