

On Partitions of $\{1, \dots, 2m + 1, L\}$ into Differences $d, \dots, d + m$: Stretched Langford Sequences.

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Abstract

A set of necessary conditions for the existence of a partition of $\{1, \dots, 2m - 1, L\}$ into differences $d, d + 1, \dots, d + m - 1$ is $(m, L) \equiv (0, 0), (1, d + 1), (2, 1), (3, d) \pmod{(4, 2)}$ and $m \geq 2d - 2$ or $m = 1$. If $m = 2d - 2$ then $L = 5d - 5$, if $m = 2d - 1$ then $4d - 2 \leq L \leq 6d - 4$ and if $m \geq 2d$ then $2m \leq L \leq 3m + d - 2$. Similar conditions for the partition of $\{1, \dots, 2m, L\} \setminus \{2\}$ into differences $d, d + 1, \dots, d + m - 1$ are $(m, L) \equiv (0, 0), (1, d + 1), (2, 1), (3, d) \pmod{(4, 2)}$, $(d, m, L) \neq (1, 1, 4), (2, 3, 8)$ and $m \geq 2d - 2$, $m = 1$ or $(d, m, L) = (3, 2, 7), (3, 3, 9)$. If $m = 2d - 2$ then $L = 5d - 5, 5d - 3$, if $m = 2d - 1$ then $4d - 1 \leq L \leq 6d - 2$ and if $m \geq 2d$ then $2m + 1 \leq L \leq 3m + d - 1$.

It is shown that for many cases when the necessary conditions hold, the set $\{1, \dots, 2m - 1, L\}$ and $\{1, \dots, 2m - 1, L\} \setminus \{2\}$ can be so partitioned. These partitions exist for all the minimum and maximum L , when $d \leq 3$, when $m = 1$ and when $m \geq 8d - 3$ ($m \geq 8d + 4$ in the hooked case). The constructions given fully solve the existence of these partitions if the necessary conditions for the existence of extended and hooked extended Langford sequences are sufficient.

1 Introduction

A Langford sequence is a partition of the set $S = \{1, 2, \dots, 2m\}$ into pairs where we restrict the differences between the elements in each pair:

Definition 1.1. A Langford sequence of defect d and m differences is a partition of the set $S = \{1, 2, \dots, 2m\}$ into pairs $\mathcal{P} = \{(a_i, b_i)\}$, $1 \leq i \leq m$ and $a_i < b_i$, such that $b_i - a_i = d - 1 + i$ for all $1 \leq i \leq m$. Equivalently, it can be written as a sequence $a\alpha_1\alpha_2\dots\alpha_{2m}$, where if $\alpha_i = \alpha_j = \delta$ then $|i - j| = \delta$.

For example, $\mathcal{P} = \{(3, 5), (1, 4), (2, 6)\}$ is a Langford sequence of defect 2 and 3 differences which can also be written as the sequence 342324. Langford [5] posed this problem for $d = 2$ and subsequently Priday [10] solved it, although slightly earlier Skolem [14] posed this problem for $d = 1$ and solved it; indeed solutions for $d = 1$ are called **Skolem sequences**.

As we shall see, there are necessary parity conditions on the existence of Langford sequences. Existence outside of these conditions is typically achieved by slightly altering the base set to $S' = \{1, 3, 4, \dots, 2m + 1\}$. The missing value, 2, is called a **hook** and the definition is modified in the obvious way. For example, $\mathcal{P} = \{(11, 13), (3, 6), (4, 8), (5, 10), (1, 7)\}$ is a hooked Langford sequence of defect 2 and 5 differences, also written as 6_345364252. In the Langford sequence literature the hook is typically put at the right hand side of the sequence, i.e. in position $2m$ rather than 2. In this article we will stretch the Langford sequence by putting a "wide-hook" at the right hand side and, when the parity of the sequence requires it, putting a standard hook at the left in position 2.

In this paper we will denote a Langford sequence of defect d and m differences by \mathcal{L}_d^m . So the sequence given in the first example, 342324, will be denoted as a \mathcal{L}_2^3 . Similarly, we will denote a hooked Langford sequence of defect d and m differences by $h\mathcal{L}_d^m$, and the sequence given in the second example, 6_345364252, will be denoted as a $h\mathcal{L}_2^5$. When writing sequences, for integers larger than 10 we denote $a = 10, b = 11, \dots, z = 35, A = 36, B = 37, \dots, Z = 71$.

For arbitrary d the following result has been obtained (where for integers $a \leq b$ we write $\{a, a + 1, \dots, b\}$ as $[a, b]$ and by writing $(x, y) \equiv (u, v) \pmod{(m, n)}$ we mean $x \equiv u \pmod{m}$ and $y \equiv v \pmod{n}$):

Theorem 1.2. (*[3, 13]*) If $m \geq 2d - 1$ then $[1, 2m]$ can be partitioned into differences $[d, d + m - 1]$ whenever $(m, d) \equiv (0, 0), (0, 1), (1, 1), (3, 0) \pmod{(4, 2)}$. If $m(m + 1 - 2d) + 2 \geq 0$ then $[1, 2m + 1] \setminus \{2\}$ can be partitioned into differences $[d, d + m - 1]$ whenever $(m, d) \equiv (1, 0), (2, 0), (2, 1), (3, 1) \pmod{(4, 2)}$.

These conditions are all necessary.

Other variants of the problem have been considered. A **k-extended Langford sequence** of defect d and m differences is a partition of $[1, 2m + 1] \setminus \{k\}$ into differences $[d, d + m - 1]$; denoted $\mathcal{L}_d^m(k)$. A **left-hooked k-extended Langford sequence** of defect d and m differences is a partition

of $[1, 2m + 2] \setminus \{2, k\}$ into differences $[d, d + m - 1]$; denoted $h\mathcal{L}_d^m(k)$. It is more usual to have the hook at position $2m + 1$ but since, in this article, we will be putting our “wide-hook” at the right, we place the standard hook at the left.

These two problems have been investigated and mostly solved in [1, 6, 7, 8, 9]. Other variants include [2, 12].

In this paper we consider the problems of constructing stretched and left-hooked stretched Langford sequences. Hooked sequences are often used in constructions by joining two sequences at the hook:

$$83743564875\mathbf{26211}$$

Extended Langford sequences can be thought of as containing a generalization of a hook; stretched sequences, with a “wide hook”, are the complementary object that could be used in such constructions:

$$67845964758\mathbf{1139232} \tag{1}$$

We will use constructive results about extended and left-hooked extended Langford sequences from [8, 9] to build stretched and left-hooked stretched Langford sequences. In Section 2 we review some definitions and results that we will use. In Section 3 we will define our problem, establish the necessary conditions for existence and show the sufficiency of these conditions in almost all cases. In each case the construction method will be similar: for a given stretched right-hook we will use an extended sequence with an appropriate k and insert a new difference, letting $s_k = (L - k)$ and $s_L = (L - k)$. In Section 4 we make some concluding remarks and discuss applications.

2 Preliminaries

The objects that we will use for our construction are the extended and left-hooked extended Langford sequences from [8, 9]. The necessary conditions for the existence of these is

Lemma 2.1. [6, 9] *The necessary conditions for a k -extended or a left-hooked k -extended Langford sequences are :*

$$(m, k) \equiv (0, 1), (1, d), (2, 0), (3, d + 1) \pmod{(4, 2)}.$$

Furthermore, for the non-hooked case $m \geq 2d - 3$ and

$$\frac{m}{2}(2d - 1 - m) + 1 \leq k \leq \frac{m}{2}(m - 2d + 5) + 1;$$

and for the left-hooked case $m + \frac{2}{m} \geq 2d - 3$ and

$$\frac{m}{2}(2d - 1 - m) \leq k \leq \frac{m}{2}(m - 2d + 5) + 2.$$

We note that although formally, an extension larger than $2m + 1$ and $2m + 2$ respectively make sense, they are not interesting from the point of view of actual sequences and so are generally not considered.

For any fixed d the sufficiency of these conditions has been established for all but a finite number of m :

Theorem 2.2. [8, 9] *The necessary conditions for the existence of extended Langford sequences are sufficient when $m \geq 2d - 1$ and $m \notin [2d + 2, 2d + 4] \cup [2d + 8, 8d - 5]$. The necessary conditions for the existence of left-hooked extended Langford sequences are sufficient whenever $m \geq 2d - 1$ and $m \notin [2d + 2, 2d + 4] \cup [2d + 8, 8d + 3]$.*

Next, in Section 3, we will use these extended sequences to build stretched Langford sequences, both non-hooked and hooked. In each case the construction method will be similar: for a given stretched right-hook we will use an extended sequence with an appropriate k and insert a new difference, (k, L) , equivalently in the sequence we observe that the left instance of the new difference is at k and the right instance is at position L .

3 Stretched Langford Sequences

Rather than partitioning the set $[1, 2m + 1] \setminus \{k\}$ into differences, we now ask to partition the set $[1, 2m - 1] \cup \{L\}$ into differences;

Definition 3.1. *For $L \geq 2m$, an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$, is a partition the set $[1, 2m - 1] \cup \{L\}$ into differences $[d, d + m - 1]$.*

Similarly, rather than partitioning the set $[1, 2m + 2] \setminus \{k, 2m + 1\}$ into differences, we can also ask to partition the set $[1, 2m] \setminus \{2\} \cup \{L\}$ into differences; we call the corresponding sequence a *left-hooked stretched Langford sequence* and denote it by $h\mathcal{L}_d^m[L]$.

We first establish the necessary conditions for these partitions to exist.

Lemma 3.2. *The following are necessary for the existence of an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$:*

1. $m \geq 2d - 2$ or $m = 1$, and
 - (a) If $m = 2d - 2$, then $L = 5d - 5$,
 - (b) If $m = 2d - 1$, then $4d - 2 \leq L \leq 6d - 4$,

(c) If $m \geq 2d$, then $2m \leq L \leq 3m + d - 2$.

2. $(m, L) \equiv (0, 0), (1, d + 1), (2, 1)$ or $(3, d) \pmod{(4, 2)}$.

Proof: If a $\mathcal{L}_d^m[L]$ exists, then there is a partition of $P = [1, 2m - 1] \cup \{L\} = \{p_1, p_2, \dots, p_{2n}\}$, with $p_1 < p_2 < \dots < p_{2n}$, into differences in $D = [d, d + m - 1] = \{d_1, d_2, \dots, d_n\}$. We have

$$\sum_{i=1}^m d_i = \sum_{i=1}^m (d + i - 1) = md + m(m - 1)/2, \quad (2)$$

$$\sum_{i=1}^{2m} p_i = \left(\sum_{i=1}^{2m-1} i \right) + L = m(2m - 1) + L. \quad (3)$$

Since each d_i can be expressed as a difference of two p_i and these pairs partition P , $\sum d_i \equiv \sum p_i \pmod{2}$ so by the above and some algebra $L \equiv md + m(m + 1)/2 \pmod{2}$; the congruences on (m, L) follow.

We have $L \geq 2m$ from the definition of an L -stretched Langford sequence, and clearly L is at most the largest difference plus the second-largest position, so $L \leq (2m - 1) + (d + m - 1) = 3m + d - 2$. If there exists a partition of P into differences in D , then Equation 2 can be expressed as a difference of two disjoint sums of m elements over P . The positive sum can never be larger than the sum of the m largest elements of P and negative sum can never be smaller than the sum of the m smaller elements of P . Thus,

$$\sum_{i=1}^n d_i \leq \sum_{i=1}^n (p_{i+n} - p_i). \quad (4)$$

This yields

$$\begin{aligned} \sum_{i=1}^m d_i &= md + m(m - 1)/2 \\ &\leq ((m + 1) + \dots + (2m - 1) + L) - (1 + \dots + m) \\ &= m^2 - 2m + L, \end{aligned} \quad (5)$$

and isolating L gives $m(2d + 3 - m)/2 \leq L$. Putting these together gives

$$\max\{2m, m(2d + 3 - m)/2\} \leq L \leq 3m + d - 2, \quad (6)$$

from which follow (1c), the lower bound of (1b) and the lower bound of (1a), i.e., $5d - 5 \leq L$. It now suffices to prove (1) and the upper bounds in (1a),(1b) under the assumptions $m \leq 2d - 1$ and $m > 1$.

Some difference $\delta \in D = [d, d + m - 1]$ is in positions $L, L - \delta$, so there exists a partition of $P' = [1, 2m - 1] \setminus \{L - \delta\}$ into differences in $D' = [d, d + m - 1] \setminus \{\delta\}$. The sum of the differences in D' is

$$\sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i = m(m + 2d - 1)/2 - \delta,$$

which is bounded below by

$$m(m + 2d - 1)/2 - (m + d - 1).$$

By Inequality 4, this must be less than the sum of the $m - 1$ largest positions minus the sum of the $m - 1$ smallest positions in P' . What this sum is exactly depends on where $L - \delta$ is in P' but it achieves its maximum, $m(m - 1)$, when $L - \delta = m$. Putting these together we get

$$\frac{m(m + 2d - 1)}{2} - (m + d - 1) \leq m(m - 1).$$

This reduces to

$$0 \leq (m - (2d - 2))(m - 1),$$

from which we can conclude that $m \geq 2d - 2$ because we can assume that $m > 1$.

To establish the upper bounds in (1a) and (1b) we consider two cases according to the position of $L - \delta$ with respect to $m - 1$. If $L - \delta \leq m - 1$ then

$$L \leq m - 1 + \delta \leq 2m + d - 2.$$

When $m = 2d - 2$, this contradicts our previous lower bound, and when $m = 2d - 1$ we get $L \leq 5d - 4 < 6d - 4$.

When $L - \delta \geq m$ Inequality 4 for P' and D' is

$$\begin{aligned} \sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i &= m(m + 2d - 1)/2 - \delta \\ &\leq (m + \dots + (2m - 1) - (L - \delta)) - (1 + \dots + (m - 1)) \\ &= m^2 - L + \delta. \end{aligned} \tag{7}$$

From (6) we have $m(2d - m + 3)/2 \leq L$, so isolating 2δ in (7) gives

$$m(2d + 1 - m) \leq L + m(2d - 1 - m)/2 \leq 2\delta \leq 2(m + d - 1). \tag{8}$$

Finally from (8) we get $L \leq 2(m + d - 1) - m(2d - 1 - m)/2$, from which the upper bounds of (1a) and (1b) follow. ■

■

In a similar vein we establish the necessary conditions for left-hooked stretched Langford sequences.

Lemma 3.3. *The following are necessary for the existence of a left-hooked L -stretched Langford sequence of defect d and m differences, $h\mathcal{L}_d^m[L]$:*

1. $m \geq 2d - 2$ or $m = 1$ or $(d, m, L) = (3, 2, 7), (3, 3, 9)$, and
 - (a) If $m = 2d - 2$, then $L = 5d - 5$, or $5d - 3$,
 - (b) If $m = 2d - 1$, then $4d - 1 \leq L \leq 6d - 2$,
 - (c) If $m \geq 2d$, then $2m + 1 \leq L \leq 3m + d - 1$;
2. $(m, L) \equiv (0, 0), (1, d + 1), (2, 1)$ or $(3, d) \pmod{(4, 2)}$,
3. $(d, m, L) \notin \{(1, 1, 4), (2, 3, 8)\}$.

Proof: The congruence on (m, L) is the same as in Lemma 3.2 since in the present case $P = [1, 2m - 1] \cup \{L\}$ is altered by changing position 2 to position $2m$, and each is even. The change in the sum of the differences is therefore precisely $2m - 2$. The change in the sums of the m largest and m smallest positions is an increase of $m - 1$ in each sum. Since the left and right hand side of Inequality 4 change by the same amount, the final lower bound, $m(2d + 3 - m)/2 \leq L$, is unchanged. The upper bound on L is one larger as the second largest position is one larger. The analog to (6) is

$$\max\{2m + 1, m(2d + 3 - m)/2\} \leq L \leq 3m + d - 1. \quad (9)$$

Now (1c) and the lower bounds in (1a),(1b) follow from (9). When $m \leq 3$, Inequality 9 gives $m \leq 4$. If $d = 4$ and $(m = 2$ or $m = 3)$ it is easy to verify that $L = 9$, or 12 respectively and that these Langford sequences do not exist. Thus the only $h\mathcal{L}_d^m[L]$ with $m \leq 3$ are:

$$h\mathcal{L}_d^1[d + 1] = d, 0^{d-1}, d \quad (d > 1);$$

$$h\mathcal{L}_1^2[5] = 2, 0, 2, 1, 1;$$

$$h\mathcal{L}_2^2[5] = 3, 0, 2, 3, 2;$$

$$h\mathcal{L}_2^2[7] = 2, 0, 2, 3, 0, 0, 3;$$

$$h\mathcal{L}_3^2[7] = 3, 0, 4, 3, 0, 0, 4;$$

$$h\mathcal{L}_1^3[7] = 3, 0, 2, 3, 2, 1, 1$$

$$(\text{or } h\mathcal{L}_1^3[7] = 2, 0, 2, 3, 1, 1, 3);$$

$$h\mathcal{L}_1^3[9] = 2, 0, 2, 1, 1, 3, 0, 0, 3;$$

$$h\mathcal{L}_2^3[10] = 3, 0, 2, 3, 2, 4, 0, 0, 0, 4;$$

$$h\mathcal{L}_3^3[9] = 4, 0, 3, 4, 5, 3, 0, 0, 5.$$

The $h\mathcal{L}_2^3[8] = h\mathcal{L}_2^3(7)$ and $h\mathcal{L}_3^3[7] = h\mathcal{L}_3^3$ do not exist because the necessary conditions for the equivalent non-stretched sequences given are not satisfied. So if $m \leq 3$, then all necessary conditions are satisfied, so we may take $4 \leq m \leq 2d - 1$ in what follows.

Some difference $\delta \in D = [d, d + m - 1]$ is in positions $L, L - \delta$, so there exists a partition of $P' = [1, 2m] \setminus \{2, L - \delta\}$ into differences in $D' = [d, d + m - 1] \setminus \{\delta\}$. The sum of the differences in D' is

$$\sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i = m(m + 2d - 1)/2 - \delta,$$

which is bounded below by

$$m(m + 2d - 1)/2 - (m + d - 1).$$

By Inequality 4, this must be less than the sum of the $m - 1$ largest positions minus the sum of the $m - 1$ smallest positions. What this sum is exactly depends on where $L - \delta$ is in P' but it achieves its maximum, $m(m - 1) + 1$, when $L - \delta = m + 1$. Putting these together we get

$$\frac{m(m + 2d - 1)}{2} - (m + d - 1) \leq m(m - 1) + 1.$$

This reduces to

$$-2 \leq (m - (2d - 2))(m - 1),$$

from which we can conclude that $m \geq 2d - 2$ because we can assume that $m \geq 4$.

To establish the upper bounds in (1a) and (1b) we consider two cases: the position of $L - \delta$ with respect to m . If $L - \delta \leq m$, then

$$L \leq m + \delta \leq 2m + d - 1.$$

When $m = 2d - 2$, this reduces to $L \leq 5d - 5 < 5d - 3$, and when $m = 2d - 1$, we get $L \leq 5d - 3 < 6d - 2$.

When $L - \delta \geq m + 1$, Inequality 4 for P' and D' is

$$\begin{aligned} \sum_{\substack{1 \leq i \leq m \\ d_i \neq \delta}} d_i &= m(m + 2d - 1)/2 - \delta \\ &\leq [(m + 1) + \cdots + 2m - (L - \delta)] - (1 + \cdots + (m - 2)) \\ &= m^2 + 2 - L + \delta. \end{aligned} \tag{10}$$

From (9) we have $m(m - 2d + 3)/2 \leq L$, so isolating 2δ in (10) gives

$$\begin{aligned} m(2d + 1 - m) - 2 &\leq L + m(2d - 1 - m)/2 - 2 \\ &\leq 2\delta \leq 2(m + d - 1). \end{aligned} \tag{11}$$

From Inequality (11) we get $L \leq 2(m+d-1) - m(2d-1-m)/2 + 2$, from which the upper bounds of (1a) and (1b) follow. Finally, when $m = 2d - 2$ the congruence on (m, L) shows that $L \neq 5d - 4$. ■

We now develop constructions for stretched Langford sequences. While a $h\mathcal{L}_d^m(k)$ is typically considered right-hooked, it is nevertheless more convenient to state the next result for a left-hooked version of the extended Langford sequence.

Lemma 3.4 (Adding difference $d+m-1$): *If there exists a $\mathcal{L}_d^{m-1}(k)$ with $k \geq m - d + 1$ then there exists a partition of $[1, 2m - 1] \cup \{k + d + m - 1\}$ into differences $d, d + 1, \dots, d + m - 1$, namely a $\mathcal{L}_d^m[k + d + m - 1]$. Similarly, if there exists a $h\mathcal{L}_d^{m-1}(k)$ with $k \geq m - d + 2$ then there exists a partition of $[1, 2m - 1] \setminus \{2\} \cup \{k + d + m - 1\}$ into differences $d, d + 1, \dots, d + m - 1$, namely a $h\mathcal{L}_d^m[k + d + m - 1]$.*

Proof: To each extended Langford sequence, $\mathcal{L}_d^{m-1}(k)$, add the difference $d + m - 1$ at positions $k, k + d + m - 1$. ■

Example 3.5. For $d = 2, m = 4$, the possible L are 8, 10 and 12. When $L = 10$ this is simply a Langford sequence:

52423543.

For $L = 10$ we start with an extended Langford sequence 2423_43 and add the pair (5, 10) to get

2423543_5.

For $L = 12$ we start with the Langford sequence 423243 and add the pair (7, 12) to get

4232435_5.

Example 3.6. To construct a decomposition of $\{1, 3, 4, 5, 6, 7, 8, 9, 10, 15\}$ with $d = 2, m = 5$ start with 5_242534_3 and add the pair (9, 15) to get

5_24253463_5.

Lemma 3.7 (Adding difference d): *If there exists a $\mathcal{L}_{d+1}^{m-1}(k)$ with $k \geq 2m - d$ then there exists a partition of $[1, 2m - 1] \cup \{k + d\}$ into differences $d, d + 1, \dots, d + m - 1$, namely a $\mathcal{L}_d^m[k + d]$.*

Similarly, if there exists a $h\mathcal{L}_{d+1}^{m-1}(k)$ with $k \geq 2m - d + 1$ then there exists a partition of $[1, 2m - 1] \setminus \{2\} \cup \{k + d\}$ into differences $d, d + 1, \dots, d + m - 1$, namely a $h\mathcal{L}_d^m[k + d]$.

Proof: To each extended Langford sequence, $\mathcal{L}_{d+1}^{m-1}(k)$, add the difference d at positions $k, k + d$. ■

From these Lemmata and the results known about extended Langford sequences [1, 6, 7, 8, 9] we can derive some sufficient conditions for stretched and left-hooked stretched Langford sequences to exist.

Theorem 3.8. *Assume the necessary conditions of Lemma 3.2 are satisfied. Any condition below is then sufficient for the existence of an L -stretched Langford sequence of defect d and m differences, $\mathcal{L}_d^m[L]$:*

1. $m = 2d - 2$ or $m = 1$;
2. $L = 2m, 2m + 1$, or $3m + d - 2$;
3. $L = 2m + 2$, $m \geq 2d - 1$ and $m \notin [2d + 2, 8d + 4]$;
4. $d = 1, 2, 3$ or 4 ;
5. $2m + 1 < L < 3m + d - 2$, $m \geq 2d$, and $m \notin [2d + 3, 2d + 5] \cup [2d + 9, 8d - 4]$;
6. $2m < L < 2m + d - 1$, $m \geq 2d + 2$, and $m \notin [2d + 5, 2d + 7] \cup [2d + 11, 8d + 4]$.

Proof: 1. If $m = 2d - 2$, then $L = 5d - 5$ by Lemma 3.2(1), and a $\mathcal{L}_d^{2d-2}[5d - 5]$ comes from a $\mathcal{L}_d^{2d-3}(2d - 2)$ (given in [8], p. 423, first paragraph) and Lemma 3.4. The case $m = 1$ is trivial.

2. We have $\mathcal{L}_d^m[2m] = \mathcal{L}_d^m$ and $\mathcal{L}_d^m[2m + 1] = h\mathcal{L}_d^m$, and these exist [8, 9]. If $L = 3m + d - 2$, then take a \mathcal{L}_d^{m-1} and add the difference $m + d - 1$ at positions $2m - 1$ and $3m + d - 2$.

3. Reversing a $\mathcal{L}_d^m[2m + 2]$ yields a $h\mathcal{L}_d^m(3)$. The current existence conditions for these are described in [9].

4. For $m \geq 2d$ or $d = 1, 2$ apply Lemma 3.4 to the known extended Langford sequences [8, 9]. It only remains to give sequences for $d = 3, 4$;

$m = 2d - 1$ and $2m + 2 = 4d \leq L \leq 6d - 4$:

$$\begin{aligned} \mathcal{L}_3^5[12] &= h\mathcal{L}_3^5(10)[8]; \\ \mathcal{L}_3^5[14] &= 4, 6, 3, 5, 4, 3, 7, 6, 5, 0, 0, 0, 0, 7; \\ \mathcal{L}_4^7[16] &= 6, 7, 8, 9, 5, a, 6, 4, 7, 5, 8, 4, 9, 0, 0, a; \\ \mathcal{L}_4^7[18] &= 9, 4, 8, 5, 7, 4, 6, a, 5, 9, 8, 7, 6, 0, 0, 0, 0, a; \\ \mathcal{L}_4^7[20] &= 7, 5, 8, 9, 4, 6, 5, 7, 4, a, 8, 6, 9, 0, 0, 0, 0, 0, a. \end{aligned}$$

5. An $\mathcal{L}_d^m(k)$ exists whenever the necessary conditions are satisfied and $m \in [2d - 1, 2d + 1] \cup [2d + 5, 2d + 7] \cup [8d - 4, \infty)$ [1, 6, 8]. Add difference $d + m$ to these extended sequences as in Lemma 3.4 to get $\mathcal{L}_d^{m+1}[L]$ for all admissible L , and with $m + 1 \in [2d, 2d + 2] \cup [2d + 6, 2d + 8] \cup [8d - 3, \infty)$. Now replace $m + 1$ by m , and take the set-complement.

6. Add difference $d - 1$ to the extended sequences in Case 5 according to Lemma 3.7 to get $\mathcal{L}_{d'}^{m'}[L]$ for $d' = d - 1$, $m' = m + 1$, $L \leq 2m' + d' - 1$ and with $m' = m + 1 \in [2d' + 2, 2d' + 4] \cup [2d' + 8, 2d' + 10] \cup [8d' + 5, \infty)$. Replace m', d' by m, d , and take the set-complement to get the result. ■

We have a similar result for left-hooked L -stretched Langford sequences.

Theorem 3.9. *Assume the necessary conditions of Lemma 3.3 are satisfied. Any condition below is then sufficient for the existence of a left-hooked L -stretched Langford sequence of defect d and m differences, $h\mathcal{L}_d^m[L]$:*

1. $m = 2d - 2$ or $m = 1$ or $(d, m, L) = (3, 2, 7), (3, 3, 9)$;
2. $L = 2m + 1$, or $3m + d - 1$;
3. $d = 1, 2, 3$;
4. $2m + 1 < L < 3m + d - 1, m \geq 2d$ and $m \notin [2d + 3, 8d + 5]$;
5. $2m + 1 < L < 2m + d, m \geq 2d + 2$ and $m \notin [2d + 5, 8d + 13]$.

Proof: 1. A family of $h\mathcal{L}_d^{2d-3}(2d - 1)$ for $d \geq 3$ is

$$\begin{aligned} &3453.4.5, \\ &567845.647.8, \\ &789ab567.8596a.b, \\ &9abcde6789.a6b7c8d.e, \\ &\vdots \end{aligned}$$

and a family of $h\mathcal{L}_d^{2d-3}(2d-3)$ for $d \geq 2$ is

$\underline{2,2,}$
 $53_435_4,$
 $7568_45764_8,$
 $b9758a_6579b86_a,$
 $db97ce8a_679bd86ca_e,$
 \vdots

Reverse these and apply Lemma 3.4 to get (resp.) $h\mathcal{L}_d^{2d-2}[5d-3]$, $d \geq 2$, and $h\mathcal{L}_d^{2d-2}[5d-5]$, $d \geq 3$. If $d = 2$, then $h\mathcal{L}_d^{2d-2}[5d-5] = h\mathcal{L}_2^2[5]$ is a reverse $\mathcal{L}_2^2(2)$ [3, 13]. The case $m = 1$ is trivial and the two sporadic sequences are in the proof of Theorem 3.3.

2. For $L = 2m + 1$ reverse a $h\mathcal{L}_d^m$ [3, 13]. For $L = 3m + d - 1$ reverse a $h\mathcal{L}_d^{m-1}$ and add the difference $m + d - 1$ at positions $2m$ and $3m + d - 1$.

3. By the above and the construction of all $h\mathcal{L}_d^m[L]$ with $m \leq 3$ in the proof of Theorem 3.3, it remains to give $h\mathcal{L}_d^m[L]$ for

(1b)' $m = 2d - 1 \geq 5$, $L \in [4d, 6d - 2]$,

(1c)' $m \geq 2d \geq 4$, $L \in [2m + 2, 3m + d - 2]$,

L of correct parity and (d, m, L) not in the list of exceptions. For $d = 1$ apply Lemma 3.4 to the required $h\mathcal{L}_1^m(k)$ to get all required $h\mathcal{L}_1^m[L]$ [14]. For $d = 2$ apply Lemma 3.4 to the required $h\mathcal{L}_2^m(k)$ to get all required $h\mathcal{L}_2^m[L]$ except $h\mathcal{L}_2^4[12] = 5, 0, 2, 3, 2, 5, 3, 4, 0, 0, 4$ [6]. For $d = 3$ apply Lemma 3.4 to the required $h\mathcal{L}_3^m(k)$ ($d = 3$) to get all required $h\mathcal{L}_3^m[L]$ except $h\mathcal{L}_3^5[12] = 5, 0, 6, 4, 7, 5, 3, 4, 6, 3, 0, 7$; $h\mathcal{L}_3^5[14] = 4, 0, 5, 6, 4, 3, 7, 5, 3, 6, 0, 0, 0, 7$; $h\mathcal{L}_3^5[16] = 6, 0, 3, 4, 5, 3, 6, 4, 7, 5, 0, 0, 0, 0, 7$ [9].

4. A $h\mathcal{L}_d^m(k)$ exists for all admissible k whenever $m \in [2d - 1, 2d + 1] \cup [8d + 5, \infty)$ [8, 9]. Reverse the sequences and add difference $d + m$ as in Lemma 3.4 to get $h\mathcal{L}_d^{m+1}[L]$ for all admissible L and $m + 1 \in [2d, 2d + 2] \cup [8d + 6, \infty)$. Now replace $m + 1$ by m and take set complements.

5. Proceed as in the preceding case, but apply Lemma 3.7 instead. ■

4 Conclusion and Applications

We construct $\mathcal{L}_d^m[L]$ sequences primarily from $\mathcal{L}_d^{m-1}(k)$ or $\mathcal{L}_{d+1}^{m-1}(k)$ sequences, with similar constructions in the hooked case. Comparing the necessary conditions for extended and stretched Langford series and the two main constructions in Lemmata 3.4 and 3.7, it is plain that once the

necessary conditions for (hooked) extended Langford sequences are shown to be sufficient, then the existence of (hooked) stretched Langford sequences will be settled [8, 9]. *A priori* this strong connection is not obvious, as position L in a (hooked) stretched Langford sequence need not be occupied by the largest or smallest difference, but it becomes evident after Lemmas 3.2 and 3.3 are carefully worked out.

Given the necessary conditions formulated in Lemmas 3.2 and 3.3, we can restate the global behavior of the central theorem 3.8 in three ways :

1. For a given d and m , where m is sufficiently large with respect to d , every possible L of the correct parity and in the correct range can be achieved.
2. For a given d and $L \geq 16d$ then almost every $8d \leq m \leq L/2$ in the correct classes modulo 4 can be achieved.
3. For a given m and $L \geq 2m$ sufficiently close to $2m$, almost every d of the correct parity in the range $L - 3m \leq d \leq m/2$ can be achieved.

The primary motivation for the definition and construction of stretched Langford sequences is the construction of partitions of an interval into collections of sets that contain given differences a required number of times.

For example using several Langford sequences to construct a longer Langford sequence. Two hooked sequences can be "hooked" together to form a longer sequence that is still a partition of the longer interval into the multi-set union of the differences from the two sequences. The "stretched hook" is complementary to the extensions in an extended Langford sequence for this same application. Here we think of the extension as a "deep" hook. For example, we can hook together an extended Langford sequence with $d = 1$, $m = 3$ and $k = 4$, and a stretched Langford sequence with $d = 4$, $m = 6$ and $L = 15$ as shown in Equation 1 to construct a Skolem sequence with nine differences.

A second example of this use of a stretched Langford sequence is in the construction of base classes of cyclic resolvable designs. If a partial base resolution class leaves an interval of points in \mathbb{Z}_v uncovered together with an isolated point at some distance from the interval, then the class can be completed with a stretched Langford sequence. An example makes the idea much more complete: A CURD(76,57,26,8) is a pairwise balanced resolvable design with blocks of size 2 and 3 such that every resolution class has 26 blocks of size 2 and 8 blocks of size 3, i.e. of type $2^{26}3^8$. Using Theorem 3.4 from [4], we could build a CURD(76,57,26,8) if we had a $\lambda = 4$ CURD with $v = 19$ and partition $1^12^63^2$ with the additional property that the blocks of size three are a replicated $\lambda = 1$ packing on 19 points. This CURD has 57 resolution classes and so could conceivably be built cyclically from three base classes.

Since the triples must be a $\lambda = 1$ packing replicated we could ask that each base resolution class contain the triples $\{0, 1, 9\}$, $\{2, 5, 7\}$ which develops into a packing on 19 points. If we also add pairs $\{4, 10\}$ and $\{6, 18\}$ we have used each difference $\{1, 2, 3, 5, 6, 7, 8, 9\}$ three times each and it remains to place the difference 4 four times and all other differences once more across the three base resolution classes. Since the points untouched in each class so far are an interval with two isolated points we can fill one of the base classes with a stretched Langford sequence with parameters $d = 1$, $m = 4$ and $L = 10$.

Only two more classes remain to be completed, rather than three! A quick search reveals that the other two base classes can be completed with

$$\begin{aligned} &\{3, 12\}, \{8, 14\}, \{11, 16\}, \{13, 17\} \\ &\{3, 11\}, \{8, 15\}, \{12, 16\}, \{13, 17\} \end{aligned}$$

The use of the stretched Langford sequence reduced the search space dramatically. For larger designs the application should be just as useful.

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