Balanced Whist Tournaments

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Abstract

Whist tournaments for v players, Wh(v) are known to exist for all $v \equiv 0, 1 \pmod 4$. In this paper a new specialization of whist tournament design, namely a *balanced whist tournament*, is introduced. We establish that balanced whist tournaments on v players, BWh(v), exist for several infinite classes of v. An adaptation of a classic construction due to R. C. Bose and J. M. Cameron enables us to establish that BWh($4n + 1$) exist whenever $4n + 1$ is a prime or a prime power. It is also established that BWh($4n$) exist for $4n = 2^k$, with $k \equiv 0 \pmod{2, 3 \text{ or } 5}$. We demonstrate that a BWh($4n + 1$) is equivalent to a conference matrix of order $4n + 2$. Consequently, a necessary condition for the existence of a BWh($4n + 1$) is that $4n + 1$ is a product of primes each of which is $\equiv 1 \pmod 4$. Thus, in particular, BWh(21) and BWh(33) do not exist. Specific examples of BWh(v) are given for $v = 4, 8, 9, 20, 24, 32$. It is also shown that a BWh(12) does not exist.

1 Introduction

Let us call a set $\mathcal{P} = \{a_1, \ldots, a_n\}$, with $n \equiv 0, 1 \pmod 4$, the set of players of a bridge tournament, and any ordered 4-tuple (a_i, a_j, a_k, a_l) of players a game. In a given game (a_i, a_j, a_k, a_l), players a_i and a_k will be called NS partners , a_j and a_l EW partners, while all the other 4 unordered couples are opponents. A round will be any set of games such that the union of the players involved in the games is the player's set $\mathcal{P}$ (except at most one, the resting player, in the $n \equiv 1$ (mod 4) case). In a round the set of players which form NS partnerships in the games involved are said to belong to the NS line, while the others (except, possibly, the resting player) belong to the EW line. Two (non resting) players are said to *play in the same line* if they are both NS or EW, otherwise they *play in opposite lines*. A whist tournament for $\mathcal{P}$ is a set of $n - 1$ (resp. n in the $n \equiv 1$ (mod 4) case) rounds such that:

1. every player partners every other player exactly once, and

2. every player opposes every other player exactly twice.

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This type of tournament is also called an individual bridge tournament or an individual doubles tennis tournament.

We focus now on the bridge tournament setting in order to make apparent how some further balancing conditions naturally arise. To fix ideas, let us consider the $4n$-player case. During the i-th round the $4n$ players are grouped in 4-tuples in order to sit at n tables and play the i-th deal of cards. Let us call $P_j^{NS}(i)$ and $P_j^{EW}(i)$ the score obtained respectively by the NS and EW pair in table (game) j at round i. The American Contract Bridge League (ACBL) scoring method for bridge tournaments consists in converting the scores in matchpoints through the following procedure. All the NS pairs' scores at round i are compared, assigning one matchpoint to a particular pair for every NS pair which scored less, and 1/2 matchpoint for each pair who scored equally. The same is done for the EW pairs. More precisely the matchpoints scored in round i by player a sitting at table j in, say, NS position, will be given by

$$ MP_i(a) = \#\{k : P_k^{NS}(i) < P_j^{NS}(i)\} + \frac{1}{2}\#\{k \neq j : P_k^{NS}(i) = P_j^{NS}(i)\} $$

It is reasonable to presume that any given tournament attracts players of different abilities. In order that a tournament be considered to be a fair tournament, one would expect a schedule of games that would smooth out (that is, balance) these disparities and provides all participants the same opportunity to maximize their score. Thus, for the sake of discussion, suppose c is a very bad player (analogous considerations hold for very good players). The following four conditions are imposed so as to ensure fairness in the tournament.

1. *Any two players must have c as a partner the same number of times*, since having c as partner will result in a bad score.

2. *Any two players must have c as opponent the same number of times*, since having c as opponent will result in a better score.

3. *Any two players must have c playing in their line, (but not at the same table), the same number of rounds*, since having c playing in the same line of a player results in one matchpoint more for that player.

4. *Any two players must have c playing in their opposite line, (but not at the same table), the same number of rounds*, since having c playing in the opposite line results in one less matchpoint.

While the first two conditions are fulfilled by the very definition of whist tournament, the others are not. This leads us to the following definition: two players a and b in a whist tournament are relative opponents if they belong to the same line, but they are not playing in the same game, while a and b are relative partners if they sit in different lines, but they are not playing in the same game. The terms used are consistent with the observation made before that, roughly

speaking, the better a partner of a performs, the higher a's score will be and the better an opponent of a performs, the lower a's score will be. Therefore in order to say that the tournament is not biased by bad (or good) players it is required that any two players will be relative opponents (and thus relative partners) the same number of times. When this is the case we say that the whist tournament is *balanced*.

The plan of the paper is as follows: in Section 2 we'll provide the preliminary definitions, relationships to other designs and nonexistence results. More precisely using the relationship between balanced whist tournaments and conference matrices we'll prove that in the $4n + 1$ setting such tournaments can exist only if $4n + 1$ is a sum of two squares. The nonexistence of a balanced whist tournament for 12 players will be addressed via explicit computations.

In Section 3 a characterization of G-cyclic balanced whist tournaments will be used to prove that the classical Bose-Cameron construction of cyclic whist tournaments leads in fact to a balanced one, thus providing examples of such tournaments when $4n + 1$ is a prime power.

In Section 4 a new notion of orthogonality between resolvable designs is introduced. This concept of orthogonality is used in the affine geometries environment to provide examples of balanced whist designs.

This notion of orthogonality will be used to refine Moore's construction for whist tournaments in Section 5. We will then construct balanced whist tournaments for $4n = 2^k$, with $k \equiv 0 \pmod{2, 3 \text{ or } 5}$. Some non prime power balanced tournaments are also described for $4n = 20$ and 24.

2   Settings and nonexistence results

The first part of this section, setting the basics for whist tournaments, is taken from [3].

Definition 2.1 *A* whist tournament *$Wh(4n)$ for $4n$ players is a schedule of games each involving two players opposing two other players, such that*

1. the games are arranged into $4n - 1$ rounds, each of n games;

2. each player plays in exactly one game in each round;

3. each player partners every other player exactly once;

4. each player opposes every other player exactly twice.

Each game is denoted by an ordered 4-tuple (a, b, c, d) in which the unordered pairs $\{a, c\}$ and $\{b, d\}$ are partner pairs. The other pairs are opponent pairs. Players a, c are NS players and b, d are EW players.

Definition 2.2 *A* whist tournament *$Wh(4n+1)$ for $4n+1$ players is a schedule of games each involving two players opposing two other players, such that*

1. *the games are arranged into $4n + 1$ rounds, each of n games;*

2. *each player plays in exactly one game in each of $4n$ rounds, but does not play in the remaining round;*

3. *each player partners every other player exactly once;*

4. *each player opposes every other player exactly twice.*

The $4n + 1$ rounds of a $Wh(4n + 1)$ are usually labelled by the player that sits out that round. That is to say, for $a \in \mathcal{P}$, Round a is the (unique) round in which a doesn't play. Player a is said to be the resting player *in round a.* *Partners, opponents and NS (EW) players are defined as in Definition 2.1.*

Theorem 2.3 *[3]*
    A $Wh(4n)$ and a $Wh(4n + 1)$ exist for all $n \geq 1$.

Definition 2.4 *The NS* line *in a given round of a $Wh(v)$ is the set of all NS players of that round, and the EW* line *the set of all EW players. Two non-resting players are said to* play in the same line *in a round if they belong to the same line of that round, and that they are said to* play in opposite lines *otherwise. A $Wh(v)$ is* balanced, *denoted by $BWh(v)$, if each pair of players plays in the same line in exactly $\lceil v/2 \rceil - 1$ rounds.*

Remark 2.5 *This latter description of balance in a whist tournament coincides with the one introduced in Section 1. We show this separately for $|\mathcal{P}| = 4n$ and $|\mathcal{P}| = 4n + 1$.*

1. *In the $4n$ case, two players a and b play in the same game in exactly three rounds: once as partners (hence in the same line) and twice as opponents (hence in opposite lines). In the remaining $4n - 4$ rounds they play in the same line exactly $2n - 2$ rounds and hence they are relative opponents $2n - 2$ times. Moreover they play in opposite lines in every other round, hence in $2n - 2$ rounds they are relative partners.*

2. *In the $4n + 1$ case, two players a and b are resting players in round a and b respectively. In the other $4n - 1$ rounds they play the same game thrice, and in the remaining $4n - 4$ rounds they are relative opponents $2n - 2$ times and relative partners all the other $2n - 2$ rounds.*

Remark 2.6 Relationships with other Designs:
    It is well known that a $Wh(4n)$ is a resolvable $(4n, 4, 3)$ BIBD and a $Wh(4n+1)$ is a near resolvable $(4n + 1, 4, 3)$ design.
    Taking the NS and EW lines in a $BWh(4n)$ as blocks of a design produces a resolvable $(4n, 2n, 2n-1)$ BIBD (sometimes called a Hadamard design), which is equivalent to a Hadamard matrix of order $4n$. The existence of such Hadamard

matrices is widely conjectured to hold for any $n \geq 1$ but a proof of this fact is not known. One would like to have a result which guarantees the existence of a BWh(4n) whenever a Hadamard matrix of order 4n exists, but such a result is in fact false in this generality since a Hadamard matrix of order 12 exists, while a BWh(12) will be proven not to exist.

On the other hand, taking the same sets in a $BWh(4n + 1)$ as blocks of a $4n + 1$ design produces a near resolvable $(4n + 1, 2n, 2n - 1)$ design. Note that the necessary conditions [2] for such a design to exist are satisfied and that such designs are equivalent to a $(\{2n\}, 2n - 1)$ frame of type 1^{4n+1} and, more importantly, to a conference matrix of order $4n + 2$, see [7].

In either of the above $(v, 2n, 2n - 1)$ designs, retention of the $BWh(v)$ partnerships demonstrates that the existence of $BWh(v)$ implies the existence of a $(2, 2n)GWhD(v)$ [1]. The converse is not valid in general since there exists a $(2, 6)GWhD(12)$.

The relationship between $BWh(4n + 1)$ and conference matrices of order $4n + 2$ leads us to the following theorem.

Theorem 2.7 *If a $BWh(4n + 1)$ exists, then $4n + 1$ must be the sum of two squares.*

Proof. The proof follows from the fact mentioned in the previous remark, namely, that a conference matrix of order $4n + 2$ can be obtained from a $BWh(4n + 1)$ through the near resolvable $(4n + 1, 2n, 2n - 1)$ design associated to NS and EW lines. Since such a matrix can exist only if $4n + 1$ is the sum of two squares (see [9]), the statement follows. $\square$

Note that it's a well known fact that $v \equiv 1 \pmod 4$ is the sum of two squares if and only if it is a product of prime powers, each of which $\equiv 1 \pmod 4$. In particular neither a $BWh(21)$ nor a $BWh(33)$ exist.

Regarding the $4n$-players case, the first, and for the time being the only, nonexistence result is the following.

Theorem 2.8 *No $BWh(12)$ exists.*

Proof. The proof of this result is basically computational: since there are only two $Wh(12)$ up to isomorphism, (see [10]), it suffices to test the balance condition for all possible $Wh(12)$ obtained through switching NS and EW positions in any subset of games of the tournaments. This was accomplished by means of a computer program. No $BWh(12)$ was obtained. $\square$

3   The $4n + 1$ result

First of all, let us recall some well known facts about finite fields.

Theorem 3.1 *For every prime p and every $k \geq 1$,*

1. *There is a unique (up to isomorphism) finite field of order p^k, which will be denoted by $GF(p^k)$.*

2. *Given an irreducible polynomial $q(x) \in \mathbb{Z}_p[x]$ of degree k, this $GF(p^k)$ is isomorphic to $\mathbb{Z}_p[x]/(q)$.*

3. *The multiplicative group $GF^*(p^k) = GF(p^k) \setminus \{0\}$ is cyclic, and every generator of this group is called a* primitive element.

4. *For every m such that $m|k$, there is a unique subfield $F_m(p^m) \trianglelefteq GF(p^k)$. Given a primitive element θ this subfield is given by*

$$\{\theta^h : h \equiv 0 \pmod{k/m}\} \cup \{0\}.$$

5. *If $p^k \equiv 1 \pmod 4$, then, for any primitive element θ the squares $\{\theta^{2i}\}$ and non-squares $\{\theta^{2i+1}\}$ are disjoint subsets each of order $(p^k-1)/2$. Moreover these sets form a difference system with multiplicity $\lambda = (p^k - 3)/2$.*

Definition 3.2 *Let G be an abelian group (with additive notation) of order $4n + 1$. A G-cyclic whist tournament is a $Wh(|G|)$ in which the players are the elements of G and the games in the g-th round are obtained by adding g to each player in the 0-th round. Such whist designs will be described, then, by giving only the first round and the group.*

Example 3.3 A GF(9)-cyclic BWh(9)
Let $GF(9)$ be the splitting field of $x^2 + 1$ over $\mathbb{Z}_3$. A primitive element is $x + 1$, and the round

$$(1, x + 1, -1, -x - 1), \quad (-x, 1 - x, x, x - 1)$$

forms the 0-th (alt. first or initial) round of a cyclic $BWh(9)$ over $GF(9)$. Renaming elements of $GF(9)$ by $ax + b = 3a + b$ we obtain the following $BWh(9)$:

$$(1, 4, 2, 8), \ (6, 7, 3, 5)$$
$$(2, 5, 0, 6), \ (7, 8, 4, 3)$$
$$(0, 3, 1, 7), \ (8, 6, 5, 4)$$
$$(4, 7, 5, 2), \ (0, 1, 6, 8)$$
$$(5, 8, 3, 0), \ (1, 2, 7, 6)$$
$$(3, 6, 4, 1), \ (2, 0, 8, 7)$$
$$(7, 1, 8, 5), \ (3, 4, 0, 2)$$
$$(8, 2, 6, 3), \ (4, 5, 1, 0)$$
$$(6, 0, 7, 4), \ (5, 3, 2, 1).$$

Theorem 3.4 *A round $\{(a_i, b_i, c_i, d_i)\}_{i=1}^n$ made of players in an abelian group G of order $4n + 1$ is the first round of a G-cyclic $BWh(4n + 1)$ if and only if*

1. *the partner differences $\{\pm(a_i - c_i), \pm(b_i - d_i)\}_{i=1}^n$ are all the non zero elements of the group G;*

2. *the opponent differences $\{\pm(a_i - b_i), \pm(b_i - c_i), \pm(c_i - d_i), \pm(d_i - a_i)\}_{i=1}^n$ are all the non zero elements of the group G, each occuring twice.*

3. *the differences between NS players and between EW players are all the non zero elements of the group G each occuring $2n - 1$ times. In other words the NS line and EW line form a $(4n + 1, 2n, 2n - 1)$ difference system.*

Proof. Conditions 1 and 2 guarantee that the given round serves as the first round of a $Wh(4n + 1)$ (for a proof of this see [4], p. 50). The third condition is clearly necessary since by Remark 2.6 the NS line and EW line form the blocks of a $(4n + 1, 2n, 2n - 1)$ BIBD. The condition is also sufficient since, if a and b are two players, $a - b \neq 0$ appears as a difference of elements $x - y$ on the same line in round 0 precisely $2n - 1$ times. Therefore a and b will be playing in the same line in the $2n - 1$ rounds $g = a - x = b - y$. $\square$

The next theorem establishes the existence of an infinite class of BWh($4n + 1$).

Theorem 3.5 *If θ is a primitive element of $GF(p^k)$ for some prime p and power k such that $p^k = 4n + 1$, then there exists an odd integer α such that the games*

$$(1, \theta^\alpha, -1, -\theta^\alpha) \times 1, \theta^2, \theta^4, \dots \theta^{2n-2}$$

form the initial round of a $GF(p^k)$-cyclic $BWh(p^k)$

Proof. That the initial round described generates a $Wh(p^k)$ follows from the classic Bose-Cameron result in [5]. Moreover since α is odd, the NS players are the squares in $GF(p^k)$, while the EW entries are the non-squares in $GF(p^k)$. Since these two sets form a $(p^k, 2n, 2n - 1)$ difference system (see Theorem 3.1, item *5)*), the claim follows from Theorem 3.4. $\square$

Therefore a product theorem would suffice to completely settle the existence problem of BWh($4n+1$). The first unknown case is, then, for $4n+1 = 9 \times 5 = 45$.

4 Orthogonal Designs

The key result for the $4n$ case depends on a careful analysis of a classical construction for Wh($4n$). One of the first methods, which goes back to Moore [12], consists in taking a resolvable $(4n, 4, 1)$ BIBD (which exists iff $4n \equiv 4 \pmod{12}$),

and, on any resolution class, construct a Wh(4) on each block. The problem with this construction is that we have no freedom in choosing the developing Wh(4). If two players are playing, say, north in the initial resolution of the BIBD, they'll be playing in the same position in each of the resulting three rounds. Hence this situation must occur, for any pair of players, the "right" number of times.

From now on a RBIBD will denote a *resolved* design rather than a resolvable (as is usually done) design, meaning that we actually fix a particular resolution, ordering the resolution classes. For such designs we'll denote by r the number of resolution classes. As usual we'll suppose the BIBD's are non trivial, i.e. the blocks are smaller than the whole set.

Definition 4.1 *Let D be a resolved $t - (v, k, \lambda)$ design. We say that a resolved $t - (v, \sigma\frac{v}{k}, \mu)$ design D' is σ-orthogonal to D if for every $j = 1, \ldots, r$ the j-th resolution class R_j of D is such that every block of the j-th resolution class R'_j of D' intersects every block of R_j in σ points. We will also say that the two resolutions are orthogonal. A set of resolved $t - (v, k, \lambda)$ designs is a set of mutually σ-orthogonal resolved t-designs if any two designs of the set are σ-orthogonal.*

Even though this definition holds in a general setting, we will consider only 1-resolvable designs, i.e. we will assume that every resolution class is actually a partition of the point set.

Example 4.2 Latin squares and 1-orthogonal $1 - (v, k, 1)$ designs.
In the particular case $t = 1$, the resolvable $1 - (v, k, \lambda)$ designs are usually called parallel configurations. *It is easy to see that a resolvable $1 - (v, k, \lambda)$ design is simply a collection of λ partitions of the v-point set in k-subsets. Therefore if $k \mid n$ there are many such designs and in order to study orthogonality in this setting one can simply let $\lambda = 1$. The connection with Latin Squares is exploited in the following two remarks:*

1. *A Latin Square $(a_{i,j})$ of order n naturally defines a $1 - (n^2, n, 1)$ design on the set of positions (i, j) in the square simply by taking the n blocks to be $\{\{(i, j) : a_{i,j} = k\} : k = 1, \ldots n\}$. This design is trivially resolvable since it has a unique resolution class. Two latin squares are orthogonal if every two blocks intersect in exactly one point. Thus they are 1-orthogonal in the sense defined above.*

2. *Not every $1 - (n^2, n, 1)$ design arises from a Latin Square via the construction above. Indeed, consider the following two $1 - (n^2, n, 1)$ designs on the set of positions (i, j) in a $n \times n$ square:*

$$\{(1, 1), (1, 2), \ldots, (1, n)\}; \{(2, 1)(2, 2), \ldots, (2, n)\}; \ldots; \{(n, 1), \ldots, (n, n)\},$$

$$\{(1, 1), (2, 1), \ldots, (n, 1)\}; \{(1, 2), (2, 2), \ldots, (n, 2)\}; \ldots; \{(1, n), \ldots, (n, n)\}.$$

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These two configurations do not arise as latin squares, but one can observe that a $1 - (n^2, n, 1)$ design is a latin square if and only if it is orthogonal to both of the above configurations. Since the labelling of the n^2 elements of the point set is clearly irrelevant, any three mutually orthogonal $1 - (n^2, n, 1)$ designs define a latin square. It is clear, then, that a maximal set of mutually 1-orthogonal $1 - (n^2, n, 1)$ designs has cardinality at most $n + 1$ and if n is a prime power there exists such a maximal set with $n + 1$ elements.

Example 4.3 Orthogonal arrays and σ-orthogonal $1 - (\sigma n^2, \sigma n, 1)$ designs. *More generally, one can consider sets of mutually σ-orthogonal $1 - (m, k, 1)$ designs only if (see Remark 4.4) $k^2 = \sigma m$, where $\sigma | k$ and $k \mid m$. Let us call $m/k = n$: one then obtains $k = \sigma n$ and $m = \sigma n^2$ and a set of h mutually σ-orthogonal $1 - (\sigma n^2, \sigma n, 1)$ designs is a $(n, h; \sigma)$-net (see [6]), which in turn is equivalent to an $OA_\sigma(h, n)$. Thus we can use the Bose-Bush bound to establish that a maximal set of σ-orthogonal $1 - (m, k, 1)$ designs has cardinality at most $\left[k \frac{m-1}{m-k} \right]$ (which is optimal in the case $\sigma = 1$, as we have seen before).*

Recall that if D is a $t - (v, k, \lambda_t)$ resolvable design, then it is also a $s - (v, k, \lambda_s)$ resolvable design for any $1 \le s \le t$, where

$$\lambda_s = \lambda_t \frac{(v - s)(v - s - 1) \dots (v - t + 1)}{(k - s)(k - s - 1) \dots (k - t + 1)}.$$

It is clear that two t-designs are orthogonal if and only if the corresponding s-designs are.

Remark 4.4 *Let us fix a RBIBD $t - (v, k, \lambda_t)$, D, and deal with the problem of the existence of a σ-orthogonal RBIBD $t - (v, h, \mu_t)$, D'. In order that such an orthogonal BIBD exists, (apart from the necessary conditions that D' is a RBIBD, see [2]), one has the following necessary conditions.*

1. Conditions on σ.

$$\sigma \mid k \, , \qquad \sigma \mid h \, , \qquad \sigma v = hk \, .$$

Indeed, each block of D' is subdivided in the v/k intersections with the blocks of the orthogonal resolution class, each having σ elements.

2. Relationships between the parameters.
If r, b and r', b' are the replication and block numbers of D and D' respectively, then, since the two BIBDs have an equal number of resolution classes, and an equal number of elements

$$r' = r \, , \qquad\qquad hb' = kb \, ,$$
$$\lambda_t(h - 1)(h - 2) \dots (h - t + 1) \ \dot{=} \ \mu_t(k - 1)(k - 2) \dots (k - t + 1).$$

(The last identity holding only if $t \ge 2$.)

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3. Bounds on σ for $t \geq 2$.
Substituting the identities above in Bose's condition for resolvable 2-RBIBDs: $b' \geq v + r - 1$, gives

$$\sigma \leq \frac{\lambda_2 k}{\lambda_2 + k - 1} \leq \min\{\lambda_2, k\}$$

with the last inequality being an equality only if $\lambda_2 = 1$. Furthermore $\sigma < k$ and hence using $\sigma \mid k$ frequently gives the best bound, ($\sigma = 1$ if, for example, k is prime).

4. $\lambda_2 = 1$ case.
Using the previous inequality, one obtains that $\lambda_2 = 1$ implies $\sigma = 1$, and a simple substitution gives:

$$h = \frac{v}{k}, \qquad \mu_2 = \frac{v - k}{k(k-1)}, \qquad b' = kr.$$

Therefore the orthogonal BIBD D' has parameters that satisfy $b' = v + r - 1$ and hence it is affine, (see [2]). This in turn implies $v \mid h^2$ and thus $k^2 \mid v$. Therefore the further necessary conditions in this case are:

$$\begin{cases} v \equiv 0 & mod\ k^2 \\ \frac{v}{k} - 1 \equiv 0 & mod\ (k-1) \end{cases}$$

Example 4.5 Kirkman Triple Systems
A $(15, 3, 1)$ Kirkman triple system has no 1-orthogonal BIBD. Indeed this would be a $(15, 5, 2)$ RBIBD, which does not exist, although all the necessary conditions (for resolvable BIBDs) are satisfied (see [2]). More generally, a Kirkman triple system on n elements, $KTS(n)$, can have a 1-orthogonal RBIBD only if $n \equiv 9$ (mod 18), by a simple application of Remark 4.4, point 4. The existence for the $n = 3^k$, $k \geq 2$ case will be solved in Theorem 4.11 in the particular case $q = 3$, $i = 1$.

Example 4.6 Balanced Whists
Consider $\sigma = 2$. A $(20, 4, 3)$ RBIBD can be obtained by developing, modulo 19, the first resolution class

$$\{\infty, 0, 12, 17\}, \{1, 6, 9, 13\}, \{2, 3, 4, 8\}, \{5, 11, 14, 15\}, \{7, 10, 16, 18\}.$$

This $(20, 4, 3)$ RBIBD has an orthogonal $(20, 10, 9)$ RBIBD. The orthogonal RBIBD is the development, modulo 19, of the first resolution class

$$\{\infty, 0, 1, 2, 5, 8, 9, 14, 16, 18\}, \{3, 4, 6, 7, 10, 11, 12, 13, 15, 17\}.$$

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Indeed, the blocks of the first round of the $(20, 4, 3)$ *RBIBD can be ordered in such a way to obtain the first round of a* $\mathbb{Z}_{19}$-*cyclic* $BWh(20)$, *with* NS *and* EW *lines being the blocks of the given* $(20, 10, 9)$ *Hadamard RBIBD.*

$$(\infty, 17, 0, 12), (1, 13, 9, 6), (8, 4, 2, 3), (5, 15, 14, 11), (16, 7, 18, 10).$$

More generally, if D *is the* $(4n, 4, 3)$ *design derived from a* $BWh(4n)$, *then the* NS *and* EW *lines, taken as blocks, yields a* $(4n, 2n, 2n - 1)$ *Hadamard design that is 2-orthogonal to* D.

Recall that an affine design is a RBIBD such that any two non-parallel blocks intersect in a $\frac{k^2}{v}$-set. If $v \mid k^2$ one can search for a set of mutually $\frac{k^2}{v}$-orthogonal (v, k, λ) RBIBDs. Recall that given a family of m subsets $\{A_i\}$ of a fixed set X, a *system of distinct representatives* (briefly, a SDR) is a m-subset $\{x_i\}$ such that $x_i \in A_i$ for every $i = 1, \ldots, m$. We'll need the following well known corollary of Hall's Theorem.

Lemma 4.7 *Let* $\{A_1, \ldots, A_n\}$ *be a family of subsets of* $X = \{1, \ldots, n\}$ *and* $1 \le k \le n - 1$. *Suppose* $|A_i| = k$ *for every* i *and that each element of* X *occurs in exactly* k *of the sets* A_i. *Then there are* k *SDRs* $\{x_i^j\}$ *for the* $\{A_i\}$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ *such that*

$$A_i = \bigcup_{j=1}^{k} \{x_i^j\}.$$

Theorem 4.8 *A* (v, k, λ) *affine design* $\mathcal{A}$ *with multiplicity coefficient* r *has* r $\frac{k^2}{v}$-*orthogonal resolutions.*

Proof. Let us fix a resolution $\mathcal{R} = (R_1, \ldots, R_r)$ of $\mathcal{A}$. We'll apply Lemma 4.7 to the set $X = \{R_i : i = 1, \ldots, r\}$, with subsets $A_i = X \setminus R_i$. Take the $r - 1$ resolutions $\mathcal{R}_j = (R_1^j, \ldots, R_r^j)$ given by the $r - 1$ SDRs $\{R_i^j\}$, $i = 1, \ldots, r$, $j = 1, \ldots, r - 1$. Since $\mathcal{A}$ is affine, given $k \ne h$, the i-th resolution classes R_i^k and R_i^h have blocks that intersect in $\frac{k^2}{v}$ elements. $\square$

The set described in the previous theorem is indeed maximal.

Theorem 4.9 *A set of mutually* $\frac{k^2}{v}$-*orthogonal affine* (v, k, λ) *designs can have at most* r *elements, where* r *is the multiplicity of the design.*

Proof. Since the (v, k, λ) is affine, it follows that $b = v + r - 1$. Suppose there exists a set of $r + 1$ $\frac{k^2}{v}$-orthogonal affine (v, k, λ) designs. Take the first resolution classes from each design and combine them to form a resolvable $1 - (v, k, r + 1)$ design. We prove that this is in fact a 2-design. Note that any two non-parallel blocks intersect in $\frac{k^2}{v}$ elements by orthogonality, thus the 1-design is affine.

Furthermore removing any one of the resolution classes gives a $1 - (v, k, r)$ design whose parameters satisfy $b = v + r - 1$ and by [11], Theorem 5.17, this implies that the $1 - (v, k, r)$ design is actually a 2-design. Take any two elements a and b and a resolution class in which they are not in the same block (this can always be done since $k < v$ and thus $\lambda < r$); removing this resolution class gives a 2-design and thus a and b are in the same block exactly λ times. Therefore the first resolution classes form together a $2 - (v, k, \lambda)$ design, which is clearly impossible since the multiplicity of this design is $r + 1$. $\square$

Consider, now, the case of Affine Geometries. As a general reference one can take [8]. If q is a prime power, recall that the *affine geometry* $AG_i(n, q)$ is the structure defined on the points of the n-dimensional $GF(q)$-vector space $GF(q)^n$ having as blocks the i-dimensional flats. This design is affine if $i = 1$ or $i = n - 1$. The *q-gaussian coefficient* $\left[{n \atop i} \right]_q$ is defined as the number of i-dimensional vector subspaces of $GF(q)^n$ and is given by the formula

$$\left[{n \atop i} \right]_q = \frac{(q^n - 1)(q^{n-1} - 1)\dots(q^{n-i+1} - 1)}{(q^i - 1)(q^{i-1} - 1)\dots(q - 1)}.$$

The structure so described provides a $(q^n, q^i, \left[{n-1 \atop i-1} \right]_q)$ RBIBD with further parameters $b = q^{n-i} \left[{n \atop i} \right]_q$ and $r = \left[{n \atop i} \right]_q$, which will be denoted again by $AG_i(n, q)$. The resolution classes of this design are constructed by taking any one of the $\left[{n \atop i} \right]_q$ i-flats through 0 and translating it in any of the q^{n-i} ways. The blocks so obtained obviously form a partition of $GF(q)^n$. Two $GF(q)$-linear vector subspaces V and V' of $G(q)^n$ are said to be *skew* if $V \cap V' = \{0\}$. Suppose that V has dimension k. Then the number of h-dimensional subspaces skew to V is $q^{hk} \left[{n-k \atop h} \right]_q$ (hence $q^{k(n-k)}$ in the particular case $h = n - k$). Moreover for any $l \le \min\{h, k\}$, the number of h-dimensional subspaces that intersect V in a subspace of dimension l is $q^{(h-l)k} \left[{n-k \atop h-l} \right]_q \left[{k \atop l} \right]_q$.

Note that in order to find a resolvable design that is σ-orthogonal to $AG_i(n, q)$ one has to constrain σ to be a divisor of q^i. Furthermore, the number of resolution classes must be the same. This suggest that a natural choice can be $AG_i(n, q)$ itself, provided σ has been suitably chosen.

Theorem 4.10 *For every $i \in \mathbb{N}$, $\frac{n}{2} \le i \le n - 1$, there is a set of $q^{(n-i)i} \left[{i \atop 2i-n} \right]_q$ mutually q^{2i-n}-orthogonal $(q^n, q^i, \left[{n-1 \atop i-1} \right]_q)$ resolvable BIBDs.*

Proof. Let us fix a resolution $AG_i(n, q)$ by numbering the i-dimensional vector subspaces $V_1, \dots, V_r$, with $r = \left[{n \atop i} \right]_q$. Apply Lemma 4.7 with $X = \{i$-dimensional subspaces of $GF(q)^n\}$ and

$$A_j = \{i \text{ dim. subspaces intersecting } V_i \text{ in a } (2i - n)\text{-dim. subspace}\}.$$

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Now, having an intersection of fixed dimension is a symmetric relation, thus the hypotheses of the lemma are satisfied. Consequently we can consider the $q^{(n-i)i} \begin{bmatrix} i \\ 2i-n \end{bmatrix}_q$ SDRs $\{V_1^j, \ldots, V_r^j\}$, $j = 1, \ldots, q^{(n-i)i} \begin{bmatrix} i \\ 2i-n \end{bmatrix}_q$. For each $h \neq k$, the translations of the corresponding subspaces V_i^k and V_i^h form two orthogonal resolutions of $AG_i(n, q)$. $\square$

Another natural choice for a RBIBD orthogonal to $AG_i(n, q)$ is indicated in the following theorem.

Theorem 4.11 *$AG_i(n, q)$ and $AG_{n-i}(n, q)$ are 1-orthogonal.*

Proof. Let us fix the natural resolution of $AG_i(n, q)$ described in the proof of Theorem 4.10. In Lemma 4.7, take $X = \{n - i$ dimensional subspaces of $GF(q)^n\}$, and $A_j = \{n - i$ dimensional subspaces of $GF(q)^n$ skew to $V_j\}$. The hypotheses of the lemma are satisfied since being skew is a symmetric relation between subspaces. Hence we can choose a SDR in such a way that V_j' is a $n - i$ flat through 0 skew to V_j, for every $j = 1, \ldots, \begin{bmatrix} n \\ i \end{bmatrix}_q$. This choice produces a resolution by translations of $AG_{n-i}(n, q)$, and since the translations of two skew subspaces intersect in exactly one point, the resolutions are orthogonal. $\square$

5 Some BWh(4n)

The main theorem of this section is a refinement of the standard construction for Wh(4n), which goes back to Moore [12], in the case that there exists a resolvable (4n, 4, 1) BIBD.

Theorem 5.1 *Suppose that a (v, k, λ) RBIBD and a $(v, \frac{v}{k}, \lambda')$ RBIBD are 1-orthogonal RBIBDs. If there exist a (k, h, μ) RBIBD and a $(k, \sigma\frac{k}{h}, \mu')$ RBIBD that are σ-orthogonal, then there exist a $(v, h, \lambda\mu)$ RBIBD and a $(v, \sigma\frac{v}{h}, \lambda\mu\frac{\sigma v - h}{h(h-1)})$ RBIBD that are σ-orthogonal.*

Proof. Let r and r' be the number of resolution classes of the (v, k, λ) and (k, h, μ) designs respectively. Consider two orthogonal resolutions of the (v, k, λ) and $(v, v/k, \lambda')$ designs. Given a resolution class of the first design, we will say that two elements in different blocks have the *same position* if they belong to the same block of the corresponding resolution class of the $(v, v/k, \lambda')$ design. By orthogonality, this relation gives exactly k different positions in each block. Now on each block of the (v, k, λ) design we will construct, orthogonally, the two (k, h, μ) and $(k, \sigma k/h, \mu')$ RBIBDS, developing the same construction on each block, according to the elements' positions. More precisely if the positions are denoted by subscripted indices, the blocks of the (v, k, λ) design will be described as

$$\{a_1, a_2, \ldots, a_k\}, \{b_1, b_2, \ldots, b_k\}, \ldots$$

and if a resolution class of the (k, h, μ) design is

$$\{i_1^1, \ldots, i_h^1\}, \ldots, \{i_1^{\frac{k}{h}}, \ldots, i_h^{\frac{k}{h}}\}$$

the corresponding blocks in the $(v, h, \lambda\mu)$ design will be

$$\{a_{i_1^1}, \ldots, a_{i_h^1}\}, \ldots, \{a_{i_1^{\frac{k}{h}}}, \ldots, a_{i_h^{\frac{k}{h}}}\}, \{b_{i_1^1}, \ldots, b_{i_h^1}\}, \ldots, \{b_{i_1^{\frac{k}{h}}}, \ldots, b_{i_h^{\frac{k}{h}}}\}, \ldots$$

This clearly gives a $(v, h, \lambda\mu)$ RBIBD. We say that two elements in a resolution class of this latter RBIBD have the *same color* if the corresponding positions belong to the same block of the orthogonal $(k, \sigma k/h, \mu')$ RBIBD. Note that having the same position implies having the same color. We claim that by declaring as blocks the subsets of points in any resolution class that have the same color provides us with the σ-orthogonal $(v, \sigma v/h, \lambda'\mu')$ RBIBD. For every color each block of the $(v, h, \lambda\mu)$ RBIBD has σ elements with that color, by orthogonality. Therefore there are $\sigma v/h$ elements in each block. Moreover, given two elements a and b and a resolution class of the initial (v, k, λ) RBIBD then either

1. they have the same position in the given resolution class, and hence same color for all the resulting r' resolution classes,
   or

2. they have different positions, say i and j and therefore will have the same color in a resulting resolution class if and only if i and j belong to the same block of the $(k, \sigma k/h, \mu')$ design μ' times.

Case 1 occurs if and only if a and b belong to the same block of the $(v, v/k, \lambda')$ orthogonal design, thus λ' times, indicating that for these occurences a and b have the same color $r'\lambda'$ times. Case 2 above occurs the remaining $r - \lambda'$ times. Thus for these occurences a and b have the same color $\mu'(r - \lambda')$ times. Repeated application of the formulas in Remark 4.4 gives

$$r'\lambda' + \mu'(r - \lambda') = \lambda\mu \frac{\sigma v - h}{h(h - 1)}.$$

Consequently, the structure so described is a $(v, \sigma v/h, \lambda\mu \frac{\sigma v - h}{h(h-1)})$ RBIBD. Moreover the two resolutions described are σ orthogonal. □

As an application of the previous theorem one obtains the following corollary.

Corollary 5.2 *Suppose there exist a $(v, 4n, 1)$ RBIBD and a $(v, \frac{v}{4n}, \lambda)$ RBIBD that are 1-orthogonal. If there exists a $BWh(4n)$ then there exists a $BWh(v)$.*

Proof. As in Example 4.6 the existence of a BWh($4n$) gives us two 2-orthogonal $(4n, 4, 3)$ and $(4n, 2n, 2n - 1)$ RBIBDs, with the blocks of the $(4n, 2n, 2n - 1)$ being the NS and EW lines. We can then apply the previous theorem obtaining, via

the construction above, a $(v, 4, 3)$ design that is 2-orthogonal to a $(v, v/2, v/2-1)$ design. Note that the $(v, 4, 3)$ design is obtained by developing the BWh($4n$) on any block of the $(v, 4n, 1)$ design. Thus the whist conditions are satisfied. Indeed, any two players a and b are partners if and only if they are in the same block of the $(v, 4n, 1)$ design and they are partners in the BWh($4n$) that is developed on that block, that is, exactly once. Therefore, being a $(v, 4, 3)$ design, a and b must be opponents exactly twice. Moreover, having the same color is equivalent to being in the same line and thus the blocks of the $(v, v/2, v/2 - 1)$ design are precisely the NS and EW lines. $\square$

The materials of this section give rise to the following existence theorem.

Theorem 5.3 *There exists a BWh(v) for $v = 2^k$, with $k = 2m$, $k = 3m$ or $k = 5m$.*

Proof. Apply Corollary 5.2 together with Theorem 4.11. In the $k = 2m$ case, we have that the only Wh(4) is balanced. Thus, letting $q = 2^2$, $n = m$, $i = 1$ in Theorem 4.11 provides us with the orthogonal RBIBDs for ingredients in Corollary 5.2. Similarly, the $\mathbb{Z}_7$-cyclic BWh(8) given by

$$(\infty, 4, 0, 5), (1, 2, 3, 6)$$

together with $q = 2^3$, $n = m$, $i = 1$ in Theorem 4.11 establishes the $k = 3m$ case. Likewise the following $\mathbb{Z}_{31}$-cyclic BWh(32) provides us with the $k = 5m$ family of BWh:

$$(\infty, 28, 0, 20), (30, 23, 12, 3), (27, 9, 17, 25), (19, 13, 18, 1),$$
$$(29, 10, 24, 14), (11, 7, 8, 5), (22, 6, 16, 15), (2, 4, 26, 21).$$

$\square$

Despite Theorem 5.3 very little is known regarding the existence of BWh($4n$). It is worth noting that trying to apply Corollary 5.2 to a general $(v, 4n, 1)$ RBIBD has little chance of success apart from the $v = 2^k$ case. Indeed in Remark 4.4, point 4) we proved that if an orthogonal RBIBD exists, it must be affine. However the only affine designs known at the moment are some affine geometries and Hadamard designs. Since no Hadamard design can be 1-orthogonal to a $(v, 4n, 1)$, the prime power situation seems to be the only fruitful one. Even if one were to constrain $4n$ to be a power of 2, one would still need to construct a BWh(2^p) for any prime $p \geq 7$. Of course there is no reason to expect $4n$ to be so constrained. Indeed, BWh($4n$) exist when $4n$ is not a prime power, as we have seen in Example 4.6 for $4n = 20$. Another specific "non-prime" BWh($4n$) is given in the following example.

Example 5.4 *The first round of a $\mathbb{Z}_{23}$-cyclic BWh(24):*

$$(\infty, 1, 0, 14), (15, 8, 13, 17), (19, 9, 4, 12),$$

$$(18, 21, 7, 20), (2, 3, 6, 10), (5, 11, 22, 16).$$

The first unknown case is, then, for $n = 28$.

References

[1] R. J. R. Abel, N. J. Finizio, M. Greig, S. J. Lewis. Generalized Whist Tournament Designs. *Disc. Math.*, **268** (2003), 1–19.

[2] R. J. R. Abel, S. C. Furino. Resolvable and Near Resolvable Designs. *CRC handbook of combinatorial design*, **I.6**, 87–94.

[3] I. Anderson. Whist Tournaments. *CRC handbook of combinatorial design*, **IV.53**, 504–508.

[4] I. Anderson. Combinatorial Designs and Tournaments. Oxford University Press, Oxford, 1997.

[5] R.C. Bose, J.M. Cameron. The bridge tournament problem and calibration designs for comparing pairs of objects *J. Res. Nat. Bureau Standards* 69 B (1965), 323–332.

[6] C. J. Colbourn. Orthogonal Arrays of Index More Than One. *CRC handbook of combinatorial design*, **II.4**, 172–178.

[7] R. Craigen. Weighing Matrices and Conference Matrices. *CRC handbook of combinatorial design*, **IV.52**, 496–504.

[8] C. Godsil. Finite Geometry. http://quoll.uwaterloo.ca/pstuff/fgeom.pdf

[9] M. Greig, H. Haanpaa, P. Kaski. On the Coexistence of Conference Matrices and Near Resolvable $2 - (2k + 1, k, k - 1)$ Designs. Preprint

[10] H. Haanpaa. Classification of whist tournaments with up to 12 players. *Disc. Appl. Math.*, 129 (2003), 299–407.

[11] D.R. Huges, F.C. Piper. Design Theory. Cambridge University Press.

[12] E. H. Moore. Tactical Memoranda I – III. *Amer. J. Math.*, 18 (1896), 264–303.