OO-Irredundance and Maximum Degree in Paths and Trees

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Abstract

A vertex set X of a simple graph is called OO-irredundant if for each $v \in X$, $N(v) - N(X - \{v\}) \neq \emptyset$. Basic results for maximal OO-irredundant set of a graph are obtained.

Keywords: OO-irredundance, irredundance

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1 Introduction

We first define four types of vertex subsets of a simple graph $G = (V, E)$. The set $X \subseteq V$ is: **CC-irredundant** (resp. **OC-irredundant, CO-irredundant, OO-irredundant**) if and only if for each $v \in X$, $N[v] - N[X - \{v\}] \neq \emptyset$ (resp. $N(v) - N[X - \{v\}] \neq \emptyset$, $N[v] - N(X - \{v\}) \neq \emptyset$, $N(v) - N(X - \{v\}) \neq \emptyset$).

For example the prefix CO is used in the name CO-irredundant set because the first neighbourhood used in its definition is Closed and the second one Open. These sets may be characterised in terms of existence of private neighbours which we now define.

For $v \in X$, the vertex t is:

an X-external private neighbour (X-epn) of v if

$$t \in V - X \quad \text{and} \quad N(t) \cap X = \{v\}.$$

an X-internal private neighbour (X-ipn) of v if

$$t \in X \quad \text{and} \quad N(t) \cap X = \{v\}.$$

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an X-self private neighbour (X-spn) of v if

$$t = v \qquad \text{and} \qquad v \text{ is isolated in } G[X].$$

The following characterisations are easily proved:

$$X \text{ is } \left.\begin{matrix} \text{CC} \\ \text{OC} \\ \text{CO} \\ \text{OO} \end{matrix}\right\} \text{-irredundant if and only if}$$

$$\text{each } v \in X \text{ has} \begin{cases} \text{an } X\text{-epn or } X\text{-spn.} \\ \text{an } X\text{-epn.} \\ \text{an } X\text{-epn, } X\text{-ipn or } X\text{-spn.} \\ \text{an } X\text{-epn or } X\text{-ipn.} \end{cases}$$

The property CC-irredundance (usually the CC prefix is omitted) has been well studied due partially to its intimate connection with domination. For a survey of this theory, the reader is directed to [14].

The other three types of sets defined above were studied in [2, 3, 11, 12, 13]. We note that OC-irredundance was originally introduced in [9] under the alias "open irredundance." The reader is referred to [3] for results concerning these sets, further generalisations and an extensive bibliography.

Some of the existing results concern the following four parameters. Let

$$\left.\begin{matrix} ccir(G) \\ ocir(G) \\ coir(G) \\ ooir(G) \end{matrix}\right\} \text{denote the smallest cardinality}$$

$$\text{of a maximal} \begin{cases} \text{CC-irredundant set.} \\ \text{OC-irredundant set.} \\ \text{CO-irredundant set.} \\ \text{OO-irredundant set.} \end{cases}$$

At this point it should be emphasised that each of the four properties is hereditary so that for each of them maximality is equivalent to 1-maximality.

Sharp lower bounds involving order and maximum degree are known for $ccir(G)$, $ocir(G)$ and $coir(G)$ [4, 6, 12] and each of these bounds has been improved for trees [7].

In this paper we obtain a characterization of maximal open-open irredundant sets and use this to determine $ooir(P_n)$ and $ooir(C_n)$ for all n. In Section 5, we discuss a maximal Open-Open Irredundance in Bipartite graphs and show that vertex and edge deletion can have a dramatic effect the parameter $ooir$. Finally, we find a lower bounds involving order and maximum degree for $ooir(G)$, which is not tight and discuss some preliminary results towards finding a sharp lower bound of this form.

Some of the proofs involve the basic partition (X, B, C, R) of V induced by the vertex subset X where

$$\begin{aligned} B &= \{w \in V - X \,\|N(w) \cap X| = 1\} \\ C &= \{w \in V - X \,\|N(w) \cap X| \geq 2\} \\ R &= \{w \in V - X \,\|N(w) \cap X| = 0\}. \end{aligned}$$

Further we define

$$\begin{aligned} Z &= \{v \in X \,|N(v) \cap X = \emptyset\} \\ Y &= X - Z. \end{aligned}$$

To end the introduction we state an observation from [13] that OO-irredundance is related to total domination just as CC-irredundance is to domination.

Proposition 1 **(i)** *The total dominating set X is minimal if and only if X is also OO-irredundant.*

(ii) *If X is minimal total dominating, then X is maximal OO-irredundant.*

2 A characterisation of maximal OO-irredundant sets

For $v \in X \subseteq V$, define

$$PN(v, X) = N(v) - N(X - \{v\}) = \{w | w \text{ is an } X\text{-epn or } X\text{-ipn of } v\}$$

Proposition 2 *Let $v \in X$ and $w \in V - X$. Then*

(i) $PN(v, X \cup \{w\}) = PN(v, X) - N(w)$
and

(ii) $PN(w, X \cup \{w\}) = N(w) - N(X)$

Proof. (i) By definition

$$\begin{aligned} PN(v, X \cup \{w\}) &= N(v) - N((X \cup \{w\}) - \{v\}) \\ &= N(v) - N((X - \{v\}) \cup \{w\}) \\ &= N(v) - [N(X - \{v\}) \cup N(w)] \\ &= [N(v) - N(X - \{v\})] - N(w) \\ &= PN(v, X) - N(w). \end{aligned}$$

(ii)

$$\begin{aligned} PN(w, X \cup \{w\}) &= N(w) - (N(X \cup \{w\}) - \{w\}) \\ &= N(w) - N(X). \end{aligned}$$

□

In the remainder of the paper we will use the abbreviation OO-irr for OO-irredundant.

Theorem 3 *The OO-irr set X is maximal if and only if for all*

$$w \in N(R) \cup N(Z), \exists v \in X \text{ such that } PN(v, X) \subseteq N(w) \quad (1)$$

Proof. Suppose that X is maximal OO-irr and contrary to the result there exists a $w \in N(Z) \cup N(R)$ such that for each $v \in X$, $PN(v, X) - N(w) \neq \emptyset$. We observe that $w \notin X$, hence by maximality $X \cup \{w\}$ is not OO-irr.

By Proposition 2(i)

$$\text{for all } v \in X,\ PN(v, X \cup \{w\}) \neq \emptyset \quad (2)$$

Suppose $w \in N(R)$ and $wu \in E$ where $u \in R$. Then u is an $(X \cup \{w\})$-epn of w so that $u \in PN(w, X \cup \{w\})$ and

$$PN(w, X \cup \{w\}) \neq \emptyset \quad (3)$$

If $w \in N(Z)$ and $wu \in E$ where $u \in Z$, then u is an $(X \cup \{w\})$-ipn of w so that again $u \in PN(w, X \cup \{w\})$ and (3) holds. By (2) and (3), $X \cup \{w\}$ is OO-irr, a contradiction.

Conversely suppose that X is OO-irr, that (1) holds for all $w \in N(R) \cup N(Z)$ and contrary to maximality there exists $u \in V - X$ such that $X \cup \{u\}$ is OO-irr. If $u \in N(R) \cup N(Z)$, then by (1) there exists $v \in X$ such that $PN(v, X) \subseteq N(u)$. Hence

$$PN(v, X \cup \{u\})) = PN(v, X) - N(u) = \emptyset$$

and therefore $X \cup \{u\}$ is not OO-irr, a contradiction.

Finally suppose that $u \notin N(Z) \cup N(R)$ and $wu \in E$. Then $w \notin Z \cup R$ and so $w \in Y \cup B \cup C$. Hence

$$N(u) \subseteq Y \cup B \cup C \subseteq N(X).$$

By Proposition 2 (ii)

$$PN(u, X \cup \{u\}) = N(u) - N(X) = \emptyset$$

and so $X \cup \{u\}$ is not OO-irr, a contradiction. □

3 Maximal OO-irr on Paths

Label the vertices of P_n, x_1, x_2, ..., x_n such that $x_i x_{i+1} \in E$ for $1 \leq i \leq n$. This labelling will be used through out the remainder of the paper. We begin with a Lemma.

Lemma 4 *For any maximal OO-irr set X of P_n,*

(i) For $n \geq 3$, $x_1 \in X$ if and only if $x_3 \notin X$

(ii) For $n \geq 3$, $x_n \in X$ if and only if $x_{n-2} \notin X$

(iii) For $n \geq 6$, if $x_2 \notin X$, then $x_4, x_6 \in X$

(iv) For $n \geq 6$, if $x_{n-1} \notin X$, then $x_{n-3}, x_{n-5} \in X$

(v) For $n \geq 6$ if $x_2 \in X$ and $x_4 \notin X$, then $x_6 \in X$

(vi) For $n \geq 6$ if $x_{n-1} \in X$ and $x_{n-3} \notin X$, then $x_{n-5} \in X$

(vii) If $x_i, x_{i+2} \notin X$, then one of the following holds:

a) $n = 7$, $i = 3$ *and* $x_1, x_7 \in X$.
b) $n \geq 9$, $i = 3$ *and* $x_1, x_7, x_9 \in X$.
c) $n \geq 9$, $i = n - 4$ *and* $x_n, x_{n-6}, x_{n-8} \in X$ *OR*
d) $n \geq 11$, $5 \leq i \leq n - 6$ *and* $x_{i-4}, x_{i-2}, x_{i+4}, x_{i+6} \in X$

Proof. We prove parts (i), (iii), (v) and (vii). The remaining follow from the symmetry of P_n.

(i) If $x_1, x_3 \in X$ then $PN(x_1, X) = \emptyset$ and hence X would not be OO-irr, a contradiction. If $x_1, x_3 \notin X$, then $x_2 \in Z \cup R$ and $N(x_1) - N(X) = \{x_2\}$. By Theorem 3 X is not maximal.

(iii) Observe $x_1 \in Z \cup R$. By the maximality of X and Theorem 3 there exists a $w \in X$ such that $PN(w, X) \subseteq N(x_2)$. It must be the case that $w = x_4$ and $PN(x_4, X) = x_3$. The result follows.

(v) If $x_6 \notin X$, then $x_5 \in Z \cup R$. By the maximality of X and Theorem 3 there exists a $w \in X$ such that $PN(w, X) \subseteq N(x_4) = \{x_3, x_5\}$. Any such w must be from the set $\{x_2, x_6\} \cap X$, however, $x_1 \in PN(x_2, X)$ and $x_6 \notin X$, a contradiction.

(vii) Observing $x_{i+1} \in Z \cup R$, by the maximality of X and Theorem 3 it must be the case that $x_{i-2}, x_{i+4} \in X$,

$$\{x_{i-3}, x_{i-1}\} \subseteq PN(x_{i-2}, X) \subseteq N(x_i) = \{x_{i-1}, x_{i+1}\}$$

and

$$\{x_{i+3}, x_{i+5}\} \subseteq PN(x_{i+4}, X) \subseteq N(x_{i+2}) = \{x_{i+1}, x_{i+3}\}.$$

It follows that $PN(x_{i-2}, X) = \{x_{i-1}\}$ (resp. $PN(x_{i+4}, X) = \{x_{i+3}\}$) and hence x_{i-4}(resp. x_{i+6}) $\in X$ or x_{i-2} (resp. x_{i+4}) is a leaf. □.

We now define a function which will be used to count the number of vertices of a maximal OO-irr set X of P_n. Let

$$\eta_X(k) = |X \cap \{x_k, x_{k-2}, x_{k-4}, \dots, x_a\}| - |(V - X) \cap \{x_k, x_{k-2}, x_{k-4}, \dots, x_a\}|$$

where

$$a = \begin{cases} 1 & \text{if } k \text{ is odd} \\ 2 & \text{if } k \text{ is even} \end{cases}$$

Lemma 5 *For any maximal OO-irr set X of P_n,*

(i)
$$|X| = \frac{n + \eta_X(n) + \eta_X(n-1)}{2}.$$

(ii) For $k \geq 3$, $\eta_X(k) = \eta_X(k-2) + 1$ if $x_k \in X$ and $\eta_X(k) = \eta_X(k-2) - 1$ if $x_k \notin X$

(iii) If $k \equiv l \mod 8$ and $k > l$, then $\eta_X(k) \geq \eta_X(l)$.

(iv) $\eta_X(1), \eta_X(2), \eta_X(5), \eta_X(9), \eta_X(10) \geq -1$, $\eta_X(3), \eta_X(4), \eta_X(7), \eta_X(8) \geq 0$, $\eta_X(6) \geq 1$

(v) $\eta_X(n-1) \geq \eta_X(n-7) + 1$.

(vi) If n is even, $\eta_X(n) \geq 0$.

Proof. (i) Let $(a, b) = (1, 2)$ if n is odd and $(a, b) = (2, 1)$ otherwise. Then

$$\begin{aligned}
\eta_X(n) + \eta_X(n-1) =& |X \cap \{x_n, x_{n-2}, \dots x_a\}| - |(V - X) \cap \{x_n, x_{n-2}, \dots, x_a\}| \\
&+ |X \cap \{x_{n-1}, x_{n-3}, \dots, x_b\}| - |(V - X) \cap \{x_{n-1}, x_{n-3}, \dots, x_b\}| \\
=& |X| - |V - X| \\
=& |X| - (|V| - |X|) \\
=& 2|X| - n.
\end{aligned}$$

(ii) follows directly from the definition of η_X.

(iii) Let $k = l + 8i$. We proceed with induction on i. Suppose $i = 1$. By Lemma 4 (vii), at least two of $x_{l+2}, x_{l+4}, x_{l+6}, x_k$ are elements of X and hence by (ii),

$$\eta_X(k) \geq \eta_X(l) + 2 - 2 = \eta_X(l).$$

Now suppose $\eta_X(l+8m) \geq \eta_X(l)$ for some $m \geq 1$ and $n \geq l+8m+8$. By Lemma 4 (vii), at least two of $x_{l+8m+2}, x_{l+8m+4}, x_{l+8m+6}, x_{l+8m+8}$ are elements of X. It follows from (ii) that

$$\eta_X(l+8(m+1)) \geq \eta_X(l+8m)+2-2 \geq \eta_X(l).$$

(iv) By definition, $\eta_X(1), \eta_X(2) \geq -1$. From (ii) and Lemma 4 (i), $\eta_X(3) \geq 0$ and hence $\eta_X(5) \geq -1$. Further from (ii) and Lemma 4 (iii) and (v), $\eta_X(4) \geq 0$ and $\eta_X(6) \geq 1$. To see $\eta_X(7), \eta_X(8) \geq 0$, observe that at least two of x_1, x_3, x_5, x_7 and at least two of x_2, x_4, x_6, x_8 are elements of X by Lemma 4 (vii). It follows from (ii) that $\eta_X(9), \eta_X(10) \geq -1$.

(v) This follows from (ii), the definition of η_X and Lemma 4 (iii) and (v).

(vi) Parts (iii) and (iv) of this Lemma show $\eta_X(n-7) \geq -1$ and hence by part (v),

$$\eta_X(n-1) \geq 0$$

and therefore

$$|X \cap \{x_{n-1}, x_{n-3}, \ldots, x_1\}| \geq |(V-X) \cap \{x_{n-1}, x_{n-3}, \ldots, x_1\}|. \qquad (4)$$

Observe that the direction of the labelling of P_n was arbitrary. When n is even, by relabelling in the opposite direction, that is, relabel the vertex x_i with x_{n-i+1}, (4) gives,

$$|X \cap \{x_2, x_4, \ldots, x_n\}| \geq |(V-X) \cap \{x_2, x_4, \ldots, x_n\}|.$$

Hence under the new labelling $\eta_X(n) \geq 0$. □

Theorem 6 *For any $n > 1$,*

$$ooir(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 0,1,4 \mod 8 \\ \lfloor \frac{n+2}{2} \rfloor & \text{otherwise} \end{cases} \qquad (5)$$

Proof. For $n = 2$, the result may be easily checked. The proof of the rest of the theorem, contains two parts. First we define an OO-irr set of P_n (dependent on the value of n) and then show there is no OO-irr set of P_n with a smaller cardinality. For a given $n > 2$, let

$$X = \begin{cases} \{x_i | i \equiv 3,4,6,7 \mod 8\} & \text{if } n \equiv 1 \mod 8 \\ \{x_i | i \equiv 4,5,6,7 \mod 8\} \cup \{x_1, x_n\} & \text{if } n \equiv 2 \mod 8 \\ \{x_i | i \equiv 3,6,7,8 \mod 8\} \cup \{x_2\} & \text{if } n \equiv 3 \mod 8 \\ \{x_i | i \equiv 1,4,6,7 \mod 8\} & \text{if } n \equiv 4 \mod 8 \\ \{x_i | i \equiv 2,3,7,8 \mod 8\} \cup \{x_4\} & \text{if } n \equiv 5 \mod 8 \\ \{x_i | i \equiv 1,2,5,6 \mod 8\} & \text{if } n \equiv 6 \mod 8 \\ \{x_i | i \equiv 2,3,4,7 \mod 8\} & \text{if } n \equiv 7 \mod 8 \\ \{x_i | i \equiv 3,4,5,6 \mod 8\} & \text{if } n \equiv 0 \mod 8 \end{cases}$$

It can be checked that in each case X is a maximal OO-irr set of P_n whose cardinality matches the appropriate value in (5).

If X is any OO-irr set of P_n, $n > 2$ and $n \equiv 2 \mod 8$, then by Lemma 5 (i), (iii), (v), (vi) and (iv)

$$|X| = \frac{n + \eta_X(n) + \eta_X(n-1)}{2} \geq \left\lceil \frac{n + 0 + \eta_X(3) + 1}{2} \right\rceil \geq \left\lceil \frac{n+1}{2} \right\rceil.$$

If X is any OO-irr set of P_n, $n > 2$ and $n \equiv i \mod 8$ where $3 \leq i \leq 9$, then by Lemma 5 (i), (iii) and (v)

$$|X| = \frac{n + \eta_X(n) + \eta_X(n-1)}{2} \geq \frac{n + \eta_X(i) + \max\{\eta_X(i-1), \eta_X(i+1) + 1\}}{2}.$$

The necessary bounds follow by applying Lemma 5 (iv). □

4 Maximal OO-irr on Cycles

Analysis for finding $ooir(C_n)$, turns out to be much simpler. We keep the notation of Section 3 by adding an edge between the leaves of a labelled path and noting $C_n = P_n \cup \{x_1 x_n\}$.

Lemma 7 *For any maximal OO-irr set X of C_n, if $x_i, x_{i+2 \mod n} \notin X$, then*

$$x_{i-4 \mod n}, x_{i-2 \mod n}, x_{i+4 \mod n}, x_{i+6 \mod n} \in X.$$

Proof. Observing $x_{i+1 \mod n} \in Z \cup R$, by the maximality of X and Theorem 3 it must be the case that $x_{i-2 \mod n}, x_{i+4 \mod n} \in X$,

$$PN(x_{i-2 \mod n}, X) \subseteq N(x_i) = \{x_{i-1 \mod n}, x_{i+1 \mod n}\}$$

and

$$PN(x_{i+4 \mod n}, X) \subseteq N(x_{i+2 \mod n}) = \{x_{i+1 \mod n}, x_{i+3 \mod n}\}.$$

It follows that $PN(x_{i-2 \mod n}, X) = \{x_{i-1 \mod n}\}$ (resp. $PN(x_{i+4 \mod n}, X) = \{x_{i+3 \mod n}\}$) and hence $x_{i-4 \mod n}$(resp. $x_{i+6 \mod n}$) $\in X$. □

Theorem 8 *For any $n > 2$,*

$$ooir(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \mod 4 \\ \lfloor \frac{n+2}{2} \rfloor & \text{otherwise} \end{cases}$$

Proof. Let X be a maximal OO-irr set of C_n. Suppose u and v are vertices at distance two in C_n which are both not in X. By Lemma 7, the other vertices at distance two and the vertices at distance four from u and v must belong to X. We conclude from this that

$$|X| \geq \left\lceil \frac{n}{2} \right\rceil .$$

Suppose $n = 4k + 2$ for some integer k. Consider only the vertices of odd (resp. even) labels, that is consider the $2k + 1$ vertices x_i where i is odd (resp. even). By Lemma 8, and the same argument as above, observe that X must contain at least half of these vertices. Hence X must have at least $k+1$ elements with odd (resp. even) labels. It follows that

$$|X| \geq 2k + 2 = \frac{n+2}{2}.$$

By Theorem 3, the set $X = \{x_i | i \equiv 1 \text{ or } 2 \mod 4\}$ is a maximal OO-irr set of C_n, for all n, showing that the bounds found above can be achieved. □

5 Bipartite Graphs and Edge and Vertex Removal

The analysis of OO-irr on paths and cycles (in particular, Lemma 4 and Lemma 7) suggests that determining whether a set is a maximal OO-irr depends, in some way on the vertices at distance two from each vertex. We now show this is the case in bipartite graphs and relate OO-irredundant sets to OC-irredundant sets, referred to below as open irredundant sets, in bipartite graphs. Recall a set X is open irredundant if every vertex in X has an X-epn.

Proposition 9 *Suppose G is a bipartite graph with bipartition $V = S \cup T$.*

(i) *If X is OO-irr, then $X_S = X \cap S$ and $X_T = X \cap T$ are both open irredundant sets.*

(ii) *If $X_S \subseteq S$ and $X_T \subseteq T$ are both open irredundant sets, then $X = X_S \cup X_T$ is OO-irr.*

Proof. (i) Suppose X is an OO-irr set of G and $v \in X_S$. By definition, the vertex v must have an X-ipn or X-epn u. The vertex u is adjacent to no other vertex of X_S. Therefore u is an X_S-epn of v. Since v was arbitrary, we conclude every vertex of X_S has an X_S-epn and hence X_S is open irredundant. The same argument shows X_T is open irredundant.

(ii) Let $v \in X_S$. Since X_S in open irredundant and G is bipartite, v has an X_S-epn, say $u \in T$. Hence u is adjacent to no other vertex of X_S and no vertex of T. If $u \in X_T$, u is an X-ipn of v and if $u \notin X_T$, u is an X-epn of v. Since v was arbitrary, we conclude every vertex of X_S has an X-ipn or an X-epn. Using the same argument we can show every vertex of X_T has the same property and hence X is OO-irr. □

Corollary 10 *Suppose G is a bipartite graph with bipartition $V = S \cup T$. Then X is maximal OO-irr, if and only if (i) $X_S = X \cap S$ is maximal (with respect to vertex addition) among all open irredundant subsets of S in G and (ii) $X_T = X \cap T$ is maximal (with respect to vertex addition) among all open irredundant subsets of T in G.*

This corollary yields that finding a maximal open irredundant set, within each set in the bipartition of G, is equivalent to finding a maximal OO-irr set of G. This will be used in a simple example later in this section.

A second topic motivated by the analysis of maximal OO-irr sets on paths and cycles is studying the effects of a vertex and edge deletion on the parameter $ooir(G)$. For example, $ooir(P_6) = ooir(P_7) = ooir(P_8) = ooir(P_9) = 4$ and $ooir(P_{10}) = 6$, hence, deletion of a leaf may have no effect on the size of the smallest maximal OO-irr set or it may decrease the size by as much as two. On the other hand $ooir(P_9 - x_3) = ooir(P_2) + ooir(P_6) = 6$. This gives an example where the deletion of a vertex may increase the size of the smallest maximal OO-irr set by two. Indeed the following example shows that $ooir(G)$ can be a very unstable parameter, under vertex and edge deletion, even for a tree.

Example. Construct the tree S_k on $2k+1$ vertices by selecting a special vertex in each of k copies of P_2 and joining an edge between each of these special vertices and a new central vertex. A picture of S_4 is shown in F 1. We will call the central vertex u and label one stem v and the corresponding leaf w.

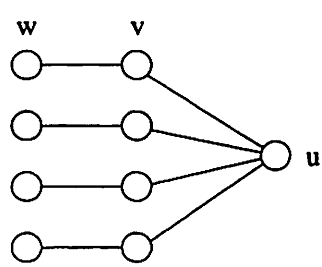

Figure 1: A picture of S_4

Lemma 11 *For all k,*

$$ooir(S_k) = k + 1.$$

Proof. The graph S_k is a bipartite graph. Let A be the bipartition containing u and B be the bipartition containing v. By Corollary 10, it is sufficient to find a smallest maximal (with respect to vertex addition) open irredundant subsets of each bipartition in S_k and checking their cardinalities add to $k+1$. The subset $\{u\}$ of A is a maximal open irredundant subset, since after the addition of any other vertex of A, say w, the new vertex would not have a $\{u, w\}$-epn in S_k. Since B itself is open irredundant (every vertex is adjacent to a leaf), this is the smallest possible. The proof is completed by observing $|\{u\}| + |B| = k + 1$. □

Now we observe that the deletion of the central vertex of S_k and the deletion of any leaf of S_k have dramatic, but opposite effects on the parameter *ooir*.

Lemma 12 *For a given k and labelling described above,*

$$ooir(S_k - \{u\}) = 2k,$$

and

$$ooir(S_k - \{w\}) = 2.$$

Proof. The first equality follows from the fact that $S_k - \{u\} \equiv kP_2$. The second equality follows from observing that $\{u, v\}$ form a maximal OO-irr set (by Theorem 3) and noting that by Corollary 10 that any maximal OO-irr set must contain at least one element in each set of the bipartition of $S_k - \{w\}$. □

This example can also be used to show the instability in the parameter *ooir* with respect to edge removal. Observing $S_k - \{vw\} = (S_k - w) \cup \{w\}$, we may use Lemma 12 to see that

$$ooir(S_k - \{vw\}) = ooir(S_k - w) = 2.$$

Further observation shows that the removal of the edge uv from $S_k - w$ is isomorphic to graph with components S_{k-1} and a singleton. Hence $ooir(S_k - w) = 2$, however,

$$ooir(S_k - w - uv) = k.$$

6 A bound on the smallest size of a maximal OO-irr set

We now proceed to attempting to find an analogous bound involving order and maximum degree as those found for $ccir(G)$, $ocir(G)$ and $coir(G)$ in [6, 4, 12].

That is we are looking for a bound of the form

$$ooir(G) \geq \frac{|V(G)|}{f(\Delta(G))}, \tag{6}$$

for some function f. This can be thought of as looking at a bound for $\frac{ooir(G)}{|V(G)|}$. That is, a bound on $ooir(G)$ as a percentage of the vertices. Results in this direction have proven difficult, despite techniques developed in finding similar bounds on the other irredundance parameters. We present some preliminary results towards finding a tight lower bound.

Lemma 13 *Every vertex in R is at distance at most two from a vertex in X.*

Proof. By Theorem 3, any vertex adjacent to another vertex in R (annihilates a vertex in X and hence) is adjacent to vertices in $N(X)$. □

Theorem 14 *For any isolate free graph with n vertices and maximum degree $\Delta(G)$,*

$$ooir(G) \geq \frac{n}{\Delta^2 - \Delta + 1}.$$

Proof. Let X be a maximal OO-irr set of G. For each $u \in X$ define r_u to be the number of vertices of R within two of u. Hence,

$$|R| \leq \sum_{u \in X} r_u. \tag{7}$$

Let $u \in Z$ and $v \in N(u)$. By Theorem 3, there exists a w such that $PN(w, X) \subseteq N(v)$. Hence in the neighbourhood of v there is u, at least one private neighbour of w (which is different from u since $u \in Z$) and at most $\Delta - 2$ vertices of R. Hence

$$r_u \leq |N(u)|(\Delta - 2) \leq \Delta^2 - 2\Delta \tag{8}$$

For $u \in X - Z$ and $v \in N(u) - X$, v is adjacent to u and at most $\Delta - 1$ vertices of R. Hence

$$r_u \leq |N(u) - X|(\Delta - 1) \leq \Delta^2 - 2\Delta + 1 \tag{9}$$

From Equations (7), (8) and (9),

$$\begin{aligned}
n &= |X| + |N(X) - X| + |R| \\
&\leq (|Z| + |Y|) + (\Delta|Z| + (\Delta - 1)|Y|) + \sum_{u \in X} r_u \\
&\leq (|Z| + |Y|) + (\Delta|Z| + (\Delta - 1)|Y|) + (\Delta^2 - 2\Delta)|Z| + (\Delta^2 - 2\Delta + 1)|Y| \\
&\leq (\Delta^2 - \Delta + 1)|Z| + (\Delta^2 - \Delta + 1)|Y| \\
&\leq (\Delta^2 - \Delta + 1)|X|
\end{aligned}$$

The theorem now follows. □

It can easily be seen that this bound is not tight. Evidence seems to suggest that this bound for all finite graphs can be improved to a bound of the form:

$$ooir(G) > \frac{n}{\frac{2}{3}\Delta^2 - O(\Delta)} + O\left(\frac{1}{\Delta}\right).$$

and that such a bound appears to be asymptotic. That is, equality could never be achieved, however for each value of Δ, there would be an infinite sequence of finite graphs G_i such that the limit of $ooir(G_i)$ would converge to the bound. We note that this is not in the form shown in Equation (6), however, the correction term at the end appears to be necessary.

Given a $\Delta > 3$, we now define an infinite graph H recursively, to represent some of the difficulties with obtaining this bound for finite graphs. Start by creating a cycle of length 3. For each vertex in this cycle, attach a new vertex and off of each vertex now at distance one from the original cycle, attach $\Delta - 1$ new vertices. Next for each vertex at distance two from the original cycle attach $\Delta - 3$ new vertices and two edges to two vertices in a copy of the graph H. This is feasible provided you embed enough additional copies of H. An illustration of the graph H is given in Figure 2. We note that for each vertex of each copy of the original cycle we add $\Delta^2 - 3\Delta + 4$ new vertices.

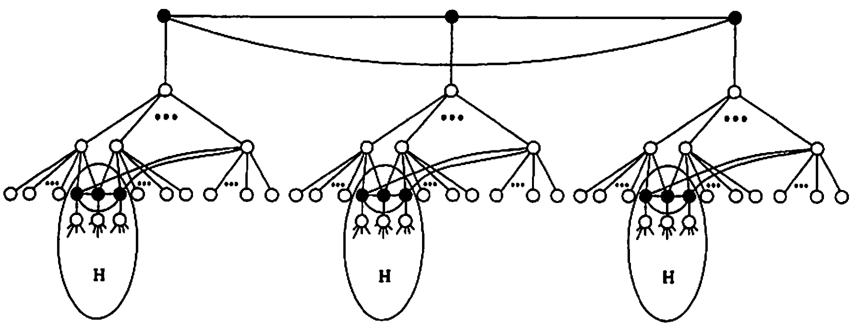

Figure 2: The recursively defined infinite graph H

Let X be the set containing the vertices of the original cycle in all copies of H. The vertices of X are black in Figure 2. By Theorem 3, the set X is a maximal OO-irr set. The graph H and the set X are infinite. However, from the point of view that we are looking for the percentage of vertices in a maximal OO-irr set, we may view (imprecisely) that for each vertex in X there are $\Delta^2 - 3\Delta + 4$ added to $V(H)$ or perhaps even more erroneously we may view this graph as having

$$\frac{|X|}{|V(H)|} = \frac{1}{\Delta^2 - 3\Delta + 4}.$$

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