Complete Simultaneous Metamorphosis of λ-fold Kite Systems

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Abstract

Let $(X, \mathcal{B})$ be a λ-fold G-decomposition of λH. Let G_i, $i = 1, \ldots, \mu$, be all nonisomorphic proper subgraphs of G without isolated vertices. Put $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$, where B_i is a subgraph of B isomorphic to G_i. A complete simultaneous metamorphosis of $(X, \mathcal{B})$ is a rearrangement, for each $i = 1, \ldots, \mu$, of the edges of $\bigcup_{B\in\mathcal{B}}(E(B) \setminus E(B_i))$ into a family $\mathcal{F}_i$ of copies of G_i with a leave L_i, such that $(X, \mathcal{B}_i \cup \mathcal{F}_i, L_i)$ is a maximum packing of λH with copies of G_i. In this paper, we give a complete answer to the existence problem of a λ-fold kite system having a complete simultaneous metamorphosis.

1 Preliminaries

Let G and H be simple finite graphs. A λ-fold *G-decomposition of λH* (λ copies of H) is a pair $(X, \mathcal{B})$ where X is the vertex set of H and $\mathcal{B}$ is a collection of graphs (blocks), each isomorphic to G, which partitions the edges of λH. If H is the complete graph K_n on n vertices then we also refer to this as a λ-fold *G-design* or *G-system of order n*. If $\lambda = 1$, we drop the term "1-fold".

The graph $K_n \setminus K_m$ has vertex set of size n containing a distinguished subset of size m; the edges set of $K_n \setminus K_m$ is the same as the edge set of K_n but with the edges between the m distinguished vertices removed. This graph is sometimes referred to as a complete graph of order n with a hole of size m.

A *packing* of λH with copies of G is a triple $(X, \mathcal{B}, L)$, where X is the vertex set of H, $\mathcal{B}$ is a collection of copies of G from the edge set of λH and L is the graph generated by the set of edges of λH not belonging to a graph of $\mathcal{B}$. The graph L is called the *leave*. If $|\mathcal{B}|$ is as large as possible, the packing $(X, \mathcal{B}, L)$

JCMCC 73 (2010), pp. 159-180

is said to be maximum ([7]). When the leave L is empty, a maximum packing of λH with copies of G coincides with a λ-fold G-decomposition of λH.

A *k-path* P_k, $k \geq 2$, is the graph $[a_1, a_2, \ldots, a_k]$ on vertices $a_1, \cdots, a_k$ and edges $\{a_i, a_{i+1}\}$, $i = 1, \ldots, k-1$.

A *k-cycle* C_k, $k \geq 3$ is the graph on vertices $a_1, a_2, \ldots, a_k$ with edges $\{a_1, a_2\}, \{a_2, a_3\}, \ldots, \{a_k, a_1\}$. A C_k will be denoted by any cyclic shift of $(a_1, a_2, \cdots, a_k)$ or $(a_k, a_{k-1}, \ldots, a_1)$. Note that $K_3 = C_3$.

A *k-star* S_k, $k \geq 2$, is the graph on vertices $a_0, a_1, \cdots, a_k$ and edges $\{a_0, a_i\}$, $i = 1, \ldots, k$. An S_k will be denoted by $[a_0; a_1, a_2, \ldots, a_k]$, where of course the order of the symbols $a_1, a_2, \ldots, a_k$ is arbitrary and a_0 is called the *center* of the star. Note that S_2 is isomorphic to P_3.

We denote by E_2 the graph on 4 vertices consisting of two disjoint edges.

Theorem 1.1. *[5] Table 1 shows the leaves of maximum packings of λK_n with triangles, where $\emptyset$ denotes the empty graph, G is a graph on n vertices of odd degrees and $(n+4)/2$ edges, D is a graph with 4 edges and even vertex degrees and a tripole is a graph consisting of $(n-4)/2$ disjoint edges and a 3-star:*

λ	$n \pmod 6$					
	0	1	2	3	4	5
=1	1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	C_4
$\equiv 0 \pmod 6$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
≥ 7 and $\equiv 1 \pmod 6$	1-factor	$\emptyset$	1-factor	$\emptyset$	tripole	D
$\equiv 2 \pmod 6$	$\emptyset$	$\emptyset$	$2P_2$	$\emptyset$	$\emptyset$	$2P_2$
$\equiv 3 \pmod 6$	1-factor	$\emptyset$	G	$\emptyset$	tripole	$\emptyset$
$\equiv 4 \pmod 6$	$\emptyset$	$\emptyset$	D	$\emptyset$	$\emptyset$	D
$\equiv 5 \pmod 6$	1-factor	$\emptyset$	tripole	$\emptyset$	tripole	$2P_2$

Table 1: Leaves of maximum packings of λK_n with triangles

It is not difficult to settle the maximum packings of λK_n with S_3s and P_4s.

Theorem 1.2. *The leaves of maximum packings of λK_n with S_3s (or with P_4s) are collections of $m = 0, 1, 2$ edges, with $m \equiv \lambda n(n-1)/2 \pmod 3$.*

A $K_3 + e$, or a *kite* is a simple graph on 4 vertices consisting of a triangle and a single edge (*tail*) sharing one common vertex (see Figure 1). We denote a kite by $(a, b, c) - d$ where (a, b, c) is the triangle having *base* $\{a, b\}$ and $\{c, d\}$ is the tail.

Theorem 1.3. *[6]A λ-fold kite-system of order n exists if and only if $\lambda n(n-1) \equiv 0 \pmod 8$, $n \geq 4$.*

Recently many authors are interested in studying the existence of kite-systems having added properties.

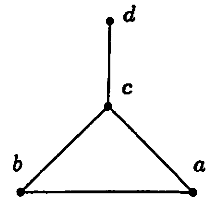

Figure 1: the kite (a,b,c)-d

Definition. Let $(X, \mathcal{B})$ be a λ-fold G-decomposition of λH. Let G_i, $i = 1, \ldots, \mu$, be non isomorphic proper subgraphs of G, each without isolated vertices. Put $\mathcal{B}_i = \{B_i \mid B \in \mathcal{B}\}$, where B_i is a subgraph of B isomorphic to G_i. A $\{G_1, G_2, \ldots, G_\mu\}$*-metamorphosis* of $(X, \mathcal{B})$ is a rearrangement, for each $i = 1, \ldots, \mu$, of the edges of $\bigcup_{B \in \mathcal{B}}(E(B) \setminus E(B_i))$ into a family $\mathcal{B}'_i$ of copies of G_i with a leave L_i, such that $(X, \mathcal{B}_i \cup \mathcal{B}'_i, L_i)$ is a maximum packing of λH with copies of G_i.

For $\mu = 1$, the above definition coincides with the definition of metamorphosis (see, for example, [2, 3, 6, 9, 10, 11, 12]). For this reason a $\{G_i, \ldots, G_\mu\}$-metamorphosis is a *simultaneous metamorphosis* introduced by P. Adams, E. Billington, E. S. Mahmoodian in [1].

We say that a $\{G_1, G_2, \ldots, G_\mu\}$-metamorphosis is *complete* if $\{G_i \mid i = 1, 2, \ldots, \mu\}$ coincides with the family of all nonisomorphic proper subgraphs of G without isolated vertices.

C.C. Lindner, G. Lo Faro and A. Tripodi [8] gave a complete answer to the existence problem of metamorphoses of a λ-fold kite system into a maximum packing of λK_n with triangles.

G. Lo Faro and A. Tripodi [13] gave also a complete answer to the existence problem of metamorphoses of a λ-fold kite system into a maximum packing of λK_n with P_4s.

The existence of metamorphoses of a λ-fold kite system into a maximum packing of λK_n with S_3s seems to be an open problem.

In this paper, we give a complete answer to the existence problem of a λ-fold kite system having a complete simultaneous metamorphosis. More precisely we prove the following

Main Theorem. *There exists a λ-fold kite system of order n having a complete simultaneous metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod 8$ and $(\lambda, n) \neq (1, 8)$. There is not a kite system of order 8 having an S_3-metamorphosis, but there is a kite system of order 8 having a $\{K_3, P_4, P_3, P_2, E_2\}$-metamorphosis.*

The most part of the paper is devoted to prove the existence of a λ-fold kite system $(X, \mathcal{B})$ having a $\{K_3, S_3, P_4\}$-metamorphosis. Section 2 is devoted to prove the existence of a λ-fold system having a $\{K_3, S_3, P_4\}$-metamorphosis. Section 3 is devoted to the proof of Main Theorem.

We will make use of the following well-known construction (we refer to [4] for further results, references and explanations not explicitly mentioned in this paper).

STANDARD WEIGHTING CONSTRUCTION. Suppose there exist:

1. an r-GDD of type $g_1^{u_1} g_2^{u_2} \dots g_h^{u_h}$;

2. a λ-fold kite system of order wg_i (or $1 + wg_i$), $i = 1, \dots, h$;

3. a λ-fold kite design of the complete r-partite graph K_w^r.

Then there is a λ-fold kite system of order $w(g_1 u_1 + \dots + g_h u_h)$ (or $1 + w(g_1 u_1 + \dots + g_h u_h)$).

Note that if the kite designs given as ingredients in the above construction have a Γ-metamorphosis with empty leaves and $\Gamma \subseteq \{K_3, S_3, P_4, P_3, P_2, E_2\}$ then the resulting kite system has a Γ-metamorphosis. Of course the same result is not always true if the leave of some metamorphosis of some ingredient is nonempty.

2 $\{K_3, S_3, P_4\}$-metamorphosis

Let $(X, \mathcal{B})$ be a λ-fold G-design having a $\{K_3, S_3, P_4\}$-metamorphosis. Then, basing on the definition of simultaneous metamorphosis, we have $\mu = 3$, $G_1 = K_3$, $G_2 = S_3$, and $G_3 = P_4$. In the following we put $\mathcal{T}, \mathcal{S}, \mathcal{P}$ instead of $\mathcal{B}'_1, \mathcal{B}'_2, \mathcal{B}'_3$ and $L_{\mathcal{T}}, L_{\mathcal{S}}, L_{\mathcal{P}}$ instead of L_1, L_2, L_3. If B is the kite $(a, b, c) - d$, we put $B_1 = (a, b, c)$, $B_2 = [c; a, b, d]$, $B_3 = [b, a, c, d]$. Moreover we omit to explicitely mention the empty leave(s), we write (a, b) instead of $\{a, b\}$ and $m(a, b)$ to denote the edge $\{a, b\}$ m times repeated. We use the subscript notation x_i to denote the ordered pair (x, i).

2.1 Kite systems

Lemma 2.1. *There is not an S_3-metamorphosis of a kite system of order 8.*

Proof Let $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}'_1, L)$ be an S_3-metamorphosis of a kite system $(\mathbb{Z}_8, \mathcal{B})$. Then $|\mathcal{B}| = 7$, so the two S_3s in $\mathcal{B}'_1$ cover 6 bases. Denote by i and j the centers of these stars. It is $i \neq j$, otherwise the vertex $i = j$ should appear as vertex of degree 2 in at least 6 kites. This is impossible. Let $\mathcal{I}$ and $\mathcal{J}$ be the sets of kites from which we picked up the bases of the stars with centers i and j, respectively. Being $|\mathcal{I}| = |\mathcal{J}| = 3$, there exists only one kite $B(i) \in \mathcal{B} \setminus \mathcal{I}$ meeting i and only one kite $B(j) \in \mathcal{B} \setminus \mathcal{J}$ meeting j. Moreover the degree of i in $B(i)$ and of j in $B(j)$ is 1. Let B be the kite of $\mathcal{B}$ covering the edge (i,j). If $B \in \mathcal{I}$ then there are at least 4 kites in $\mathcal{B}$ having j as a vertex of degree $d(j) \geq 2$. This is impossible. Analogously we obtain that $B \notin \mathcal{J}$. Then $\mathcal{B} \setminus (\mathcal{I} \cup \mathcal{J}) = \{B\}$ and $B = B(i) = B(j)$, a contradiction, because the degree of i in $B(i)$ and of j in $B(j)$ is 1. □

The following example gives a simultaneous metamorphosis of a kite system of order 8 into a maximum packing of K_8 with K_3s and P_4s and into a (not maximum) packing with S_3s having two 2-stars as leave.

Example 2.1. Let $X = \cup_{i=0}^{3}\{\alpha_i, \beta_i\}$ and let $\mathcal{B} = \{(\alpha_0, \alpha_1, \alpha_3) - \beta_1$, $(\beta_0, \beta_1, \alpha_2) - \alpha_1$, $(\beta_2, \alpha_2, \alpha_0) - \beta_0$, $(\beta_0, \alpha_1, \beta_3) - \beta_1$, $(\alpha_1, \beta_2, \beta_1) - \alpha_0$, $(\alpha_3, \alpha_2, \beta_3) - \alpha_0$, $(\beta_0, \alpha_3, \beta_2) - \beta_3\}$. Then $(X, \mathcal{B})$ is a kite system of order 8 having

- a K_3-metamorphosis with $\mathcal{T} = \{(\alpha_0, \beta_1, \beta_3)\}$ and $L_\mathcal{T} = \{(\beta_2, \beta_3), (\alpha_3, \beta_1), (\alpha_0, \beta_0), (\alpha_1, \alpha_2)\}$;
- a P_4-metamorphosis with $\mathcal{P} = \{[\beta_3, \alpha_1, \alpha_3, \beta_2], [\beta_3, \alpha_2, \beta_1, \beta_2]\}$ and $L_\mathcal{P} = (\alpha_0, \alpha_2)$.
- a metamorphosis into a packing of K_8 with S_3s such that $\mathcal{S} = \{[\beta_0; \alpha_3, \alpha_1, \beta_1]\}$ and $L_\mathcal{S} = \{[\alpha_1; \alpha_0, \beta_2], [\alpha_2; \alpha_3, \beta_2]\}$.

In order to handle the remaining cases we need the following example:

Example 2.2. Let $K_{4,4,4}$ be the complete tripartite graph with partition classes $V_1 = \{\alpha_0, \ldots, \alpha_3\}$, $V_2 = \{\beta_0, \ldots, \beta_3\}$ and $V_3 = \{\gamma_0, \ldots, \gamma_3\}$. Let $\mathcal{B} = \{(\gamma_3, \alpha_1, \beta_3) - \alpha_3, (\beta_1, \alpha_1, \gamma_1) - \alpha_3, (\alpha_1, \beta_2, \gamma_2) - \alpha_0, (\alpha_2, \beta_3, \gamma_2) - \beta_1, (\gamma_0, \alpha_2, \beta_1) - \alpha_0, (\alpha_2, \beta_2, \gamma_1) - \beta_3, (\gamma_2, \alpha_3, \beta_0) - \alpha_2, (\alpha_3, \beta_1, \gamma_3) - \alpha_2, (\beta_2, \alpha_3, \gamma_0) - \alpha_1, (\alpha_0, \gamma_1, \beta_0) - \alpha_1, (\alpha_0, \beta_2, \gamma_3) - \beta_0, (\alpha_0, \beta_3, \gamma_0) - \beta_0\}$. Then $(V_1 \cup V_2 \cup V_3, \mathcal{B})$ is a kite-decomposition of $K_{4,4,4}$ having a $\{K_3, S_3, P_4\}$-metamorphosis with

- $\mathcal{T} = \{(\alpha_3, \beta_3, \gamma_1), (\alpha_0, \beta_1, \gamma_2), (\alpha_1, \beta_0, \gamma_0), (\alpha_2, \beta_0, \gamma_3)\}$;
- $\mathcal{S} = \{[\alpha_1; \gamma_3, \beta_1, \beta_2], [\alpha_2; \gamma_0, \beta_3, \beta_2], [\alpha_3; \beta_1, \gamma_2, \beta_2], [\alpha_0; \gamma_1, \beta_3, \beta_2]\}$;
- $\mathcal{P} = \{[\alpha_2, \beta_1, \gamma_3, \beta_2], [\beta_3, \alpha_1, \gamma_1, \beta_2], [\beta_2, \gamma_2, \beta_3, \gamma_0], [\gamma_0, \alpha_3, \beta_0, \gamma_1]\}$.

Lemma 2.2. *For n=32,40,48,56,64,80 there exist kite systems of order n having a $\{K_3, S_3, P_4\}$-metamorphosis.*

Proof Suppose at first $n = 32, 40, 48, 56, 64$. The existence of a 3-GDD $(S, \mathcal{G}, \mathcal{U})$ of type 2^4, $2^3 4^1$, 2^6, $2^4 6^1$ and $2^3 4^1 6^1$ is well-known [4]. Apply the standard weighting construction by giving weight $w = 4$ and placing in each expanded block a copy of the kite-decomposition in Example 2.2 and in each expanded group a copy of the kite-designs in Examples 2.1, 4.1, 4.2, 4.3, 4.4. Starting from any 3-GDD, the result is a kite system of order n having a K_3-metamorphosis but not a $\{K_3, S_3, P_4\}$-metamorphosis: the kite systems induced by the expanded groups of size 8 cannot have an S_3-metamorphosis (see Lemma 2.1). Moreover the leaves produced in their P_4-metamorphoses don't share any vertex. Now we present a procedure that, starting from a suitable 3-GDD $(S, \mathcal{G}, \mathcal{U})$, shows how to rearrange the leaves and some blocks of $\mathcal{S} \cup \mathcal{P}$ in order to construct new S_3s and P_4s. We will write $\{\underline{a}, b\}$ if, inflating by 4 the group $\{a, b\} \in \mathcal{G}$, we apply Example 2.1 in order to produce an S_3-metamorphosis with leave $\{[a_1; a_0, b_2], [a_2; a_3, b_2]\}$ and a P_4-metamorphosis with leave $\{(a_0, a_2)\}$.

Step 1 (Building S_3s). Suppose that $(S, \mathcal{G}, \mathcal{U})$ contains 3 groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and 3 blocks (x, a, w), (x, c, y), (x, e, z) such that $x \notin \{a, b, c, d, e, f\}$. Using the standard weighting construction we produce (from above groups and blocks) the following S_2s and S_3s: $[a_1; a_0, b_2]$, $[a_2; a_3, b_2]$, $[c_1; c_0, d_2]$, $[c_2; c_3, d_2]$, $[e_1; e_0, f_2]$, $[e_2; e_3, f_2]$, $[x_1; w_3, a_1, a_2]$, $[x_1; y_3, c_1, c_2]$, $[x_1; z_3, e_1, e_2]$. It is easy to rearrange the edges of these stars to construct the following S_3s: $[a_1; a_0, b_2, x_1]$, $[a_2; a_3, b_2, x_1]$, $[c_1; c_0, d_2, x_1]$, $[c_2; c_3, d_2, x_1]$, $[e_1; e_0, f_2, x_1]$, $[e_2; e_3, f_2, x_1]$, $[x_1; y_3, z_3, w_3]$.

Step 2 (Building P_4s). Suppose that $(S, \mathcal{G}, \mathcal{U})$ contains 3 groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and 2 blocks (a, c, t), (e, u, t). Using the standard weighting construction we produce (from above groups and blocks) the following P_2s and P_4s: $(a_0, a_2), (c_0, c_2), (e_0, e_2)$, $[a_2, c_1, t_3, c_2]$ and $[e_2, u_1, t_3, u_2]$. It is easy to rearrange the edges of these paths to construct the following P_4s: $[a_0, a_2, c_1, t_3]$, $[e_0, e_2, u_1, t_3]$, $[c_0, c_2, t_3, u_2]$.

Case $n = 32$. Take the 3-GDD of type 2^4 with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{\underline{h}, g\}\}$ and blocks $\mathcal{U} = \{(g, a, d), (g, c, f), (g, e, b), (b, h, c), (e, d, h), (b, d, f), (c, e, a), (h, f, a)\}$.

Apply Step 1 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (g, a, d), (g, c, f), (g, e, b). To complete the S_3-metamorphosis take $[h_1; h_0, g_2]$, $[h_2; h_3, g_2]$, $[b_1; c_3, h_1, h_2]$ and form the stars $[h_1; h_0, g_2, b_1]$, $[h_2; h_3, g_2, b_1]$ and leave $\{(b_1, c_3)\}$.

Apply Step 2 to groups $\{\underline{c}, d\}$, $\{\underline{e}, f\}$, $\{\underline{h}, g\}$ and blocks (c, e, a), (h, f, a). The result is the required P_4-metamorphosis having leave (a_0, a_2).

Case $n = 40$. Take the 3-GDD of type $2^4 4^1$ with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{c}, d\}, \{\underline{e}, f\}, \{x, y, z, t\}\}$ and $\mathcal{U} = \{(x, a, d), (x, c, f), (x, e, b), (a, c, z), (e, d, z), (f, b, z), (y, a, f), (y, b, d), (y, e, c), (e, d, z), (t, a, e), (t, b, c), (t, f, d)\}$.

Apply Step 1 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (x, a, d), (x, c, f), (x, e, b).

Apply Step 2 to groups $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$ and blocks (a, c, z), (e, d, z).

Case $n = 48$. Take the 3-GDD of type 2^6 with $\mathcal{G} = \{\{\underline{a}, b\}, \{\underline{e}, f\}, \{\underline{c}, d\}, \{\underline{g}, h\}, \{\underline{m}, p\}, \{\underline{q}, r\}\}$ and $\mathcal{U} = \{(p, a, f), (p, e, h), (p, c, g), (a, e, r), (c, f, r), (a, d, h), (a, c, m), (a, g, q), (b, d, f), (b, c, h), (b, m, q), (b, r, p), (d, p, q), (d, r, g), (h, m, r), (e, g, b), (e, m, d), (e, q, c), (g, m, f), (q, h, f)\}$.

Apply Step 1 to the following sets of groups and blocks:

- $\{\underline{a},b\}$, $\{\underline{e},f\}$, $\{\underline{c},d\}$, (p,a,f), (p,e,h), (p,c,g);
- $\{\underline{g},h\}$, $\{\underline{m},p\}$, $\{\underline{q},r\}$, (e,g,b), (e,m,d), (e,q,c).

Apply Step 2 to the following sets of groups and blocks:

- $\{\underline{a},b\}$, $\{\underline{e},f\}$, $\{\underline{c},d\}$, (a,e,r), (c,f,r);
- $\{\underline{g},h\}$, $\{\underline{m},p\}$, $\{\underline{q},r\}$, (g,m,f), (q,h,f).

Case $n=56$. Take the 3-GDD of type $2^4 6^1$ with $\mathcal{G}=\{\{\underline{a},b\}, \{\underline{c},d\}, \{\underline{e},f\}, \{\underline{g},h\}, \{1,2,3,4,5,6\}\}$ and $\mathcal{U}=\{(1,a,d), (1,c,h), (1,e,b), (a,c,3), (e,h,3), (1,g,f)$ $(2,a,e), (2,b,d), (2,c,g), (2,f,h), (3,d,g), (3,b,f), (4,a,f), (4,c,e), (4,b,g), (4,d,h)$ $(5,a,g), (5,c,f), (5,d,e), (5,b,h), (6,a,h), (6,b,c), (6,e,g), (6,d,f)\}$.

Apply Step 1 to the groups $\{\underline{a},b\}$, $\{\underline{c},d\}$, $\{\underline{e},f\}$ and blocks $(1,a,d)$, $(1,c,h)$, $(1,e,b)$. To complete the S_3-metamorphosis take $[g_1; g_0, h_2]$,
$[g_2; g_3, h_2]$, $[1_1; f_3, g_1, g_2]$ and form the stars $[g_1; g_0, h_2, 1_1]$, $[g_2; g_3, h_2, 1_1]$ and the leave $\{(1_1, f_3)\}$.

Apply Step 2 to the groups $\{\underline{a},b\}$, $\{\underline{c},d\}$, $\{\underline{e},f\}$ and blocks $(a,c,3)$, $(e,h,3)$. The result is a P_4-metamorphosis having leave $\{(g_0,g_2)\}$.

Case $n=64$. Take the 3-GDD of type $2^3 4^1 6^1$ with $\mathcal{G}=\{\{\underline{a},b\}, \{\underline{d},c\}, \{\underline{f},e\}, \{x,y,z,t\}, \{1,2,3,4,5,6\}\}$ and $\mathcal{U}=\{(1,a,x), (1,d,y), (1,f,z), (a,d,6), (f,t,6), (1,c,b), (1,e,t), (2,x,e), (2,y,a), (2,z,b), (2,t,d), (2,c,f)$ $(3,x,d), (3,y,e), (3,z,c), (3,t,b), (3,a,f)$ $(4,x,f), (4,y,b), (4,z,a), (4,t,c), (4,d,e), (5,x,c), (5,y,f), (5,z,d), (5,t,a), (5,b,e), (6,x,b), (6,y,c), (6,z,e), (a,c,e), (b,d,f)\}$.

Apply Step 1 to the groups $\{\underline{a},b\}$, $\{\underline{d},c\}$, $\{\underline{f},e\}$, and blocks $(1,a,x)$, $(1,d,y)$, $(1,f,z)$.

Apply Step 2 to the groups $\{\underline{a},b\}$, $\{\underline{d},c\}$, $\{\underline{f},e\}$, and blocks $(a,d,6)$, $(f,t,6)$.

To complete the proof we prove the case $n=80$. We can proceed as above by applying the standard weighting construction to the 3-GDD of type $2^6 8^1$ with $\mathcal{G}=\{\{\underline{a},b\}, \{\underline{c},d\}, \{\underline{e},f\}, \{\underline{g},h\}, \{\underline{n},m\}, \{\underline{p},q\}, \{1,2,3,4,5,6,7,8\}\}$ and $\mathcal{U}=\{(1,a,d), (1,c,h), (1,e,q), (a,c,3), (e,b,3), (2,e,n), (2,f,p), (2,m,q), (2,d,h), (2,b,c), (2,a,g), (3,d,f), (3,g,m), (3,n,q), (3,h,p), (4,e,m), (4,n,p), (4,a,h), (4,b,d), (4,c,g), (4,f,q), (5,d,m), (5,c,e), (5,a,q), (5,g,p), (5,b,f), (5,h,n), (6,a,f), (6,c,n), (6,d,e), (6,g,q), (6,b,p), (6,h,m), (7,a,m), (7,c,p), (7,d,q), (7,e,g), (7,b,n), (7,f,h), (8,a,n), (8,c,q), (8,e,h), (8,f,g), (8,b,m), (c,f,m), (a,e,p), (b,h,q), (1,g,b), (1,n,f), (1,p,m), (g,n,d), (p,8,d)\}$.

Apply Step 1 to the following sets of groups and blocks:

- $\{\underline{a},b\}$, $\{\underline{c},d\}$, $\{\underline{e},f\}$, $(1,a,d)$, $(1,c,h)$, $(1,e,q)$;
- $\{\underline{g},h\}$, $\{\underline{n},m\}$, $\{\underline{p},q\}$, $(1,g,b)$, $(1,n,f)$, $(1,p,m)$.

Apply Step 2 to the following sets of groups and blocks:

- $\{\underline{a}, b\}$, $\{\underline{c}, d\}$, $\{\underline{e}, f\}$, $(a, c, 3)$, $(e, b, 3)$;
- $\{\underline{g}, h\}$, $\{\underline{n}, m\}$, $\{\underline{p}, q\}$, (g, n, d), $(p, 8, d)$.

□

Theorem 2.3. *There exists a kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 8$, $n \geq 9$. There exists a kite system of order 8 having a $\{K_3, P_4\}$-metamorphosis.*

Proof The proof of the necessary part is in Theorem 1.3, so we prove only the sufficient part. The proof for $n = 8, 9, 16, 17, 24, 32, 40, 48, 56, 64, 80$ follows from Examples 2.1, 4.1, 4.2, 4.3 and Lemma 2.2. For the remaining $n \geq 25$, apply the standard weighting construction by giving weight $w = 4$ to a 3-GDD as shown in Table 2. L_T, L_S, L_P are obtained by joining the leaves from the metamorphoses on each expanded group. □

n	k	3-GDD of type	L_T	L_S, L_P
24k	≥ 3	6^k	1-factor	$\emptyset$
24k+1	≥ 1	2^{3k}	$\emptyset$	$\emptyset$
24k+8	≥ 4	$6^{k-1}8$	1-factor	P_2
24k+9	≥ 1	2^{3k+1}	$\emptyset$	$\emptyset$
24k+16	≥ 3	6^k4	tripole	$\emptyset$
24k+17	≥ 1	$2^{3k}4$	C_4	P_2

Table 2: $\lambda = 1$ ($\emptyset$ denotes the empty graph)

Remark 2.1. Note that in Theorem 2.3 we have:

- for $n \equiv 8 \pmod{24}$, $n \geq 32$, L_T is an 1-factor which contains the edge (a_0, a_1), $L_S = (b_1, c_3)$ and $L_P = (a_0, a_2)$;
- for $n \equiv 17 \pmod{24}$, $L_T = (1, 2, 3, 16)$, $L_S = (2, 16)$, $L_P = (8, 16)$.

2.2 2-fold kite systems

Example 2.3. Let $2K_{2,2,2}$ be two copies of the complete tripartite graph with partition classes $V_1 = \{a_0, a_1\}$, $V_2 = \{b_0, b_1\}$ and $V_3 = \{c_0, c_1\}$. Let $\mathcal{B} = \{(c_0, a_0, b_1) - c_1, (b_0, a_0, c_1) - a_1, (a_0, c_1, b_1) - a_1, (b_0, a_0, c_0) - a_1, (b_0, a_1, c_0) - b_1, (b_0, c_1, a_1) - b_1\}$. Then $(V_1 \cup V_2 \cup V_3, \mathcal{B})$ is a 2-fold kite-decomposition of $2K_{2,2,2}$ having a $\{K_3, S_3, P_4\}$-metamorphosis with:

- $\mathcal{T} = \{(a_1, b_1, c_1), (a_1, b_1, c_0)\}$;
- $\mathcal{S} = \{[a_0; b_0, c_1, c_0], [b_0; a_0, a_1, c_1]\}$;
- $\mathcal{P} = \{[c_0, a_0, b_1, c_1], [a_0, c_1, a_1, c_0]\}$.

Theorem 2.4. *There exists a 2-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 4$, $n \geq 4$.*

Proof The proof of the necessary part is in Theorem 1.3, so we prove only the sufficient part. The proof for $n = 4, 5, 8, 9$ follows from Examples 4.5, 4.6, 4.7, 4.8.

Let $n \equiv 0 \pmod 4$, $n \geq 12$. Put $n = 4k$. Let $(S, \mathcal{G}, \mathcal{U})$ be a 3-GDD of type 2^k if $k \equiv 0, 1 \pmod 3$, $2^{k-2}4^1$ if $k \equiv 2 \pmod 3$. Apply the standard weighting construction by giving weight $w = 2$. By Examples 2.3, 4.5 and 4.7 we obtain the proof.

Let $n \equiv 1 \pmod 4$, $n \geq 13$. Put $n = 1+4k$. Let $(S, \mathcal{G}, \mathcal{U})$ be a 3-GDD of type 2^k if $k \equiv 0, 1 \pmod 3$, $2^{k-2}4$ if $k \equiv 2 \pmod 3$ having groups $G_1 = \{1, 2\}, G_2 = \{3, 4\}, \dots, G_k = \{2k-1, 2k\}$ or $G_1 = \{1, 2\}, G_2 = \{3, 4\}, \dots, G_{k-1} = \{2k-3, 2k-2, 2k-1, 2k\}$, respectively. Let $X = \{\infty\} \cup (S \times \mathbb{Z}_2)$, then $|X| = 4k+1 = n$. We define a 2-fold kite system as follows:

1. For each G_i, let $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ be a copy of the 2-fold kite-system in Example 4.6 obtained by renaming its vertices as follows: $0 \to \infty, 1 \to (2i-1)_0, 2 \to (2i)_0, 3 \to (2i-1)_1, 4 \to (2i)_1$, with $1 \leq i \leq k$ if $k \equiv 0, 1 \pmod 3$ and $1 \leq i \leq k-2$ if $k \equiv 2 \pmod 3$; in the latter case, for $i = k-1$, take a copy of the system in Example 4.8 by renaming its vertices as follows: $j \to (2k-4+j)_0$, if $1 \leq j \leq 4$, $j \to (2k-8+j)_1$, if $5 \leq j \leq 8$, $0 \to \infty$.

2. For each $U = (a, b, c) \in \mathcal{U}$, let $((a \times \mathbb{Z}_2) \cup (b \times \mathbb{Z}_2) \cup (c \times \mathbb{Z}_2), \mathcal{B}_U)$ be the 2-fold kite-system, given in Example 2.3.

Let $\mathcal{B} = (\bigcup_{G \in \mathcal{G}} \mathcal{B}_G) \cup (\bigcup_{U \in \mathcal{U}} \mathcal{B}_U)$. Then $(X, \mathcal{B})$ is a 2-fold kite system of order n. Now we show that

- $(X, \mathcal{B})$ has a K_3-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a K_3-metamorphosis whose leave L_T^i is $\{2((2i-1)_1, (2i)_1)\}$ (the empty set) if the size of the starting group G_i is 2 (4 respectively). The set of tails of the blocks of the 2-fold kite system placed on $(a, b, c) \in \mathcal{U}$ is $\{2(a_1, b_1), (a_1, c_1), (b_1, c_1), (a_1, c_0), (b_1, c_0)\}$. Using three of them construct the triangle (a_1, b_1, c_0). The edges (a_1, b_1), (a_1, c_1), (b_1, c_1) can be assembled with $\bigcup L_T^i$ as follows:

  1. if $k \equiv 0 \pmod 3$, $(\bigcup_{i=1}^k L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\}) = \{(1_1, 2_1), (3_1, 4_1), \dots, ((2k-1)_1, (2k)_1)\} \cup K_{2k}$, where K_{2k} is the complete graph on vertex set $S \times \{1\}$. Since an 1-factor is the padding of a minimum covering with triangles of order $2k \equiv 0 \pmod 6$ (see [7]), there exists a K_3-decomposition of $(\bigcup_{i=1}^k L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\})$ with empty leave;
  2. if $k \equiv 1 \pmod 3$, $(\bigcup_{i=1}^k L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\}) = \{((2i-1)_1, (2i)_1, (2k)_1), ((2i-1)_1, (2i)_1, (2k-1)_1) \mid 1 \leq i \leq k-1\} \cup \{2((2k-1)_1, (2k)_1)\} \cup K_2^{k-1}$, where K_2^{k-1} is the complete $(k-1)$-partite graph with partition classes $G_1, G_2, \dots, G_{k-1}$. Since there exists a 3-GDD

of type 2^{k-1} (see [4]), with $k \equiv 1 \pmod 3$, $(\bigcup_{i=1}^{k} L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\})$ is decomposable into triangles with leave $\{2((2k-1)_1, (2k)_1)\}$;

3. if $k \equiv 2 \pmod 3$, $(\bigcup_{i=1}^{k-2} L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\}) = \{((2i-1)_1, (2i)_1, (2k)_1), ((2i-1)_1, (2i)_1, (2k-1)_1) \mid 1 \leq i \leq k-2\} \cup K_2^{k-1}$, where K_2^{k-1} is the complete $(k-1)$-partite graph with partition classes $G_1, G_2, \dots, G_{k-2}, \{(2k-3)_1, (2k-2)_1\}$. Since there exist a 3-GDD of type 2^{k-1} (see [4]), with $k \equiv 2 \pmod 3$, $(\bigcup_{i=1}^{k-1} L_T^i) \cup (\bigcup_{U \in \mathcal{U}} U \times \{1\})$ is decomposable into triangles with empty leave.

- $(X, \mathcal{B})$ has an S_3-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a S_3-metamorphosis whose leave L_S^i is $\{(\infty, (2i-1)_0), (\infty, (2i)_0)\}$, for $1 \leq i \leq k$ or $1 \leq i \leq k-2$ (if $k \equiv 2 \pmod 3$). These edges can be assembled into 3-stars $[\infty; i_0, (i+1)_0, (i+2)_0]$ with $i \equiv 1 \pmod 3$. The leave is empty if $k \equiv 0, 2 \pmod 3$ and $\{(\infty, (2k-1)_0), (\infty, (2k)_0)\}$ if $k \equiv 1 \pmod 3$.

- $(X, \mathcal{B})$ has a P_4-metamorphosis. To prove it note that $(\{\infty\} \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ has a P_4-metamorphosis whose leave L_P^i is $\{(\infty, (2i)_1), ((2i)_0, (2i-1)_1)\}$, for $1 \leq i \leq k$ or $1 \leq i \leq k-2$ (if $k \equiv 2 \pmod 3$). For each i, with $i \neq k$ if $k \equiv 1 \pmod 3$, remove the path $[(2i)_0, (2i)_1, \infty, (2i-1)_1]$ and let Γ the set of edges covered by these paths and by $\bigcup L_P^i$. Construct the following paths, for $i \equiv 1 \pmod 3$: $[(2i)_0, (2i)_1, \infty, (2i+4)_1]$, $[(2i+2)_0, (2i+2)_1, \infty, (2i)_1]$, $[(2i+3)_1, (2i+4)_0, (2i+4)_1, \infty]$, $[(2i)_0, (2i-1)_1, \infty, (2i+3)_1]$, $[(2i+2)_0, (2i+1)_1, \infty, (2i+2)_1]$. The above paths cover all edges in Γ if $k \equiv 0, 2 \pmod 3$ and all edges in $\Gamma \setminus \{(\infty, (2k)_1), ((2k)_0, (2k-1)_1)\}$ if $k \equiv 1 \pmod 3$. It follows that $L_P = \emptyset$ if $k \equiv 0, 2 \pmod 3$ and $L_P = \{(\infty, (2k)_1), ((2k)_0, (2k-1)_1)\}$ if $k \equiv 1 \pmod 3$.

□

Remark 2.2. Note that the nonemty leaves of the 2-fold kite systems having a $\{K_3, S_3, P_4\}$-metamorphosis constructed in this section are as follows:

- if $n \equiv 5 \pmod{12}$, $L_T = 2(a, b)$, $L_S = [c; e, d]$, $L_P = \{(c, b), (a, e)\}$;
- if $n \equiv 8 \pmod{12}$, $L_T = 2(a, b)$, $L_S = [c; e, d]$, $L_P = \{(c, b), (f, e)\}$.

2.3 3-fold kite systems

Theorem 2.5. *There exists a 3-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \equiv 0, 1 \pmod 8$, $n \geq 8$.*

Proof The necessary part is in Theorem 1.3. The sufficiency for $n = 8$ is given in Example 4.9. Now construct on the same set X of size $n \geq 9$ a copy $(X, \mathcal{B}_1)$ of the kite system given in Section 2.1 and a copy $(X, \mathcal{B}_2)$ of the 2-fold kite system given in Section 2.2. It is clear that $(X, \mathcal{B}_1)$ and $(X, \mathcal{B}_2)$ have a $\{K_3, S_3, P_4\}$-metamorphosis. Denote by $\mathcal{T}^i, \mathcal{S}^i, \mathcal{P}^i, L_T^i, L_S^i, L_P^i$ the sets $\mathcal{T}, \mathcal{S}, \mathcal{P}, L_T, L_S, L_P$ corresponding to $(X, \mathcal{B}_i)$, $i = 1, 2$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 3-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis. To prove this it is sufficient to put $\mathcal{T} = \mathcal{T}^1 \cup \mathcal{T}^2$, $\mathcal{S} = \mathcal{S}^1 \cup \mathcal{S}^2$, $\mathcal{P} = \mathcal{P}^1 \cup \mathcal{P}^2$ and:

- for $n \equiv 1, 9 \pmod{24}$, $n \geq 9$, $L_T^1 = L_T^2 = L_S^1 = L_S^2 = L_P^1 = L_P^2 = \emptyset$. Then $L_T = L_S = L_P = \emptyset$;
- for $n \equiv 0 \pmod{24}$, $n \geq 24$, $L_S^1 = L_P^1 = L_S^2 = L_P^2 = \emptyset$, $L_T^1 =$ 1-factor, $L_T^2 = \emptyset$. Then $L_S = L_P = \emptyset$, $L_T = L_T^1$ is an 1-factor;
- for $n \equiv 16 \pmod{24}$, $L_S^1 = L_P^1 = L_S^2 = L_P^2 = \emptyset$, L_T^1 is a tripole and $L_T^2 = \emptyset$. Then $L_S = L_P = \emptyset$ and L_T is a tripole;
- for $n \equiv 17 \pmod{24}$, by Remarks 2.1 and 2.2 the leaves are of the type: $L_T^1 = \{(1,2,3,16)\}$, $L_S^1 = \{(2,16)\}$, $L_P^1 = \{(16,8)\}$, $L_T^2 = \{2(a,b)\}$, $L_S^2 = \{[d,c,e]\}$, $L_P^2 = \{(c,b),(a,e)\}$. Construct the required 2-fold kite system by renaming c,a,b,d,e as follows: $c \to 16, a \to 1, b \to 3, d \to 0, e \to 8$. The leaves can be reassembled into the triangles $(1,2,3)$, $(1,3,16)$, the star $[16;2,0,8]$, the path $[3,16,8,1]$. Then $L_T = L_S = L_P = \emptyset$;
- for $n \equiv 8 \pmod{24}$, $n \geq 32$, by Remarks 2.1 and 2.2 the leaves are of the type: $L_T^1 =$ 1-factor containing the edge (a_0, a_1), $L_S^1 = \{(b_1, c_3)\}$, $L_P^1 = \{(a_0, a_2)\}$, $L_T^2 = \{2(a,b)\}$, $L_S^2 = \{(c,d),(c,e)\}$, $L_P^2 = \{(c,b),(f,e)\}$. Construct the required kite system by renaming b_1, c_3, a_1, a_0, a_2 as follows: $b_1 \to c$, $c_3 \to f$, $a_1 \to a$, $a_0 \to b$, $a_2 \to e$. The leaves can be assembled into the star $[c;d,e,f]$ and the path $[c,b,e,f]$. Then $L_S = L_P = \emptyset$, L_T contains the 3-times repeated edge $3(a,b)$ and an 1-factor on the vertices $X \setminus \{a,b\}$.

□

2.4 4-fold kite systems

Example 2.4 ($4(K_6 \setminus K_2)$). Let $X = \{\infty_1, \infty_2, 0, 1, 2, 3\}$, $\mathcal{B} = \{(1,2,\infty_1) - 3$, $(2,3,\infty_1) - 1$, $(0,3,\infty_2) - 2$, $(1,\infty_1,0) - \infty_2$, $(\infty_2,1,2) - 3$, $(\infty_1,0,3) - \infty_2$, $(\infty_2,0,1) - 3$, $(3,1,\infty_1) - 2$, $(3,\infty_2,2) - 0$, $(2,\infty_2,0) - \infty_1$, $(3,\infty_2,1) - 2$, $(2,\infty_1,0) - 3$, $(1,0,2) - 3$, $(0,3,1) - \infty_2\}$. Then $(X, \mathcal{B})$ is a 4-fold kite-system of order 6 with hole $\{\infty_1, \infty_2\}$ having:

- a K_3-metamorphosis with $\mathcal{T} = \{(\infty_1,1,2), (\infty_1,0,3), (\infty_2,0,2), (\infty_2,1,3)\}$ and leave $\{(2,3),(2,3)\}$;
- an S_3-metamorphosis with $\mathcal{S} = \{[1;\infty_2,2,3], [\infty_2;0,2,3], [3;0,2,\infty_2], [0;1,\infty_1,3]\}$ and leave $\{(\infty_1,1), (\infty_1,2)\}$;
- a P_4-metamorphosis with $\mathcal{P} = \{[0,\infty_1,2,1], [\infty_1,0,3,1], [\infty_2,1,\infty_1,3], [3,\infty_2,2,0]\}$ and leave $\{(0,\infty_2),(0,1)\}$.

Example 2.5 ($4(K_7 \setminus K_3)$). Let $X = \{\infty_1, \infty_2, \infty_3, 4, 5, 6, 7\}$, $\mathcal{B} = \{(\infty_1,7,4) - \infty_2$, $(\infty_1,5,6) - \infty_2$, $(5,\infty_2,7) - \infty_1$, $(6,\infty_3,7) - 5$, $(5,\infty_3,4) - 6$, $(7,\infty_2,4) - \infty_1$, $(5,\infty_2,6) - \infty_1$, $(5,7,\infty_1) - 4$, $(6,7,\infty_3) - 4$, $(\infty_3,5,4) - 7$, $(7,\infty_1,4) - 6$, $(\infty_1,5,6) - \infty_3$, $(5,\infty_3,7) - \infty_2$, $(6,7,\infty_2) - 5$, $(5,\infty_2,4) - 6$, $(5,\infty_3,4) - \infty_2$, $(5,\infty_1,6) - \infty_2$, $(\infty_3,7,6) - 4\}$. Then $(X, \mathcal{B})$ is a 4-fold kite-system of order 7 with hole $\{\infty_1, \infty_2, \infty_3\}$ having:

- a K_3-metamorphosis with $\mathcal{T} = \{(\infty_2,4,6), (\infty_2,4,6), (\infty_1,4,6), (\infty_3,4,6), (\infty_1,4,7), (\infty_2,5,7)\}$;
- an S_3-metamorphosis with $\mathcal{S} = \{[7;\infty_1,5,6], [7;\infty_1,\infty_2,6], [5;\infty_1,\infty_2,\infty_3], [5;\infty_1,\infty_2,\infty_3], [5;\infty_1,\infty_2,\infty_3], [\infty_3;5,6,7]\}$;
- a P_4-metamorphosis with $\mathcal{P} = \{[4,7,\infty_2,6], [\infty_2,4,\infty_3,7], [\infty_2,4,\infty_3,7], [6,5,4,\infty_1], [6,\infty_1,7,\infty_3], [\infty_2,7,6,5]\}$.

Theorem 2.6. *For every $n \geq 4$ there exists a 4-fold kite system having a $\{K_3,S_3,P_4\}$-metamorphosis. Moreover for $n \equiv 2,5 \pmod 6$ L_T is either a 4-cicle or a $2P_3$.*

Proof If $n \equiv 0,1 \pmod 4$, let $(X,\mathcal{B})$ be the 2-fold kite system of order n, having a $\{K_3,S_3,P_4\}$-metamorphosis, constructed in Section 2.2. By Remark 2.2, for $n \equiv 5,8 \pmod{12}$ it is $L_T^1 = \{2(a,b)\}$, $L_S^1 = \{(c,d),(c,e)\}$, $L_P^1 = \{(c,b),(e,f)\}$ with $|\{a,b,c,d,e\}| = 5$ and $|\{b,c,e,f\}| = 4$ (note that for $n \equiv 5 \pmod{12}$ in L_P^1 it is $f = a$). Let $\mathcal{B}'$ be the block set obtained by changing b with e in each block of $\mathcal{B}$. Then $(X,\mathcal{B}')$ is a 2-fold kite system having a $\{K_3,S_3,P_4\}$-metamorphosis with empty leaves or, for $n \equiv 5,8 \pmod{12}$, $L_T^2 = \{2(a,e)\}$, $L_S^2 = \{(c,d),(c,b)\}$, $L_P^2 = \{(c,e),(b,f)\}$. Then $(X,\mathcal{B}\cup\mathcal{B}')$ is a 4-fold kite system of order n having a $\{K_3,S_3,P_4\}$-metamorphosis with empty leaves or, for $n \equiv 5,8 \pmod{12}$, $L_T = \{2[b,a,e]\}$, $L_S = \{(c,d)\}$, $L_P = \{(b,f)\}$ and $[c;b,e,d] \in \mathcal{S}$, $[f,e,c,b] \in \mathcal{P}$.

If $n \equiv 2,3 \pmod 4$, $n = 4k+s$, $k \geq 3$ and $s \in \{2,3\}$, let $S = \{1,2,\ldots,2k\}$, $R_s = \{\infty_1,\ldots,\infty_s\}$ and $(S,\mathcal{G},\mathcal{U})$ a 3-GDD of type 2^k(if $k \equiv 0,1 \pmod 3$) or $2^{k-2}4$ (if $k \equiv 2 \pmod 3$), with groups $G_1 = \{1,2\}, G_2 = \{3,4\},\ldots,G_k = \{2k-1,2k\}$ or $G_1 = \{1,2,3,4\}, G_2 = \{5,6\},\ldots,G_{k-1} = \{2k-1,2k\}$, respectively. Set $X = R_s \cup (S \times \mathbb{Z}_2)$ and define a collection $\mathcal{B}$ of kites as follows:

1. Let $(R_s \cup (G_1 \times \mathbb{Z}_2), \mathcal{B}_{G_1})$ be a copy of the 4-fold kite system of order $2|G_1| + s$ given in Examples 4.11,4.12,4.14,4.15 having a $\{K_3,S_3,P_4\}$-metamorphosis with leaves L_T^1, L_S^1, L_P^1; put $\mathcal{B}_{G_1} \subseteq \mathcal{B}$.

2. For every $U = (x,y,z) \in \mathcal{U}$, let $(S_U,\mathcal{B}_U)$ be a copy of the $2K_{2,2,2}$ kite-decomposition of Example 2.3; put $2\mathcal{B}_U \subseteq \mathcal{B}$.

3. For every $G_i \in \mathcal{G}$, with $i > 1$, construct a 4-fold kite system $(R_s \cup (G_i \times \mathbb{Z}_2), \mathcal{B}_{G_i})$ of order $2|G_i| + s$ with a hole of size s by taking a copy of the designs in Examples 2.4, 2.5 and renaming the vertices $0,1,2,3$ of Example 2.4 and $4,5,6,7$ of Example 2.5 as follows:

   - if $s = 2, k \equiv 0,1 \pmod 3$: $0 \to (2i-1)_0$, $1 \to (2i)_0$, $2 \to (2i-1)_1$, $3 \to (2i)_1$;
   - if $s = 2, k \equiv 2 \pmod 3$: $0 \to (2i+1)_0$, $1 \to (2i+2)_0$, $2 \to (2i+1)_1$, $3 \to (2i+2)_1$;
   - if $s = 3, k \equiv 0,1 \pmod 3$: $4 \to (2i-1)_1$, $5 \to (2i)_1$, $6 \to (2i-1)_0$, $7 \to (2i)_0$;

- if $s = 3, k \equiv 2 \pmod 3$: $4 \to (2i+1)_1$, $5 \to (2i+2)_1$, $6 \to (2i+1)_0$, $7 \to (2i+2)_0$.

Denote the leaves by L_T^i, L_S^i, L_P^i. Put $\mathcal{B}_{G_i} \subseteq \mathcal{B}$. Then $(X, \mathcal{B})$ is a 4-fold kite system of order $4k+s$. The metamorphoses are obtained as follows: apply the metamorphoses showed in steps 1. 2. 3. to the blocks of $\mathcal{B}_{G_1}, 2\mathcal{B}_U$ and $\mathcal{B}_{G_i}, i > 1$, respectively. In order to complete our metamorphoses and so to obtain the leaves L_T, L_S, L_P, proceed as follows:

- For $s = 3$ and $k \equiv 0, 1 \pmod 3$, we have $L_T^i = L_S^i = L_P^i = \emptyset$, for all $G_i \in \mathcal{G}$. Then $L_T = L_S = L_P = \emptyset$.
- For $s = 3$ and $k \equiv 2 \pmod 3$, we have $L_T^i = L_S^i = L_P^i = \emptyset$, for $i > 1$, then $L_T = L_T^1, L_S = L_S^1, L_P = L_P^1$.
- For $s = 2$ and $k \equiv 0, 1 \pmod 3$, we have $L_T^1 = L_S^1 = L_P^1 = \emptyset$. Moreover
  - in the K_3-metamorphosis, it is $\bigcup_{i=2}^k L_T^i = \{2(3_1, 4_1), 2(5_1, 6_1), \ldots, 2((2k-1)_1, (2k)_1)\}$. Remove from $2\mathcal{B}_U$ the blocks $2(x_1, y_1, z_1)$ for each $(x, y, z) \in \mathcal{U}$. These blocks and the edges in $\bigcup_{i=2}^k L_T^i$ cover the graph $2(K_{2k} \setminus K_2)$ on vertex set $S \times \{1\}$ with the hole $\{1_1, 2_1\}$. For $k \equiv 0 \pmod 3$, take a decomposition $(S \times \{1\}, \mathcal{T}')$ of $2K_{2k}$ into triangles (see Section 2.2) such that $\{(1_1, 2_1, y_1), (1_1, 2_1, z_1)\} \subseteq \mathcal{T}'$, with $y_1 \neq z_1$. Delete the edges $2(1_1, 2_1)$. The result is a maximum packing of $2(K_{2k} \setminus K_2)$ with triangles having the 4-cycle $(1_1, y_1, 2_1, z_1)$ as leave; we have $L_T = \{(1_1, y_1, 2_1, z_1)\}$. For $k \equiv 1 \pmod 3$, take a decomposition of $2K_{2k}$ on vertex set $S \times \{1\}$ with leave $\{2(1_1, 2_1)\}$. The result is a decomposition of $2(K_{2k} \setminus K_2)$ into triangles. Then we have $L_T = \emptyset$.
  - in the S_3-metamorphosis, it is $\bigcup_{i=2}^k L_S^i = \{(\infty_1, 4_0), (\infty_1, 3_1), (\infty_1, 6_0), (\infty_1, 5_1), \ldots, (\infty_1, (2k)_0), (\infty_1, (2k-1)_1)\}$. The edges of $\bigcup_{i=2}^k L_S^i$ can be assembled into stars $[\infty_1; i_0, (i+2)_0, (i+4)_0]$, $i = 4 + 6h, h \geq 0$ and $[\infty_1; i_1, (i+2)_1, (i+4)_1]$, $i = 3 + 6h, h \geq 0$. It is easy to verify that $L_S = \emptyset$, if $k \equiv 1 \pmod 3$, or $L_S = \{(\infty_1, (2k)_0)\}$, if $k \equiv 0 \pmod 3$.
  - in the P_4-metamorphosis, it is $L_P^i = \{(\infty_2, (2i-1)_0), ((2i-1)_0, (2i)_0)\}$, $i \geq 2$. For every $i = 2, 3, \ldots, k$ remove the path $[(2i-1)_0, \infty_1, (2i-1)_1, (2i)_0]$. Let Γ be the set of edges covered by these paths and by $\bigcup_{i=2}^k L_P^i$. Construct the following paths with $i \equiv 2 \pmod 3$: $[\infty_1, (2i-1)_1, (2i)_0, (2i-1)_0]$, $[\infty_1, (2i+1)_1, (2i+2)_0, (2i+1)_0]$, $[\infty_1, (2i+3)_1, (2i+4)_0, (2i+3)_0]$, $[\infty_1, (2i+1)_0, \infty_2, (2i+1)_0]$, $[(2i+1)_0, \infty_1, (2i+3)_0, \infty_2]$. The above paths cover all edges in Γ if $k \equiv 1 \pmod 3$ or all edges in $\Gamma \setminus \{((2k-1)_0, (2k)_0)\}$ if $k \equiv 0 \pmod 3$. It follows that $L_P = \emptyset$ for $k \equiv 1 \pmod 3$ and $L_P = \{((2k-1)_0, (2k)_0)\}$ for $k \equiv 0 \pmod 3$.
- $s = 2, k \equiv 2 \pmod 3$. $L_T^1 = L_S^1 = L_P^1 = \emptyset$; leaves of the other groups are:

- in the K_3-metamorphosis, it is $\bigcup_{i=2}^{k-1} L_T^i = \{2(5_1,6_1),\ 2(7_1,8_1),\ \dots,\ 2((2k-1)_1,(2k)_1)\}$. Remove from $2\mathcal{B}_U$ the blocks $2(x_1,y_1,z_1)$ for each $(x,y,z) \in \mathcal{U}$. These blocks and the edges in $\bigcup_{i=2}^{k-1} L_T^i$ cover the graph $2K_{2k}$ on vertex set $S \times \{1\}$ with the hole $\{1_1,2_1,3_1,4_1\}$. Then a maximum packing of $2(K_{2k} \setminus K_4)$ with triangles with leave empty (see [5]) completes the K_3-metamorphosis.
- in the S_3-metamorphosis, $\bigcup_{i=2}^{k-1} L_S^i = \{(\infty_1,6_0),\ (\infty_1,5_1),\ (\infty_1,8_0),\ (\infty_1,7_1),\ \dots,\ (\infty_1,(2k)_0),\ (\infty_1,(2k-1)_1)\}$. The edges of $\bigcup_{i=2}^{k-1} L_S^i$ can be assembled into the 3-stars $[\infty_1; i_0,(i+2)_0,(i+4)_0]$, $i = 6h$, $h \geq 1$ and $[\infty_1; i_1,(i+2)_1,(i+4)_1]$ with $i = 5+6h, h \geq 0$. Therefore the leave is empty.
- in the P_4-metamorphosis, it is $L_P^i = \{(\infty_2,(2i+1)_0),\ ((2i+1)_0,(2i+2)_0)\}, 2 \leq i \leq k-1$. For every $i = 2,3,\dots,k-1$ remove the path $[(2i+1)_0,\infty_1,(2i+1)_1,(2i+2)_0]$. Let Γ be the set of edges covered by these paths and by $\bigcup_{i=2}^{k-1} L_P^i$. Construct the following paths with $i \equiv 2 \pmod 3$: $[\infty_1,(2i+1)_1,(2i+2)_0,(2i+1)_0]$, $[\infty_1,(2i+3)_1,(2i+4)_0,(2i+3)_0]$, $[\infty_1,(2i+5)_1,(2i+6)_0,(2i+5)_0]$, $[\infty_1,(2i+1)_0,\infty_2,(2i+3)_0]$, $[(2i+3)_0,\infty_1,(2i+5)_0,\infty_2]$. The above paths cover all edges in Γ, because $k \equiv 2 \pmod 3$ and so $2k-4 \equiv 0 \pmod 3$. Therefore the leave is empty.

□

Remark 2.3. The nonempty leaves of $\{K_3,S_3,P_4\}$-metamorphoses constructed in this section are

- if $n \equiv 5 \pmod 6$ or $n \equiv 8 \pmod{12}$, $L_T = 2[a,e,b]$, $L_S = (f,c)$, $L_P = (e,c)$;
- if $n \equiv 2 \pmod{12}$ and $n \geq 14$, $L_T = (a,b,c,d)$, $L_S = (e,f)$, $L_P = (f,g)$.

2.5 λ-fold kite systems

Lemma 2.7. *For every $n \geq 4$ there exists a 12-fold kite system of order n having $\{K_3,S_3,P_4\}$-metamorphosis with empty leaves.*

Proof If $n \equiv 0,1,3,4 \pmod 6$, combine 3 copies of the 4-fold kite system constructed in Section 2.4. If $n \equiv 5 \pmod 6$ or $n \equiv 8 \pmod{12}$, let $(X,\mathcal{B}_1)$ be the 4-fold kite system of order n constructed in Section 2.4. By Remark 2.3, we can suppose $L_T^1 = \{2[d,a,e]\}$, $L_S^1 = \{(b,c)\}$, $L_P^1 = \{(a,c)\}$. Applying the permutation $\varphi = (a,d,e,b)$ ($\psi = (e,d,b)$) to the vertices of X, we obtain the 4-fold kite system $(X,\mathcal{B}_2)$ ($(X,\mathcal{B}_3)$ respectively) having a $\{K_3,S_3,P_4\}$-metamorphosis with $L_T^2 = \{2[e,d,b]\}$, $L_S^2 = \{(a,c)\}$, $L_P^2 = \{(d,c)\}$ and $L_T^3 = \{2[b,a,d]\}$, $L_S^3 = \{(e,c)\}$, $L_P^3 = \{(a,c)\}$. Then $(X,\mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a 12-fold kite system having a $\{K_3,S_3,P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles $2(a,d,e), 2(a,b,d)$, the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[c;a,b,e]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[d,c,a,e]$.

If $n \equiv 2 \pmod{12}$, $n \geq 14$, let $(X, \mathcal{B}_1)$ be the 4-fold kite system of order n given in Section 2.4. By Remark 2.3, we can suppose $L_T^1 = \{(a,b,c,d)\}$, $L_S^1 = \{(e,f)\}$, $L_P^1 = \{(f,g)\}$. Let h, m be two vertices distinct from a,b,c,d,e,f,g. Applying the permutation $\varphi = (f,g,h)$ and changing b with c, we obtain a 4-fold kite system $(X, \mathcal{B}_2)$ of order n with $L_T^2 = \{(a,c,b,d)\}$, $L_S^2 = \{(e,g)\}$, $L_P^2 = \{(g,h)\}$. Applying to $(X, \mathcal{B}_1)$ the permutation $\varphi = (b,c,d)$ and changing g with m and f with h, we obtain a 4-fold kite system $(X, \mathcal{B}_3)$ of order n with $L_T^3 = \{(a,c,d,b)\}$, $L_S^3 = \{(e,h)\}$, $L_P^3 = \{(h,m)\}$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3)$ is a 12-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles (a,c,b), (a,c,d), (c,d,b), (a,b,d), the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[e; f,g,h]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[m,h,g,f]$. □

Theorem 2.8. *There exists a λ-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis if and only if $n \geq 4$, $\lambda n(n-1) \equiv 0 \pmod 8$, $(\lambda, n) \neq (1,8)$. There exists a kite system of order 8 having a $\{K_3, P_4\}$-metamorphosis*

Proof The necessity follows from Theorem 1.3. For $1 \leq \lambda \leq 4$ the proof follows from Sections 2.1, 2.2, 2.3, 2.4. Let $\lambda \geq 5$ and $n \geq 4$, with $\lambda n(n-1) \equiv 0 \pmod 8$. If $n = 8$ and $\lambda = 5, 7$ the proof follows from Examples 4.16 and 4.18. If $n = 5$ and $\lambda = 6$, the proof follows from Example 4.17.

Let F_n be a 1-factor of K_n containing the edges $(a,d), (b,c)$. Define the following set of edges: $T_n = [a; b,c,d] \cup (F_n \setminus \{(a,d),(b,c)\})$, $2P_3 = 2[b,a,c]$, $C_4 = (a,b,d,c)$ and $2P_2 = 2(b,c)$. Put $A = 2P_3 \cup F_n = (a,b,c) \cup T_n$, $C = 2P_3 \cup 2P_2 = 2(a,b,c)$; $F = 2[a,c,b] \cup 2[a,b,c] = 2(a,b,c) \cup 2P_2$, $H = (a,b,c,d) \cup (a,d,b,c) = \{(a,b,d),(a,c,d)\} \cup 2P_2$.
Let $5 \leq \lambda \leq 11$. Combine a suitable λ_1-fold kite system having $\{K_3, S_3, P_4\}$-metamorphosis (with leaves L_T^1, L_S^1, L_P^1) and a suitable λ_2-fold kite-system having $\{K_3, S_3, P_4\}$-metamorphosis (with leaves L_T^2, L_S^2, L_P^2), for suitable values of λ_1 and λ_2, and replace the leaves where it is necessary (see Table 3).
For example, for $\lambda = 6$ and $n \equiv 5, 8 \pmod{12}, n \geq 8$, let $(X, \mathcal{B}_1)$ be a copy of the 2-fold kite-system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis given in Section 2.2 and let $(X, \mathcal{B}_2)$ be a copy of the 4-fold kite-system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis given in Section 2.4. Therefore, by Remarks 2.2, 2.3, we can suppose $L_T^1 = 2(a,b)$, $L_S^1 = [c; d,e]$, $L_P^1 = \{(c,b),(e,f)\}$ and $L_T^2 = 2[a,e,b]$, $L_S^2 = (f,c)$, $L_P^2 = (e,c)$. Then $(X, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 6-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the edges of $L_T^1 \cup L_T^2 \cup L_T^3$ into the triangles $2(a,e,b)$, the edges of $L_S^1 \cup L_S^2 \cup L_S^3$ into the star $[c; d,e,f]$, the edges of $L_P^1 \cup L_P^2 \cup L_P^3$ into the path $[f,e,c,b]$ ($f = a$ if $n \equiv 5 \pmod{12}$).

For $\lambda = 12$ the proof follows from Lemma 2.7. Let $\lambda \equiv 1 \pmod 6, \lambda \geq 13$. Write $\lambda = 6k + 7$ and combine k copies of a 6-fold kite-system having a $\{K_3, S_3, P_4\}$-metamorphosis with a 7-fold kite-system having a $\{K_3, S_3, P_4\}$-metamorphosis. For each $\lambda = 12k + h$, with $0 \leq h \leq 11$ and $h \neq 1, 7$, combine k copies of a 12-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis with an h-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. □

λ	$n \geq 4$	λ_1	λ_2	L_T^1	L_T^2	$L_T^1 \cup L_T^2$	L_T	L_S^1	L_S^2	L_S	L_P^1	L_P^2	L_P
5	0 (mod 24)	1	4	F_n	$\emptyset$	F_n	F_n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	1, 9 (mod 24)	1	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	16 (mod 24)	1	4	T_n	$\emptyset$	T_n	T_n	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
5	8 (mod 24), $n \neq 8$	1	4	F_n	$2P_3$	A	T_n	P_2	P_2	S_2	P_2	P_2	E_2
5	17 (mod 24)	2	3	$2P_2$	$\emptyset$	$2P_2$	$2P_2$	S_2	$\emptyset$	S_2	E_2	$\emptyset$	E_2
6	0, 1, 4, 9 (mod 12)	2	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
6	5, 8 (mod 12), $n \neq 5$	2	4	$2P_2$	$2P_3$	C	$\emptyset$	S_2	P_2	$\emptyset$	E_2	P_2	$\emptyset$
7	17 (mod 24)	3	4	$\emptyset$	$2P_3$	$2P_3$	$2P_3$	$\emptyset$	P_2	P_2	$\emptyset$	P_2	P_2
7	0, 1, 9, 16 (mod 24)	1	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
7	8 (mod 24), $n \neq 8$	1	6	L_T^1	$\emptyset$	L_T^1	L_T^1	P_2	$\emptyset$	P_2	P_2	$\emptyset$	P_2
8	0, 1 (mod 3)	4	4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
8	2 (mod 3)	4	4	$2P_3$ or C_4	$2P_3$ or C_4	F or H	$2P_2$	P_2	P_2	S_2	P_2	P_2	E_2
9	0, 1 (mod 8)	3	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
10	0, 1, 4, 9 (mod 12)	4	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
10	5, 8 (mod 12)	4	6	L_T^1	$\emptyset$	L_T^1	L_T^1	P_2	$\emptyset$	P_2	P_2	$\emptyset$	P_2
11	0, 1, 9, 16 (mod 24)	5	6	L_T^1	$\emptyset$	L_T^1	L_T^1	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
11	8, 17 (mod 24)	5	6	L_T^1	$\emptyset$	L_T^1	L_T^1	S_2	$\emptyset$	S_2	E_2	$\emptyset$	E_2

Table 3: $\lambda = 5, 6, 7, 8, 9, 10, 11$ ($\emptyset$ denotes the empty graph)

3 Proof of Main Theorem

Theorem 3.1. *Every λ-fold kite system of order 4 has an E_2-metamorphosis.*

Proof Let $(\mathbb{Z}_4, \mathcal{B})$ be a λ-fold kite system of order 4. Then $|\mathcal{B}| = \frac{3\lambda}{2}$. For each $B = (a,b,c) - d \in \mathcal{B}$, let $B_1 = \{(a,b),(c,d)\}$ and $B_1' = \{(a,c),(b,c)\}$. Let L be the graph $(\mathbb{Z}_4, \bigcup_{B\in\mathcal{B}} B_1')$. We denote by $d_G(x)$ the degree of the vertex x in the graph G. It is easy to check that for every $x \in \mathbb{Z}_4$ and for every $B \in \mathcal{B}$, $d_{B_1'}(x) = d_B(x) - 1$. Then $d_L(x) = d_{\lambda K_4}(x) - |\mathcal{B}| = 3\lambda - \frac{3\lambda}{2} = \frac{3\lambda}{2}$, for every $x \in \mathbb{Z}_4$. Therefore L is a regular graph. Suppose that the edge (x,y) appears α times in L. Let $\{z,t\} = \mathbb{Z}_4 \setminus \{x,y\}$. Then (z,t) appears α times in L, otherwise L couldn't be regular. Using the edges $(x,y),(z,t)$ costruct α E_2s. Since each B_1 is an E_2, the E_2-metamorphosis is trivially completed. □

Theorem 3.2. *Every λ-fold kite system of order n, with $n \geq 10$ if $\lambda \geq 2$ has an E_2-metamorphosis.*

Proof Let $(X, \mathcal{B})$ be a λ-fold kite system of order n. Then $|\mathcal{B}| = \frac{\lambda n(n-1)}{8}$. For each $B = (a,b,c) - d \in \mathcal{B}$, let $B_1 = \{(a,b),(c,d)\}$ and $B_1' = \{(a,c),(b,c)\}$. Let $L = \bigcup_{B\in\mathcal{B}} B_1'$. The degree of each vertex of L is at most $\left\lfloor \lambda\frac{2(n-1)}{3} \right\rfloor$. Combine at random the edges of L into E_2s. The result is a set $\mathcal{E}$ of E_2s and a graph G having $2h \geq 0$ edges. For $h = 0$ the theorem is proved. Let $h > 0$. Then every two edges of G share a common vertex. Let $\mathcal{E}_{vw}' = \{E \in \mathcal{E} \mid E$ is not incident in $v, w\}$, $\mathcal{E}_{vw} = \{E \in \mathcal{E} \mid E$ is incident in v and $w\}$, $\mathcal{E}_v = \{E \in \mathcal{E} \mid E$ incident in $v\}$, $\mathcal{E}_v' = \{E \in \mathcal{E} \mid E$ is not incident in $v\}$. The following two cases arise:

Case 1. G is a star, possibly with repeated edges. Let $G = S_{2h} = [0; v_1, v_2, \ldots, v_{2h}]$.

Case 1a. Let $v_1 = v_2 = \ldots = v_{2h} = 1$. Then $|\mathcal{E}_{01}'| = |\mathcal{E}| - |\mathcal{E}_0| - |\mathcal{E}_1| + |\mathcal{E}_{01}| \geq |\mathcal{E}| - 2(\left\lfloor \lambda\frac{2(n-1)}{3} \right\rfloor - 2h) + (\lambda - 2h) \geq \lambda\frac{n(n-1)}{8} - h - \frac{4}{3}\lambda(n-1) + 4h + \lambda - 2h = \lambda(n-1)(\frac{3n-32}{24}) + h + \lambda$. It follows $|\mathcal{E}_{01}'| > h$ for $n \geq 10$. Choose h blocks $E \in \mathcal{E}_{01}'$. Combining each of this block with two edges $(0,1)$ we complete the E_2-metamorphosis.

Case 1b. Let $|\{v_1, v_2, \ldots, v_{2h}\}| \geq 2$. Take v_i, v_j with $v_i \neq v_j$. Note that $|\mathcal{E}_0'| = |\mathcal{E}| - |\mathcal{E}_0| \geq \lambda\frac{n(n-1)}{8} - h - (\left\lfloor \lambda\frac{2(n-1)}{3} \right\rfloor - 2h) \geq \lambda\frac{n(n-1)}{8} - \lambda\frac{2(n-1)}{3} + h = \lambda\frac{(n-1)(3n-16)}{24} + h$. Then $|\mathcal{E}_0'| > \lambda + h$, for $n \geq 7$. Choose a block $\overline{E} \in \mathcal{E}_0'$ not containing the edge (v_i, v_j). It is possible to rearrange the edges $(0, v_i), (0, v_j)$ of S_{2h} with the edges of $\overline{E}$ in order to form two new E_2s. Remove $(0, v_i), (0, v_j)$ from S_{2h}, substitue $\overline{E}$ with the new E_2s in $\mathcal{E}$ and reapply the procedure, that will stop when S_{2h} is empty.

Case 2. G is a triangle with repeated edges. Suppose G contains the edges $m(0,1)$, $p(1,2)$, $q(2,0)$. Since $m + p + q = 2h$, at least one of m, p, q must be even. Suppose $m = 2k$. Then $|\mathcal{E}_{01}'| = |\mathcal{E}| - |\mathcal{E}_0| - |\mathcal{E}_1| + |\mathcal{E}_{01}| \geq |\mathcal{E}| - (\left\lfloor \lambda\frac{2(n-1)}{3} \right\rfloor - (m+q)) - (\left\lfloor \lambda\frac{2(n-1)}{3} \right\rfloor - (m+p)) + (\lambda - m) \geq \lambda\frac{n(n-1)}{8} - h - \frac{4}{3}\lambda(n-1)\lambda + 2h = \lambda(n-1)(\frac{3n-32}{24}) + h + \lambda$. Then $|\mathcal{E}_{01}'| > h > k$, for $n \geq 10$. Choose k blocks

$E \in \mathcal{E}'_{01}$. Combine each of this block with two edges $(0,1)$. The left edges make a star $S_{2(h-k)} = [2; 0, 0, \ldots, 1, 1, \ldots]$ that we can assemble as in Case 1.

For $\lambda = 1$, only subcase 1b holds and $n \geq 8$, so every kite system has an E_2-metamorphosis. □

Proof of Main Theorem. Every λ-fold G-design has P_2-metamorphoses. Let $B = (a,b,c) - d$ be a block of a λ-fold kite system $(X, \mathcal{B})$. Decompose B into the two paths $[a,b,c]$ and $[a,c,d]$. Then every λ-fold kite system $(X,\mathcal{B})$ has a P_3-metamorphosis. Let $n \geq 4$, λ such that $\lambda n(n-1) \equiv 0 \pmod 8$. Let $(X, \mathcal{B})$ be the λ-fold kite system of order n having a $\{K_3, S_3, P_4\}$-metamorphosis (see Theorem 2.8) or, if $(\lambda, n) = (1, 8)$, having a $\{K_3, P_4\}$-metamorphosis. Then $(X, \mathcal{B})$ has a $\{K_3, S_3, P_4, P_3, P_2\}$-metamorphosis or, if $(\lambda, n) = (1, 8)$, a $\{K_3, P_4, P_3, P_2\}$-metamorphosis. $(X, \mathcal{B})$ has also an E_2-metamorphosis. This follows from Theorems 3.1 and 3.2 for $n = 4$, $n \geq 10$ and for $\lambda = 1, \forall n \equiv 0, 1 \pmod 8$. For the remaining values of n and λ, the E_2-metamorphosis of $(X, \mathcal{B})$ follows easily from the proof of Theorems 2.4, 2.5, 2.6, 2.8 and from the observation that the starting designs (see Examples 4.6, 4.7, 4.8, 4.9, 4.16, 4.18, 4.17) have also an E_2-metamorphosis. □

4 Appendix: small cases

The following are λ-fold kite-systems of order n having a $\{K_3, S_3, P_4\}$-metamorphosis. Except otherwise specified, the vertex set is $\mathbb{Z}_n$.

Example 4.1 ($\lambda = 1, n = 9$). $\mathcal{B} = \{(1,3,4)-8,\ (1,0,6)-5,\ (1,8,7)-5,\ (1,2,5)-0,\ (2,7,4)-6,\ (0,2,8)-6,\ (3,2,6)-7,\ (8,3,5)-4,\ (3,7,0)-4\}$; $\mathcal{T} = \{(5,7,6),\ (8,4,6),\ (4,5,0)\}$; $\mathcal{S} = \{[1;8,3,0], [2;0,1,7], [3;2,8,7]\}$; $\mathcal{P} = \{[7,8,2,6], [4,3,5,2], [6,0,7,4]\}$.

Example 4.2 ($\lambda = 1, n = 16$). $\mathcal{B} = \{(4,0,9) - 2,\ (11,0,5) - 13,\ (3,0,1) - 15,\ (1,5,10) - 3,\ (12,1,6) - 14,\ (4,1,2) - 15,\ (6,2,11) - 4,\ (13,2,7) - 0,\ (0,2,14) - 15,\ (7,3,12) - 5,\ (3,14,8) - 1,\ (6,3,4) - 15,\ (8,4,13) - 6,\ (12,8,2) - 10,\ (7,8,10) - 15,\ (14,9,5) - 15,\ (7,9,6) - 15,\ (8,9,11) - 15,\ (6,10,0) - 8,\ (14,10,4) - 12,\ (10,12,9) - 15,\ (11,7,1) - 9,\ (11,10,13) - 15,\ (11,14,12) - 15,\ (9,13,3) - 11,\ (13,12,0) - 15,\ (1,13,14) - 7,\ (4,5,7) - 15,\ (5,6,8) - 15,\ (2,5,3) - 15\}$;
$\mathcal{T} = \{(2,10,15),\ (3,11,15),\ (7,14,15),\ (1,9,15),\ (0,8,15),\ (5,12,15),\ (6,13,15)\}$;
$\mathcal{S} = \{[0;4,3,11],\ [1;5,12,4],\ [2;6,13,0],\ [3;14,7,6],\ [8;4,12,7],\ [9;14,7,8],\ [10;6,14,12],\ [11;7,10,14],\ [13;9,12,1],\ [5;4,6,2]\}$;
$\mathcal{P} = \{[9,0,5,10],\ [1,0,10,13],\ [6,1,2,11],\ [7,2,14,8],\ [12,3,4,13],\ [2,8,10,4],\ [5,9,6,8],\ [11,9,12,0],\ [1,7,5,3],\ [12,14,13,3]\}$;
$L_T = \{[4;11,12,15],\ (1,8),\ (2,9),\ (3,10),\ (6,14),\ (0,7),\ (5,13)\}$.

Example 4.3 ($\lambda = 1, n = 17$). $\mathcal{B} = \{(1,4,14) - 3,\ (1,5,7) - 15,\ (1,10,8) - 14,\ (15,2,5) - 10,\ (2,6,8) - 3,\ (2,11,9) - 1,\ (3,4,6) - 14,\ (3,7,9) - 15,\ (3,10,12) - 15,\ (11,4,7) - 12,\ (10,4,2) - 3,\ (8,4,13) - 10,\ (8,5,12) - 14,\ (5,11,3) - 1,\ (5,9,14) - 2,\ (6,13,9) - 12,\ (6,12,1) - 16,\ (6,10,15) - 1,\ (7,13,2) - 1,\ (13,11,1) - 0,\ (15,13,3) - 16,\ (16,5,4) - 9,\ (16,10,11) - 14,\ (16,6,7) - 10,\ (16,13,12) - 2,\ (9,16,8) - 11,\ (16,14,15) - 11,\ (0,5,6) - 11,\ (0,11,12) - 4,\ (0,8,7) - 14,\ (0,14,13) - 5,\ (4,0,15) - 8,\ (0,9,10) - 14,$

$(16,2,0)-3\}$;
$\mathcal{T}=\{(1,3,0), (7,10,14), (8,11,15), (9,12,4), (8,14,3), (9,15,1), (10,13,5), (11,14,6), (12,15,7), (2,14,12)\}$;
$\mathcal{S}=\{[4;8,10,11], [2;6,11,15], [6;10,12,13], [5;8,9,11], [0;4,9,14], [0;5,8,11], [1;4,5,10], [3;4,7,10], [13;7,11,15], [16;5,6,10], [16;9,13,14]\}$;
$\mathcal{P}=\{[14,4,6,8], [7,5,2,0], [8,10,12,5], [7,4,2,13], [4,13,9,14], [9,11,1,12], [15,10,11,12], [0,15,14,13], [4,5,6,7], [10,9,7,8], [12,13,3,11]\}$;
$L_{\mathcal{T}}=(1,2,3,16)$; $L_{\mathcal{S}}=(8,16)$; $L_{\mathcal{P}}=(8,16)$.

Example 4.4 ($\lambda=1, n=24$). $\mathcal{B}=\{(3,1,10)-11, (4,1,9)-10, (5,1,11)-12, (4,2,11)-23, (5,2,10)-22, (2,12,6)-7, (5,3,12)-13, (6,3,11)-22, (3,7,13)-0, (6,4,13)-14, (7,4,12)-0, (8,4,14)-0, (5,14,7)-8, (5,8,13)-1, (5,9,15)-1, (8,6,15)-19, (9,6,14)-2, (10,6,16)-23, (9,7,16)-14, (10,7,15)-14, (11,7,17)-14, (10,8,17)-20, (11,8,16)-4, (12,8,18)-16, (11,9,18)-19, (12,9,17)-5, (13,9,19)-5, (12,10,19)-20, (13,10,18)-5, (14,10,20)-6, (13,11,20)-21, (14,11,19)-6, (15,11,21)-7, (21,12,14)-18, (15,12,20)-7, (12,16,22)-5, (15,13,22)-17, (16,13,21)-6, (13,17,23)-7, (15,16,2)-9, (17,15,0)-11, (15,18,23)-14, (17,16,3)-8, (19,16,0)-18, (16,20,1)-12, (18,17,4)-10, (19,17,1)-2, (21,17,2)-13, (20,18,2)-3, (21,18,1)-14, (22,18,3)-14, (21,19,3)-4, (2,19,22)-14, (4,19,23)-12, (20,22,4)-5, (20,23,3)-15, (0,20,5)-6, (22,21,8)-19, (21,23,5)-16, (21,0,4)-15, (22,23,9)-21, (22,0,6)-18, (22,1,7)-18, (0,23,10)-21, (1,23,6)-17, (2,23,8)-20, (0,1,8)-9, (0,2,7)-19, (3,0,9)-20\}$;
$\mathcal{T}=\{(1,2,13), (2,3,14), (3,4,15), (4,5,16), (5,6,17), (6,7,18), (7,8,19), (8,9,20), (9,10,21), (10,11,22), (11,12,23), (12,13,0), (14,15,1), (14,16,23), (14,17,22), (14,18,0), (18,19,5), (19,20,6), (20,21,7)\}$;
$\mathcal{S}=\{[1;3,4,5], [2;4,5,12], [3;5,6,7], [4;6,7,8], [5;14,8,9], [6;8,9,10], [7;9,10,11], [8;10,11,12], [9;11,12,13], [10;12,13,14], [11;13,14,15], [12;21,15,16], [13;15,16,17], [15;16,17,18], [16;17,19,20], [17;18,19,21], [18;20,21,22], [19;21,2,4], [20;22,23,0], [21;22,23,0], [22;23,0,1], [23;0,1,2], [0;1,2,3]\}$;
$\mathcal{P}=\{[1,10,2,11], [1,9,15,6], [1,11,3,12], [6,12,4,13], [4,14,7,16], [13,8,17,9], [14,6,16,8], [15,7,17,23], [8,18,9,19], [19,10,18,23], [10,20,11,19], [11,21,13,22], [14,12,20,1], [22,16,2,7], [15,0,16,3], [4,17,1,18], [17,2,18,3], [3,19,22,4], [19,23,5,20], [3,23,9,0], [13,7,1,8], [21,8,23,10], [4,0,6,23]\}$;
$L_{\mathcal{T}}=\{(1,12), (2,9), (3,8), (4,10), (13,14), (15,19), (16,18), (17,20), (21,6), (22,5), (23,7), (0,11)\}$.

Example 4.5 ($\lambda=2, n=4$). $\mathcal{B}=\{(3,0,1)-2, (0,1,2)-3, (0,2,3)-1\}$, $\mathcal{T}=(1,2,3)$, $\mathcal{S}=[0;1,2,3]$, $\mathcal{P}=[0,1,2,3]$.

Example 4.6 ($\lambda=2, n=5$). $\mathcal{B}=\{(1,0,4)-3, (2,0,4)-3, (1,2,3)-0, (1,3,0)-2, (1,4,2)-3\}$, $\mathcal{T}=\{(0,2,3)\}$, $\mathcal{S}=\{[1;2,3,4]\}$, $\mathcal{P}=\{[2,4,0,3]\}$, $L_{\mathcal{T}}=2(4,3)$, $L_{\mathcal{S}}=[0;1,2]$, $L_{\mathcal{P}}=\{(2,3),(0,4)\}$.

Example 4.7 ($\lambda=2, n=8$). $\mathcal{B}=\{(6,7,1)-5, (4,7,5)-3, (7,2,3)-1, (4,6,2)-1, (3,0,6)-5, (0,1,4)-3, (2,5,0)-7, (0,5,1)-2, (6,5,3)-1, (3,2,4)-1, (7,0,2)-5, (1,7,6)-2, (4,5,7)-3, (4,6,0)-3\}$
$\mathcal{T}=\{(1,3,4), (2,5,6), (3,7,0), (1,3,5)\}$,
$\mathcal{S}=\{[7;1,4,0], [6;7,5,4], [0;1,3,5], [2;7,3,5]\}$,
$\mathcal{P}=\{[1,7,5,0], [0,6,2,3], [3,5,1,4], [2,0,6,7]\}$,
$L_{\mathcal{T}}=2(1,2)$, $L_{\mathcal{S}}=[4;5,6]$, $L_{\mathcal{P}}=\{(4,2),(7,5)\}$.

Example 4.8 ($\lambda = 2, n = 9$). It is sufficient doubling the blocks of the above kite-system of order 9.

Example 4.9 ($\lambda = 3, n = 8$). $\mathcal{B} = \{(7,4,2)-3,\ (6,7,3)-1,\ (7,5,1)-2,\ (4,6,5)-2,\ (1,0,4)-3,\ (2,0,6)-1,\ (5,0,3)-7,\ (1,0,2)-5,\ (6,1,3)-2,\ (5,3,4)-2,\ (7,0,5)-1,\ (2,7,6)-5,\ (4,1,7)-0,\ (4,6,0)-3,\ (4,6,2)-1,\ (7,4,1)-3,\ (4,5,3)-2,\ (6,7,5)-2,\ (3,0,6)-1,\ (2,0,7)-3,\ (1,5,0)-4\}$.
$\mathcal{T} = \{(2,3,1),\ (2,1,3),\ (2,3,4),\ (3,7,0),\ (1,6,5)\}$;
$\mathcal{S} = \{[4;6,7,5],\ [7;6,2,0],\ [0;2,1,3],\ [5;7,3,1],\ [6;1,7,4],\ [0;1,5,2],\ [4;6,1,7]\}$;
$\mathcal{P} = \{[7,3,1,4],\ [5,6,0,3],\ [4,2,6,0],\ [3,5,1,7],\ [2,0,5,7],\ [3,4,0,6],\ [6,7,0,5]\}$;
$L_T = \{3(2,5),\ (1,6),\ (3,7),\ (4,0)\}$.

Example 4.10 ($\lambda = 4, n = 5$). $\mathcal{B} = \{(1,2,0)-4, (1,3,0)-4, (2,3,4)-1, (2,4,1)-3, (2,0,3)-4, (1,2,0)-3, (1,4,0)-3, (2,4,3)-1, (2,3,1)-4, (2,0,4)-3\}$;
$\mathcal{T} = \{2(1,3,4)\}$; $\mathcal{S} = \{[1;2,4,3],\ [2;0,3,4],\ [2;0,4,3]\}$; $\mathcal{P} = \{[2,0,3,4],\ [1,4,0,3],\ [1,3,4,0]\}$; $L_T = 2[3,0,4]$; $L_S = (1,2)$; $L_P = (0,2)$.

Example 4.11 ($\lambda = 4, n = 6$). $X = \mathbb{Z}_5 \cup \{\infty\}$, $\mathcal{B} = \{(i, 2+i, \infty) - (i+1),\ (i+1, 2+i, i) - \infty,\ (2+i, 4+i, i) - (i+1) | i \in \mathbb{Z}_5\}$;
$\mathcal{T} = \{(i, 1+i, \infty) | i \in \mathbb{Z}_5)\}$; $\mathcal{S} = \{[1;2,4,3], [0;2,3,1], [2;0,4,3], [3;0,1,4], [4;2,0,1]\}$;
$\mathcal{P} = \{[\infty, 2+i, i, 4+i]\}$.

Example 4.12 ($\lambda = 4, n = 7$). $\mathcal{B} = \{(1,0,4)-2,\ (1,5,6)-2,\ (5,0,2)-1,\ (0,6,3)-2,\ (3,5,4)-6,\ (0,2,4)-1,\ (5,2,6)-1,\ (5,0,1)-2,\ (0,6,3)-1,\ (5,3,4)-6,\ (1,0,4)-3,\ (1,5,6)-3,\ (5,0,3)-1,\ (0,6,2)-3,\ (2,5,4)-6,\ (5,0,4)-2,\ (5,1,6)-2,\ (0,1,2)-5,\ (0,6,3)-2,\ (1,3,4)-6,\ (2,1,3)-5\}$;
$\mathcal{T} = \{(2,4,6), (1,4,6), (3,4,6), (2,4,6), (1,2,3), (1,2,3), (2,3,5)\}$;
$\mathcal{S} = \{[0;1,5,6], [5;1,3,2], [0;2,5,6], [5;1,2,3], [0;1,5,6], [1;2,3,5], [0;1,5,6]\}$;
$\mathcal{P} = \{[4,0,2,6], [0,4,5,6], [4,3,6,1], [4,2,6,3], [1,0,3,4], [6,3,1,2], [6,5,4,0]\}$.

Example 4.13 ($\lambda = 4, n = 8$). Take two copies of the 2-fold kite-system of order 8. In one of them change 5 with 2. The result is a 4-fold kite-system of order 8 having a $\{K_3, S_3, P_4\}$-metamorphosis, with
$\mathcal{T} = \{(1,3,4),\ (2,5,6),\ (3,7,0),\ (1,3,5),\ (1,3,4),\ (5,2,6),\ (3,7,0),\ (1,3,2)\}$;
$\mathcal{S} = \{[7;1,4,0], [6;7,5,4], [0;5,1,3], [2;7,3,5], [7;1,4,0], [6;7,2,4], [0;2,1,3], [5;7,3,2], [4;2,6,5]\}$;
$\mathcal{P} = \{[1,7,5,0], [0,6,2,3], [3,5,1,4], [2,0,6,7], [6,7,2,0], [0,6,5,3], [3,2,1,7], [6,0,5,4], [4,2,7,5]\}$;
$L_T = 2[2,1,5]$; $L_S = (6,4)$; $L_P = (1,4)$.

Example 4.14 ($\lambda = 4, n = 10$). $X = \mathbb{Z}_9 \cup \{\infty\}$, $\mathcal{B} = \{(i, 4+i, \infty) - (i+1),\ (i+6, 4+i, i) - \infty,\ (3+i, 5+i, 1+i) - i,\ (4+i, 1+i, i) - (3+i),\ (2+i, 1+i, i) - (3+i) | i \in \mathbb{Z}_9\}$;
$\mathcal{T} = \{(i, 1+i, \infty) | i \in \mathbb{Z}_9\} \cup (2\{(0,3,6),\ (1,4,7),\ (2,5,8)\})$;
$\mathcal{S} = \{[0;2,1,7],\ [8;0,1,7],\ [2;1,4,3],\ [3;4,1,5],\ [5;4,6,7],\ [6;4,7,8]\} \cup \{[4+i; i, 6+i, 1+i] | i \in \mathbb{Z}_9\}$;
$\mathcal{P} = \{[\infty, 4+i, i, 1+i] | i \in \mathbb{Z}_9\} \cup \{[0,5,6,2],\ [1,5,4,8],\ [1,6,7,3],\ [1,2,7,8],\ [3,8,0,1],\ [2,3,4,0]\}$.

Example 4.15 ($\lambda = 4, n = 11$). Let $S = \{x_j | j \in \mathbb{Z}_5\}$ and $X = S \cup \mathbb{Z}_5 \cup \{\infty\}$.

1. Let (S,$\mathcal{B}'$) be a copy of the 4-fold kite-system of order 5 above constructed with the leaves L_T, L_S, L_P.

2. For each $i \in \mathbb{Z}_5$, let $\mathcal{B}_i = \{(i-1, i+1, x_i) - \infty$, $(i-2, i+2, x_i) - (i+1)$, $(x_i, i, \infty) - (i+1)$, $(i, \infty, x_i) - (i+2)$, $(i-1, i+1, x_i) - i$, $(x_i, i-2, i+2) - i$, $(\infty, i, x_i) - (i-2)$, $(i-2, i+2, x_i) - (i-1)$, $(x_i, i+1, i-1) - (i-2)\}$. It is easy to check that $(X, \cup \mathcal{B}_i)$ is a 4-fold kite-system of order 11 with a hole of size 5 on S having:

   - a K_3-metamorphosis with $\mathcal{T} = \{(x_i, 1+i, \infty), (i, i+2, x_i), (i-1, i-2, x_i)\}$ and empty leave;
   - an S_3-metamorphosis with $\mathcal{S} = \{[x_i; i-2, i, i+1], 0 \le i \le 4\} \cup \{[i-1; i, i+1, i+3], 0 \le i \le 3\} \cup \{[1; 3, 4, \infty], [2; 0, 4, \infty], [3; 0, 4, \infty], [3; 0, 2, \infty], [\infty; 0, 2, 4], [\infty; 0, 1, 4]\}$ and empty leave;
   - a P_4-metamorphosis with $\mathcal{P} = \{[i+1, x_i, i+2, i-2], [i, \infty, x_i, i+2], [i, x_i, i+1, i-1]\}$ and empty leave.

Then $(X, \mathcal{B}' \cup (\bigcup \mathcal{B}_i))$ is a 4-fold kite system of order 11 having $\{K_3, S_3, P_4\}$-metamorphosis with leaves $L_{\mathcal{T}}, L_{\mathcal{S}}, L_{\mathcal{P}}$.

Example 4.16 ($\lambda = 5, n = 8$). Let $\mathcal{B}_1 = \{(1,2,5) - 6$, $(1,7,4) - 6$, $(1,3,6) - 5$, $(2,7,3) - 5$, $(6,0,2) - 4$, $(5,0,7) - 6$, $(4,3,0) - 1$, $(2,5,4) - 6$, $(2,7,6) - 3$, $(2,1,3) - 4$, $(5,7,1) - 4$, $(3,0,5) - 4$, $(4,0,7) - 3$, $(6,1,0) - 2$, $(7,2,5) - 1$, $(7,6,1) - 3$, $(7,4,3) - 5$, $(2,6,4) - 5$, $(3,0,2) - 1$, $(5,0,6) - 3$, $(1,4,0) - 7\}$. Then $(\mathbb{Z}_8, \mathcal{B}_1)$ is a 3-fold kite system having

- a K_3-metamorphosis with $\mathcal{T} = \{(5,6,4), (3,4,6), (5,3,1), (5,6,3), (4,1,2)\}$ and leave $L_{\mathcal{T}} = \{(7,6), (7,3), (7,0), (0,1), (0,2), (5,4)\}$;
- an S_3-metamorphosis with $\mathcal{S} = \{[2;6,1,7], [1;6,3,7], [2;5,1,7], [4;3,1,7], [0;5,6,3], [0;5,3,4], [7;5,6,2]\}$;
- a P_4-metamorphosis with $\mathcal{P} = \{[5,2,0,7], [4,7,3,6], [3,0,1,7], [7,0,5,4], [7,6,1,3], [5,2,0,4], [3,4,6,0]\}$.

Let $(\mathbb{Z}_8, \mathcal{B}_2)$ be the above 2-fold kite system. Then $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 5-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. Note that we can rearrange the leaves of the K_3-metamorphosis of $(\mathbb{Z}_8, \mathcal{B}_1)$ and $(\mathbb{Z}_8, \mathcal{B}_2)$ into the triangle $(1,2,0)$ and the leave $\{[7;6,3,0], (1,2), (5,4)\}$.

Example 4.17 ($\lambda = 6, n = 5$). Let $(\mathbb{Z}_5, \mathcal{B}_1)$ be the above 4-fold kite system of order 5 and $(\mathbb{Z}_5, \mathcal{B}_2)$ be the above 2-fold kite system when we change 0 with 2. Then $(\mathbb{Z}_5, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 6-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis, with:
$\mathcal{T} = \{(2,0,3), 2(1,3,4), 2(0,3,4)\}$;
$\mathcal{S} = \{[1;0,3,4], [1;2,4,3], [2;0,1,4], [2;0,4,3], [2;1,0,3]\}$;
$\mathcal{P} = \{[0,4,2,3], [2,0,3,4], [1,4,0,3], [1,3,4,0], [3,0,2,4]\}$.

Example 4.18 ($\lambda = 7, n = 8$). Let $(\mathbb{Z}_8, \mathcal{B}_1)$ be the above 4-fold kite-system of order 8 and $(\mathbb{Z}_8, \mathcal{B}_2)$ be the above 3-fold kite system. Then $(\mathbb{Z}_8, \mathcal{B}_1 \cup \mathcal{B}_2)$ is a 7-fold kite system having a $\{K_3, S_3, P_4\}$-metamorphosis. We can rearrange the leaves of the K_3-metamorphosis of $(\mathbb{Z}_8, \mathcal{B}_1)$ and $(\mathbb{Z}_8, \mathcal{B}_2)$ into the triangles $2(1,2,5)$ and the leave $\{(2,5), (1,6), (3,7), (4,0)\}$.

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