Dynamics of the P_3 intersection graph

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Abstract

The P_3 intersection graph $P_3(G)$ of a graph G is the intersection graph of all induced 3-paths in G. In this paper, we prove that any P_3-convergent graph is $P_3^n(G)$-complete for some $n \geqslant 1$. Also we prove that there are no P_3-fixed graphs. The touching number, periodicity and connectivity of $P_3(G)$ are also studied.

1 Introduction

Graph dynamics deals with the study of convergence, divergence, fixedness, periodicity etc of graph operators. Ore [4] and Harary [2] initiated this study.

The H-intersection graph $Int_H(G)$ is the intersection graph of all subgraphs of G that are isomorphic to H [5]. In [1], Akiyama and Chvátal have characterized the graphs for which $IntP_3(G)$ is perfect. We have defined and discussed some properties of $P_3(G)$ in [3]. The P_3 intersection graph of G, $P_3(G)$ has the induced paths on three vertices in G as its vertices and two distinct vertices in $P_3(G)$ are adjacent if the corresponding induced 3-paths in G intersect. We have characterized the graphs G such that $P_3(G)$ is bipartite and obtained forbidden subgraph characterizations for $P_3(G)$ being chordal, complete and H-free for any finite graph H. Graph parameters such as chromatic number, domination number, independence number,

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diameter and radius of $P_3(G)$ were also studied.

For a detailed discussion on other graph operators, the reader may refer to [5].

We shall now list some graph dynamical terminology from [5] as applied to $P_3(G)$. For any graph G, the k^{th} iterated graph is iteratively defined as $P_3{}^2(G) = P_3(P_3(G))$ and $P_3{}^k(G) = P_3(P_3{}^{k-1}(G))$ for $k > 2$. We say that G is P_3-convergent if $\{P_3{}^k(G),\ k \in N\}$ is finite. Otherwise G is P_3-divergent. A graph G is P_3-periodic if there is some natural number k with $G = P_3{}^k(G)$. The smallest such number k is called the period of G. A graph is P_3-fixed if $k = 1$. A graph G is P_3-convergent if and only if G is P_3-periodic, or there is some integer $k \geqslant 1$ such that $P_3{}^k(G)$ is P_3-periodic. A graph G is P_3-mortal if for some $k \in N$, $P_3{}^k(G) = \phi$, the empty graph. A graph G is $P_3{}^n$-complete if $P_3{}^n(G)$ is a complete graph. $P_3(G)$ is the empty graph for any graph G which is the union of complete graphs. Hence, we consider here only non complete graphs.

The touching number of a cycle is the cardinality of the set of all edges having exactly one of its vertices on the cycle. For every integer $k \geqslant 3$, the k-touching number $t_k(G)$ of a graph G is the supremum of all touching numbers of C_k, provided G contains some C_k. If G does not contain any C_k then $t_k(G)$ is undefined. The vertex touching number of an induced C_k is the cardinality of the set of all vertices which are adjacent to exactly one vertex of it. The vertex touching number of a graph $vt_k(G)$ is the supremum of all vertex touching numbers of induced C_k provided G contains some induced C_k.

All the graphs considered here are finite, undirected and simple. We denote by P_k (respectively C_k), a path (respectively cycle) on k vertices. The graph obtained by deleting any edge of K_n is denoted by $K_n - \{e\}$. A k-clique is a clique of size k.

For a graph G, a subset V' of $V(G)$ is a k-vertex cut of G if the number of components in $G - V'$ is greater than that of G and $|V'| = k$. A vertex v of G is a cut vertex of G if $\{v\}$ is a vertex cut of G. If G has no cut vertices, then G is a block. A graph is a geodetic graph if for every pair of vertices there is a unique path of minimum length between them. It is known [6] that a graph is geodetic if and only if each of its block is geodetic.

Whitney's theorem [8] is as follows. Let G be a simple graph with at least three vertices. Then G is 2-connected if and only if for each pair of distinct vertices u and v of G there are two internally disjoint $u - v$ paths in G. It can be generalized as follows. A simple graph G is n-connected if and only if given any pair of distinct vertices u and v of G, there are at least n internally disjoint $u - v$ paths in G.

For all other basic concepts and notations not mentioned here, we refer [7].

In this paper we prove that all P_3-convergent graphs converges to complete graphs and hence all such graphs are P_3-mortal. We prove that there does not exist any P_3-periodic and hence no P_3-fixed graphs. The touching number and connectivity of $P_3(G)$ are also discussed. We also prove that the only geodetic P_3 graph is the complete graph.

2 Dynamical properties

If $a_1 - a_2 - a_3$ is an induced 3-path in G then the corresponding vertex in $P_3(G)$ is denoted by $a_1a_2a_3$.

Theorem 1 (3). *Let G be a connected graph, then $\omega(P_3(G)) \geqslant \omega(G) - 1$.*

Lemma 2. *If $P_3(G)$ is not complete, then $\omega(P_3(G)) \geqslant \omega(G)$. Equality holds if and only if G is one of the following.*
1. G is a complete graph with two pendant vertices attached to any two of its distinct vertices.
2. The graph G is as in figure 2(a) or 2(b).

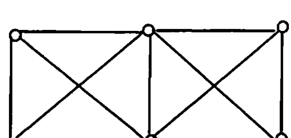

Figure : 2(a)

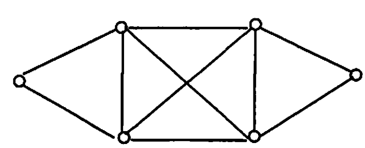

Figure : 2(b)

Proof. Let $\omega(G) = k$. So consider a $K_k \subseteq G$. Let the vertices of the K_k be $\{u_1, u_2, ..., u_k\}$. Since G is not complete, $K_k \subset G$.
Let $v \in V(G) - V(K_k)$ be a vertex which is adjacent to t vertices $u_1, u_2, ..., u_t$

of K_k. Then the vertices of the form vu_iu_j, $1 \leqslant i \leqslant t$, $t+1 \leqslant j \leqslant k$ form a $K_{t(k-t)}$ in $P_3(G)$, where $t(k-t) \geqslant k-1$. If $t(k-t) = k-1$, then $t = 1$ or $k-1$.

Case1: $t = k-1$.

Let v be adjacent to $u_1, u_2, ..., u_{k-1}$. Since $P_3(G)$ is not complete, there must exist more vertices in G. Consider the case when there exists a vertex w adjacent to some u_i, $i = 1, 2, ..., k-1$, but not adjacent to u_k. If w is not adjacent to v, then vu_iu_k ; $i = 1, 2, ..., k-1$, wu_iv and wu_iu_k will form a K_{k+1} in $P_3(G)$. If w is adjacent to v, then since v is adjacent to $k-1$ vertices of K_k, w can be adjacent to at most $k-2$ vertices of K_k. If w is not adjacent to u_{k-1}, then vu_iu_k ; $i = 1, 2, ..., k-1$, wvu_{k-1} and $wu_iu_k, i = 1, 2...k-2$ will form at least a K_{k+1} in $P_3(G)$. Similarly if w is adjacent to u_k, we can find at least a K_{k+1} in $P_3(G)$.

Therefore consider the case when such a w does not exist. Since $P_3(G)$ is not complete, the vertex v must have a neighbor x having an induced P_3 which is independent of the u_is and v. Then $vu_iu_k;i = 1, 2, ...k-1$ and xvu_i; $i = 1, 2...k-1$ will form at least a K_{k+1} in $P_3(G)$. Thus in this case, $\omega(P_3(G)) \geqslant \omega(G) + 1$.

Case2: $t = 1$.

Let v be adjacent to u_1 alone. Since $P_3(G)$ is not complete, there must exist an induced P_3 which is independent of u_1 and v. Let there exist w adjacent to some u_is. If w is also adjacent to u_1, then there exists at least a K_{k+1} in $P_3(G)$. If w is adjacent to more than one vertex of K_k, then also at least a K_{k+1} is contained in $P_3(G)$. If w is adjacent to exactly one u_i, $i \neq 1$, then $\omega(P_3(G)) = k$ except when $k = 3$ [But when $k = 3$, $P_3(G)$ is complete]. This is the graph mentioned in 1 of the statement. Now, if v or w has a neighbor, then $\omega(P_3(G)) > k$. If more than two vertices are adjacent to the u_is, then also $\omega(P_3(G)) > k$.

So consider the case when u_is have no neighbor other than v. Since $P_3(G)$ is not complete, v must have a neighbor x having an induced P_3 consisting of x but none of v or the u_is. In this case also we can find at least a K_{k+1} in $P_3(G)$.

If $t(k-t) = k$, then we get $k = 4$ and $t = 2$. So let v be adjacent to u_1 and u_2. Since $P_3(G)$ is not complete, it must contain more vertices. If there exists a vertex w which is adjacent to v and if w is not adjacent to u_1 or u_2, then $\omega(P_3(G)) > 4$.

So consider the case when w is adjacent with v, u_1, u_2 [w can be adjacent only to these two vertices of K_4]. Then the graph is the graph shown in Figure : 2(a) of the Lemma.

For the graph G in Figure : 2(a), $\omega(P_3(G)) = \omega(G) = 4$. If we join one more vertex to this graph, then $\omega(P_3(G)) > \omega(G)$.
Consider the case when w is not adjacent to v. Then $\omega(P_3(G)) > \omega(G))$ except when w is adjacent to both u_3 and u_4. Then the graph is the graph shown in Figure : 2(b) of the Lemma.
For this graph, $\omega(P_3(G)) = \omega(G))$. As in the above case, if this graph contains more vertices, then $\omega(P_3(G)) > \omega(G)$. Hence the lemma is proved. □

Theorem 3. *There are no P_3-periodic graphs.*

Proof. If a graph G is P_3-fixed, then $\omega(P_3(G)) = \omega(G)$. From Lemma 2, $\omega(P_3(G)) = \omega(G)$ only for the above three types of graphs. But, none of them are fixed under P_3. Thus there does not exist any graph with period one. Again, from Lemma 2, $\omega(P_3^2(G)) > \omega(G)$. Hence $P_3^n(G) \neq G$ for any $n > 1$. □

Corollary 3.1. *There are no P_3-fixed graphs.*

Theorem 4. *If a graph G is P_3-convergent, then it is $P_3^n(G)$-complete for some $n \geqslant 1$.*

Proof. Let G be a P_3-convergent graph. If G is none of the three graphs mentioned in Lemma-2, then $\omega(P_3(G)) > \omega(G)$. So if it converges, then it converges to some complete graph. If G is one among the three graphs mentioned in Lemma-2, then by Theorem 3, $P_3(G) \neq G$ and $\omega(P_3^2(G)) > \omega(G)$. This indicates that in both the cases, the clique size goes on increasing. Also we know that if $P_3^k(G)$ is a complete graph, then $P_3^{k+1}(G) = \phi$. Hence if G converges, it converges to some complete graph. □

Note: By Theorem 4, it follows that all the P_3-convergent graphs are P_3-mortal graphs.

Open problem: Are there any P_3-divergent graphs?

Theorem 5. *For any graph G, $t_k(P_3(G)) \geqslant 8\, vt_k(G)$, $k \geqslant 4$ and $t_3(P_3(G)) \geqslant 9\, vt_k(G)$ for any $k > 3$.*

Proof. Let $x_1, x_2, ..., x_k, x_1$ be an induced C_k in G. Let y be a touching vertex of this cycle which is adjacent to x_i alone. Then $x_1x_2x_3$, $x_2x_3x_4$, ..., $x_{k-1}x_kx_1$, x_kx_1, x_2 forms a C_k in $P_3(G)$. Also yx_ix_{i+1} and yx_ix_{i-1} are two vertices in $P_3(G)$ such that yx_ix_{i+1} is adjacent to $x_ix_{i+1}x_{i+2}$, $x_{i-1}x_ix_{i+1}$, $x_{i-2}x_{i-1}x_i$, $x_{i+1}x_{i+2}x_{i+3}$ and yx_ix_{i-1} are all adjacent to $x_ix_{i+1}x_{i+2}$, x_{i-1} x_ix_{i+1}, $x_{i-2}x_{i-1}x_i$ and $x_{i-3}x_{i-2}x_{i-1}$. All these eight edges are touching edges to the C_k in $P_3(G)$. Hence $t_k(P_3(G)) \geqslant 8\ vt_k(G)$, $k \geqslant 4$.

Again, let $C_k = x_1, x_2, ..., x_k, x_1$ be an induced C_k in G. If y is a touching vertex of the cycle which touches x_i alone, then yx_ix_{i+1}, $x_{i-1}x_ix_{i+1}$ and yx_ix_{i-1} induce a C_3 in $P_3(G)$. Then $x_ix_{i+1}x_{i+2}$, $x_{i-2}x_{i-1}x_i$ and $x_{i+1}x_{i+2}x_{i+3}$ are all adjacent to all the three vertices of the C_3 and hence the result. □

Note: The bounds in the above theorem are strict.
If G is 4 - pan, then, $P_3(G) = K_6$. But, $vt_4(G) = 1$; $t_4(P_3(G)) = 8$; $t_3(P_3(G)) = 9$.

Theorem 6. *If G is k-connected, then $P_3(G)$ is $(k-1)$-connected. Further $\kappa(P_3(G)) = \kappa(G) - 1$ if and only if G is $K_k - \{e\}$.*

Proof. Let G be k-connected. Then by Whitney's theorem [8], for any two vertices u and v, there exists at least k internally disjoint $u - v$ paths.
Let $u_1u_2u_3$ and $v_1v_2v_3$ be any two distinct vertices in $P_3(G)$. In G, if u_i and v_j are connected by a path which contain at least one vertex other than these six vertices, then correspondingly there exists at least one path in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$. So it is enough to consider the paths in G involving one or more of these six vertices.

Case 1: Let u_i and v_j are not adjacent for some $i, j \in \{1, 2, 3\}$.
Then corresponding to any path in G joining u_i and v_j, there exists a path in $P_3(G)$ joining $u_1u_2u_3$ and $v_1v_2v_3$. So $\kappa(P_3(G)) \geqslant \kappa(G)$.

Case 2: Let u_i be adjacent to v_j.
Case 2a: All u_is and v_js are distinct.
Then in between any u_i and v_j in G, there exists at most five internally disjoint u_i-v_j paths involving those six vertices only. But there exists six internally disjoint paths joining $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ which are of the form $u_1u_2u_3$, $u_1v_ju_3$, $v_1v_2v_3$; j= 1,2,3 and $u_1u_2u_3$, $v_1u_jv_3$, $v_1v_2v_3$.
Hence in this case also $\kappa(P_3(G)) \geqslant \kappa(G)$.
Case 2b: u_is and v_js share a common vertex.
In this case there exists more paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ than the minimum number of paths between any u_i and v_j in G. Hence $\kappa(P_3(G)) \geqslant \kappa(G)$.
Case 2c: u_is and v_js share two common vertices.
In this case also, except when they form $K_4 - \{e\}$, the number of internally disjoint paths between $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$ is greater than or equal

to the minimum number of internally disjoint paths between any u_i and v_j in G. If there exists one more vertex then $\kappa(P_3(G)) < \kappa(G)$ only when the newly adjoined vertex is adjacent to all the four vertices. If we adjoin more vertices to this graph also, $\kappa(P_3(G)) < \kappa(G)$ only when the adjoined vertices are adjacent to all the other existing vertices. Hence $\kappa(P_3(G)) < \kappa(G)$ only when G is $K_k - \{e\}$. For $K_k - \{e\}$, $\kappa(P_3(G))= \kappa(G)$ - 1. □

Theorem 7. *For any connected graph G, $P_3(G)$ is a block.*

Proof. Suppose that $w = xyz$ is a cut vertex in $P_3(G)$. Then there exists two non adjacent vertices $u_1u_2u_3$ and $v_1v_2v_3$ such that the only path joining them is $u_1u_2u_3$, w, $v_1v_2v_3$. Then u_is and v_js are distinct and some u_i, v_j $= x$, y or z. Thus we can find at least one more path joining $u_1u_2u_3$ and $v_1v_2v_3$ which is a contradiction to the fact that xyz is a cut vertex. □

Theorem 8. *For all connected graphs, the only geodetic P_3-intersection graphs are the complete graphs.*

Proof. Let $P_3(G)$ be non-complete. Consider two non-adjacent vertices $u_1u_2u_3$ and $v_1v_2v_3$ in $P_3(G)$. Since G is connected, we may choose $v_1v_2v_3$ such that u_i adjacent to some v_j for $i, j \in$ 1,2,3. Then there exists at least two disjoint paths of length two connecting $u_1u_2u_3$ and $v_1v_2v_3$ and hence $P_3(G)$ cannot be a geodetic graph. Hence the result. □

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