Offensive Alliances in Bipartite Graphs

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Abstract

For a graph $G = (V, E)$, a non-empty set $S \subseteq V$ is a global offensive alliance (respectively, global strong offensive alliance) if for every vertex v in $V - S$, at least half of the vertices in its closed neighborhood are in S (respectively, a strict majority of its closed neighborhood are in S). The global offensive alliance number $\gamma_o(G)$ (respectively, global strong offensive alliance number $\gamma_{\hat{o}}(G)$) is the minimum cardinality of a global offensive alliance (respectively, global strong offensive alliance) of G. In this paper, we determine an upper bound on each parameter for bipartite graphs without isolated vertices. More precisely, we show that $\gamma_o(G) \leq (n - \ell + s)/2$ and $\gamma_{\hat{o}}(G) \leq (n + \ell)/2$, where n, ℓ and s are the order, the number of leaves and support vertices of G, respectively. Moreover, extremal trees attaining each bound are characterized.

Keywords: global offensive alliances, global strong offensive alliances, bipartite graphs, trees.

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1 Introduction

In a graph $G = (V, E)$ of order n, the *neighborhood* of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$. If S is a subset of vertices, its neighborhood is $N(S) = \cup_{v \in S} N_G(v)$. The *closed neighborhoods* of v and S are $N[v] = N(v) \cup \{v\}$ and $N[S] = N(S) \cup S$. The *degree* of a vertex v of G denoted by $\deg_G(v)$ is the size of its neighborhood. A vertex of degree one is called a *pendent vertex* or a *leaf* and its neighbor is called a *support* vertex. If v is a support vertex, then L_v will denote the set of the leaves attached at v. A support vertex v is called *strong* if $|L_v| > 1$. We also denote the set of leaves of a graph G by $L(G)$, the set of support vertices by $S(G)$, and let $|L(G)| = \ell(G)$, $|S(G)| = s(G)$. If $T = P_2$ then $\ell(P_2) = s(P_2) = 2$. If u is a vertex of a rooted tree T, we denote by T_u the subtree of root u. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A double star with respectively p and q leaves attached at each support vertex is denoted by $S_{p,q}$. A *subdivided star* is obtained from a star $K_{1,q}$ by subdividing each edge by exactly one vertex. For a graph $G = (V, E)$, a set S is a *dominating set* if every vertex in $V - S$ has at least a neighbor in S.

In this paper we are interested in two types of offensive alliances, namely global offensive alliances and global strong offensive alliances defined as follows: A dominating set S is called a *global offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| \geq |N[v] - S|$ and is a *global strong offensive alliance* if for every $v \in V - S$, $|N[v] \cap S| > |N[v] - S|$. The *global offensive alliance number* $\gamma_o(G)$ (respectively, *global strong offensive alliance number* $\gamma_{\hat{o}}(G)$) is the minimum cardinality of a global offensive alliance (respectively, global strong offensive alliance) of G. Every graph has a global (strong) offensive alliance, since $S = V$ is such a set. We abbreviate global offensive alliance as *goa* and global strong offensive alliance as *gsoa*. Alliances in graphs were introduced by Hedetniemi, Hedetniemi, and Kristiansen in [6].

In this paper, we show that every bipartite graph without isolated vertices satisfies $\gamma_o(G) \leq (n - \ell(G) + s(G))/2$ and $\gamma_{\hat{o}}(G) \leq (n + \ell(G))/2$. For each upper bound we characterize the class of extremal trees.

2 Global offensive alliances

We begin by a couple of observations.

Observation 1 *If G is a connected graph of order at least three, then there is $\gamma_o(G)$-set that contains all the support vertices.*

Observation 2 *Let T be a tree obtained from a nontrivial tree T' by attaching a path $P_2 = yx$, with an edge edge yz at a vertex z of T', where z is either a support vertex or a leaf. Then $\gamma_o(T) = \gamma_o(T')+1$.*

Proof. By Observation 1 there is a $\gamma_o(T)$-set S that contains all the support vertices. Hence $y \in S$. If z is a support of T', then $z \in S$ and $S - \{y\}$ is a goa of T'. If z is a leaf of T' then S contains either z or its unique neighbor in T' and hence $S - \{y\}$ is a goa of T'. In both cases, $\gamma_o(T') \leq \gamma_o(T) - 1$. The equality follows from the fact that every $\gamma_o(T')$-set can be extended to a goa of T by adding y. ■

Since each bipartition of a bipartite graph G without isolated vertices is a goa, $\gamma_o(G) \leq n/2$. Our next result improves this upper bound.

Theorem 3 *For every bipartite graph G without isolated vertices, $\gamma_o(G) \leq (n - \ell(G) + s(G))/2$.*

Proof. The result can be easily checked if $diam(G) \in \{1, 2\}$. Thus assume that $diam(G) \geq 3$ and consider the bipartite graph G' obtained from G by removing all its leaves. Then G' is nontrivial and admits a bipartition A, B. Let $A' = A - S(G)$ and $B' = B - S(G)$. Then $S(G) \cup A'$ and $S(G) \cup B'$ are two goa of G, implying that

$$\gamma_o(G) \leq s(G) + \min\{|A'|, |B'|\} \leq s(G) + (n - s(G) - \ell(G))/2.$$

■

Next we are interested in characterizing the trees that attain the bound in Theorem 3. For this purpose we introduce the family $\mathcal{G}$

of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is the path P_2, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the three operations listed below.

- Operation $\mathcal{O}_1$: Add a vertex attached by an edge at any support vertex of T_i.
- Operation $\mathcal{O}_2$: Add a path $P_2 = xy$ and join x by an edge to a support vertex z of T_i.
- Operation $\mathcal{O}_3$: Add $p \geq 1$ path(s) P_2 and join a vertex of each path by an edge to a leaf u of T_i adjacent to a support vertex w that is not a strong one with the condition that $p = 1$ if w has degree at least three.

Theorem 4 *Let T be a nontrivial tree of order n and with $\ell(T)$ leaves and $s(T)$ support vertices. Then $\gamma_o(T) = (n - \ell(T) + s(T))/2$ if and only if $T \in \mathcal{G}$.*

Proof. Let $T \in \mathcal{G}$. We proceed by induction on the number of operations $\mathcal{O}_i$ performed to construct T. Clearly if $T_1 = P_2$, then $\gamma_o(T) = (n - \ell(T) + s(T))/2$. Assume that the property is true for all trees of $\mathcal{G}$ constructed with $k - 1 \geq 0$ operations and let T be a tree of $\mathcal{G}$ constructed with k operations. Thus T is obtained by performing either Operation $\mathcal{O}_1$, $\mathcal{O}_2$ or $\mathcal{O}_3$ on a tree T' obtained by $k - 1$ operations. Let S be a $\gamma_o(T)$-set.

If T is obtained from T' by using Operation $\mathcal{O}_1$, then $\gamma_o(T) = \gamma_o(T')$. Since $n = n' + 1$, $\ell(T) = \ell(T') + 1$, $s(T) = s(T')$, by induction on T', we obtain $\gamma_o(T) = (n - \ell(T) + s(T))$.

If the last operation performed on T' is $\mathcal{O}_2$, then by Observation 2, $\gamma_o(T) = \gamma_o(T') + 1$. Since $n = n' + 2$, $\ell(T) = \ell(T') + 1$, $s(T) = s(T') + 1$, by induction on T', we obtained the desired result on $\gamma_o(T)$.

Assume now that the last operation performed on T' is $\mathcal{O}_3$ and let X be the set of new support vertices of T obtained by adding p paths P_2. Thus $|X| = p$, where $p = 1$ if $\deg_{T'}(w) \geq 3$. By Observation 1, there is a $\gamma_o(T)$-set S that contains X. Now if w has degree two in T', then S contains one of u or w, say w. Likewise if w has degree at least three in T', then $p = 1$ and without loss of generality, $w \in S$. Hence $S - X$ is a goa of T', and $\gamma_o(T') \leq \gamma_o(T) - |X|$. Furthermore, $\gamma_o(T) \leq \gamma_o(T') + |X|$ since every $\gamma_o(T')$-set can be extended to a goa of T by adding the set X, implying that $\gamma_o(T) = \gamma_o(T') + p$. Now since $n = n' + 2p$, $\ell(T) = \ell(T') + p - 1$, $s(T) = s(T') + p - 1$, by induction on T', we have $\gamma_o(T) = (n - \ell(T) + s(T))$.

To prove the converse we proceed by induction on the order of T. If $n = 2$ then $T = P_2$ which belongs to $\mathcal{G}$. If $n = 3$ then $T = P_3$ which belongs to $\mathcal{G}$ since it is obtained from P_2 by using Operation $\mathcal{O}_1$. Let $n \geq 4$ and assume that every tree T' of order n' less than n satisfying $\gamma_o(T') = (n' - \ell(T') + s(T'))/2$ is in $\mathcal{G}$. Let T be a tree of order n with $\gamma_o(T) = (n - \ell(T) + s(T))/2$. Since all stars satisfy $\gamma_o(T) = (n - \ell(T) + s(T))/2$ and are obtained from P_2 by using operation $\mathcal{O}_1$. Thus all stars belong to $\mathcal{G}$ and so we suppose that T has diameter at least three.

If T contains a strong support vertex, say t, then let T' be the tree obtained from T by removing a leaf adjacent to t. Then t remains a support vertex of T', and $\gamma_o(T') = \gamma_o(T)$. Since $n' = n - 1, \ell(T') = \ell(T) - 1$, and $s(T') = s(T)$, we obtain $\gamma_o(T') = (n' - \ell(T') + s(T'))/2$. By induction on T', $T' \in \mathcal{G}$. Thus $T \in \mathcal{G}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Henceforth, we can assume that T contains no strong support vertex.

Notice that in the proof of Theorem 3, we showed that $S(T) \cup A'$ and $S(T) \cup B'$ are two goa of T where $\min\{|A'|, |B'|\} \leq (n - \ell(T) - s(T))/2$. Therefore if $\gamma_o(T) = (n - \ell(T) + s(T))/2$ then $S(T) \cup A'$ and $S(T) \cup B'$ are two $\gamma_o(T)$-sets.

Root T at a vertex r, and let v be a support vertex of maximum distance from r and v' be the leaf-neighbor of v. Let u, w be the parent of v and u in the rooted tree, respectively. If w is a leaf, then $T = P_4$ and T belongs to $\mathcal{G}$ because it is obtained from P_2 by using Operation $\mathcal{O}_2$. So assume that w is not a leaf. We distinguish between three cases.

Case 1. u is a support vertex. Let $T' = T - \{v, v'\}$. Then $n' = n - 2, s(T') = s(T) - 1, \ell(T') = \ell(T) - 1$, and by Observation 2, $\gamma_o(T') = \gamma_o(T) - 1$. It follows that $\gamma_o(T') = (n' - \ell(T') + s(T'))/2$ and by induction on T', $T' \in \mathcal{G}$. Thus $T \in \mathcal{G}$ since it is obtained from T' by using Operation $\mathcal{O}_2$.

Case 2. u is not a support vertex but has at least one child say $b \neq v$ as a support vertex. Thus $\deg_T(u) \geq 3$. All children of u are support vertices and so are in $S(T)$. We now claim that $\deg_T(w) = 2$. Suppose to the contrary that $\deg_T(w) \geq 3$. Let us consider the bipartition of T' as considered in the proof of Theorem 3 and recall that $S(T) \cup A'$ and $S(T) \cup B'$ are two $\gamma_o(T)$-sets. Since u is neither a leaf nor a support vertex, then without loss of generality $u \in B'$. Thus $w \in A' \cup S(T)$ and all its neighbors are in $B' \cup S(T)$. Then $(B' - \{u\}) \cup S(T)$ is a goa of T of size $(n - \ell(T) + s(T))/2 - 1$, a contradiction. Thus $\deg_T(w) = 2$. Now let $T' = T - (T_u - \{u\})$. Clearly T' has order equal to at least four, otherwise T is a subdivided star with $\gamma_o(T) < (n - \ell(T) + s(T))/2$. Thus $n' = n - 2(\deg_T(u) - 1)$, $\ell(T') = \ell(T) - \deg_T(u) + 2$, and $s(T') = s(T) - \deg_T(u) + 2$. Since every $\gamma_o(T')$-set can be extended to a goa of T by adding the set $N(u) - \{w\}$, $\gamma_o(T) \leq \gamma_o(T') + \deg_T(u) - 1$. It follows from Theorem 3:

$$\begin{aligned}(n - \ell(T) + s(T))/2 = \gamma_o(T) &\leq \gamma_o(T') + \deg_T(u) - 1 \\ &\leq (n' - \ell(T') + s(T'))/2 + \deg_T(u) - 1 = \\ &= (n - \ell(T) - s(T))/2.\end{aligned}$$

Thus $\gamma_o(T') = (n' - \ell(T') + s(T'))/2$ and w is a support vertex of degree two in T' having u as a unique leaf. Applying the inductive hypothesis to T', we have $T' \in \mathcal{G}$. Thus $T \in \mathcal{G}$ since it is obtained from T' by using Operation $\mathcal{O}_3$.

Case 3. $\deg_T(u) = 2$. Based on the previous cases, we may assume that every child of w has degree at most two. Also since $T = P_5$ does not satisfy $\gamma_o(T) = (n - \ell(T) + s(T))/2$, we may assume that $n \geq 6$. We claim that every child (if any) of w besides u is a support vertex. Suppose first that w is a support vertex. Since $u \in B'$, w and $v \in S(T)$, $(B' - \{u\}) \cup S(T)$ is a goa of T of size $(n - \ell(T) + s(T))/2 - 1$, a contradiction. Likewise, assume that $b_1 b_2 b_3$

is a pendant path P_3 of T_w attached to w by b_1. Then b_2 is a support vertex, and $u, b_1 \in B'$. It follows that $(B' - \{u, b_1\}) \cup \{w\} \cup S(T)$ is a goa of T' of size $(n - \ell(T) + s(T))/2 - 1$, a contradiction.

Now let $T' = T - \{v, v'\}$. Then $n' = n - 2, \ell(T') = \ell(T)$ and $s(T') = s(T)$. By Observation 2, $\gamma_o(T) = \gamma_o(T') + 1$. It follows that $\gamma_o(T') = (n' - \ell(T') + s(T'))/2$. Now by induction on T', we have $T' \in \mathcal{G}$ and hence $T \in \mathcal{G}$ since it is obtained from T' by using Operation $\mathcal{O}_3$. This completes the proof. ■

3 Global strong offensive alliances

Observation 5 *For any graph G, the leaves of G are contained in every $\gamma_{\hat{o}}(G)$-set.*

Recall that the *2-domination number* of a graph G denoted by $\gamma_2(G)$ is the minimum cardinality of a subset S of V that dominates every vertex of $V - S$ at least twice. Observe that if S is a global strong offensive alliance of a graph G, then every vertex of $V - S$ has at least two neighbors in S, that is, S 2-dominates G. Hence we have the following:

Observation 6 *For every graph G, $\gamma_{\hat{o}}(G) \geq \gamma_2(G)$.*

In [2], Blidia, Chellali and Volkmann showed that every bipartite graphs G without isolated vertices, satisfies $\gamma_2(G) \leq (n + \ell(G))/2$, while extremal trees achieving this upper bound have been characterized by Blidia, Chellali and Favaron [1].

Theorem 7 *For every bipartite graph G without isolated vertices, $\gamma_{\hat{o}}(G) \leq (n + \ell(G))/2$.*

Proof. It is a routine matter to check the result if $diam(G) \in \{1, 2\}$. Thus assume that $diam(G) \geq 3$ and let A, B the bipartition of G' as considered in the proof of Theorem 3. Then every vertex of

A (resp. B) has at least two neighbors in $B \cup L(T)$ (resp. $A \cup L(T)$). Thus $A \cup L(T)$ and $B \cup L(T)$ are two gsoa of G and so

$$\gamma_{\hat{o}}(G) \leq \ell(G) + \min\{|A|, |B|\} \leq \ell(G) + (n - \ell(G))/2.$$

■

In order to characterize the nontrivial trees attaining the upper bound in Theorem 7, we define the family $\mathcal{F}$ of all trees T that can be obtained from a sequence $T_1, T_2, \ldots, T_k$ ($k \geq 1$) of trees, where T_1 is the path P_2, $T = T_k$, and if $k \geq 2$, T_{i+1} is obtained recursively from T_i by one of the two operations listed below.

- Operation $\mathcal{F}_1$: Add a star $K_{1,p}$, $p \geq 3$, centered at a vertex v, join v by an edge to a leaf u of T_i and add at most one new vertex adjacent to u.
- Operation $\mathcal{F}_2$: Add a star $K_{1,p}$, $2 \geq p \geq 1$ centered at x, join x by an edge to a leaf y of T_i, and add $j \geq 0$ new vertices adjacent to y.

Theorem 8 *Let T be a nontrivial tree of order n and with $\ell(T)$ leaves. Then $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$ if and only if $T \in \mathcal{F}$.*

Proof. We first prove the sufficient condition by induction on the number $k - 1$ of operations performed to construct T from $T_1 = P_2$. If $k = 1$, then $T = P_2$, and so $\gamma_{\hat{o}}(T) = (n + \ell(T))/2 = 2$. This establishes the basis case.

Assume now that $k \geqslant 2$ and that the result holds for all trees in $\mathcal{F}$ that can be constructed from less than $k-1$ operations, and let T be a tree of $\mathcal{F}$ constructed with k operations. We consider two cases.

Case 1. the last operation, performed on a tree T' obtained by $k - 1$ operations, is $\mathcal{F}_1$.

By Observation 5, u is in every $\gamma_{\hat{o}}(T')$-set, such a set can be extended to a global strong offensive alliance of T by adding the set

$L_v \cup L_u$, where $|L_u| = 1$ if a new vertex is added adjacent to u, and $|L_u| = 0$ else. Thus $\gamma_{\hat{o}}(T) \leq \gamma_{\hat{o}}(T') + |L_v| + |L_u|$. Again by Observation 5, if S is any $\gamma_{\hat{o}}(T)$-set, then S contains $L_v \cup L_u$ and one of v, u, say u. Hence $S - (L_v \cup L_u)$ is a global strong offensive alliance of T', and $\gamma_{\hat{o}}(T') \leq |S| - |L_v| - |L_u|$. It follows that $\gamma_{\hat{o}}(T) = \gamma_{\hat{o}}(T') + |L_v| + |L_u|$. Now since $n = n' + |L_v| + |L_u| + 1$ and $\ell(T) = \ell(T') + |L_v| + |L_u| - 1$, by induction on T' we obtain $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$.

Case 2. the last operation, performed on a tree T' obtained by $k-1$ operations, is $\mathcal{F}_2$.

As before, it can be seen that $\gamma_{\hat{o}}(T) = \gamma_{\hat{o}}(T') + |L_v| + j$, with $1 \leq |L_v| \leq 2$. Since $n = n' + |L_v| + j + 1$ and $\ell(T) = \ell(T') + |L_v| + j - 1$, by induction on T' we obtain $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$.

We now prove the converse by induction on the order of T. If $n = 2$ then $T = P_2$ which belongs to $\mathcal{F}$. Since P_3 does not satisfy $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$, the only tree of order four that satisfies $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$ is P_4 which belongs to $\mathcal{F}$, and is obtained from T_1 by using Operation $\mathcal{F}_2$.

Let $n \geq 5$ and assume that every tree T' of order $n' < n$ satisfying $\gamma_{\hat{o}}(T') = (n' + \ell(T'))/2$ is in $\mathcal{F}$. Consider a tree T of order n with $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$. Since no tree of diameter two satisfies $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$, we suppose that T has diameter at least three.

In the proof of Theorem 7, we have seen that $L(T) \cup A$ and $L(T) \cup B$ are two gsoa of T where $\min\{|A|, |B|\} \leq (n - \ell(T))/2$. It follows that if $\gamma_{\hat{o}}(T) = (n + \ell(T))/2$ then $L(T) \cup A$ and $L(T) \cup B$ are two $\gamma_{\hat{o}}(T)$-sets.

Root now T at r and let v be a support vertex of maximum distance from r. Let u, w be the parent of v and u, respectively. We first show that u has no child besides v as a support vertex. Suppose not, and let $b \neq v$ be a child of u as a support. Without loss of generality we assume that $v, b \in A$ and hence $u \in B$. Then $L(T) \cup (A - \{v, b\}) \cup \{u\}$ is a gsoa of size $(n + \ell(T))/2 - 1$, a contradiction. Thus every child (if any) of u besides v is a leaf. We distinguish between two cases:

Case 1. $|L_v| \geq 3$. If w is a leaf, then T is a double star $S_{p,q}$ where $p = |L_v|$ and $1 \leq q = |L_u| \leq 2$. Since T is obtained from T_1 by using Operation $\mathcal{F}_1$, $T \in \mathcal{F}$. Thus assume that w has degree at least two. Now if $|L_u| \geq 2$ then $v, w \in A$ and so $(A - \{v\}) \cup L(T)$ is a gsoa of size less than $(n + \ell(T))/2$, a contradiction. Thus $|L_u| \leq 1$. Let $T' = T-(L_v \cup L_u \cup \{v\})$. Then T' is nontrivial, $n' = n-(|L_v|+|L_u|+1)$ and $\ell(T') = \ell(T) - (|L_v| + |L_u|) + 1$. Since u is in every $\gamma_{\hat{o}}(T')$-set, such a set can be extended to a gsoa of T by adding the set $L_v \cup L_u$, and hence $\gamma_{\hat{o}}(T) \leq \gamma_{\hat{o}}(T') + (|L_v| + |L_u|)$. Thus by Theorem 7:

$$\begin{aligned}(n + \ell(T))/2 = \gamma_{\hat{o}}(T) &\leq \gamma_{\hat{o}}(T') + (|L_v| + |L_u|) \\ &\leq (n' + \ell(T'))/2 + (|L_v| + |L_u|) = (n + \ell(T))/2.\end{aligned}$$

It follows that $\gamma_{\hat{o}}(T') = (n' + \ell(T'))/2$ and by induction on T', we have $T' \in \mathcal{F}$. Now since T is obtained from T' by using Operation $\mathcal{F}_1$, $T \in \mathcal{F}$.

Case 2. $1 \leq |L_v| \leq 2$. If T is a double star then $T \in \mathcal{F}$ and is obtained from T_1 by using Operation $\mathcal{F}_2$. So assume that T is not a double star. Let $T' = T - (L_v \cup L_u \cup \{v\})$. Then T' is nontrivial, $n' = n - (|L_v| + |L_u| + 1)$ and $\ell(T') = \ell(T) - (|L_v| + |L_u|) + 1$. As shown before, $\gamma_{\hat{o}}(T) \leq \gamma_{\hat{o}}(T') + (|L_v| + |L_u|)$ and by Theorem 7 it follows that $\gamma_{\hat{o}}(T') = (n' + \ell(T'))/2$. By induction on T', $T' \in \mathcal{F}$ and hence $T \in \mathcal{F}$ since it is obtained from T' by using Operation $\mathcal{F}_2$. ∎

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