# Acyclic Orientations and Monotonicity in Graphs

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**Abstract**

A maximal directed path in an acyclic orientation of a graph $G$ is a path $a_1 \to a_2 \to \dots \to a_k$ such that id $a_1$ = od $a_k = 0$. The compression of $G$ is the smallest integer $k$ such that, for any acyclic orientation of $G$, there is a maximal directed path of length at most $k$. We characterize graphs with compression 1 and 2 and determine the compression of trees.

**Keywords:** vertex ordering, acyclically oriented graph, directed path, compression

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## 1 Introduction

A *vertex ordering* of a graph $G = (V, E)$ is a one-to-one function $f$ from $V$ to $\mathbb{Z}^+$. Every vertex ordering of $G$ corresponds to an acyclic orientation $D$ of $G$ – the edge $ab$ becomes the arc $(a, b)$ if $a < b$. We define the *compression* $\sigma(G)$ to be the smallest integer $k$ such that, for any acyclic orientation (or vertex ordering) of $G$, there is a maximal directed path of length at most $k$. For example, orient the path $P_n$ from one leaf to the other. There is only one maximal directed path, namely $1, \dots, n$, and it has length $n - 1$. Thus $\sigma(P_n) \geq n - 1$. Any other orientation of $P_n$ has a maximal directed path of length at most $n - 1$, so $\sigma(P_n) = n - 1$.

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Also, in any vertex ordering $f : V(K_n) \to \{1, ..., n\}$, $n \geq 2$, of $K_n$, the vertices labelled 1 and $n$ are adjacent, thus $1, n$ is a maximal increasing path of length 1. Therefore $\sigma(K_n) = 1$. We show in Section 3 that for a connected graph $G$, $\sigma(G) = 1$ if and only if $G = K_n$, $n \geq 2$, and that $\sigma(G) = 2$ if and only if $\operatorname{diam} G = 2$. In Section 4 we determine the compression of all trees; the formula involves the distances from branch vertices to leaves of the tree. Open problems are mentioned in Section 5.

## 2 Definitions and background

We generally follow the notation and terminology of [4]. The study of monotone paths in edge ordered graphs was introduced by Chvátal and Komlós [5] in 1971. An *edge ordering* of the graph $G = (V, E)$ is a one-to-one function $f$ from $E$ to the set $\mathbb{Z}^+$ of positive integers, and an *$f$-increasing path* is a path of $G$ for which $f$ increases along the edge sequence. The *altitude* $\alpha(G)$ of $G$ is the greatest integer $k$ such that $G$ has an $f$-increasing path of length $k$ for each edge-ordering $f$ of $G$.

Early work on the topic included bounds for $\alpha(K_n)$ by Graham and Kleitman [8], an asymptotic upper bound for $\alpha(K_n)$ by Calderbank, Chung and Sturtevant [3], and a forbidden subgraph characterization of graphs with altitude two by Bialostocki and Roditty [1]. After a hiatus of more than a decade, the topic was revived in the 21$^{st}$ century by Roditty, Shoham and Yuster [9] and Yuster [10]. More recent work includes [6] and [7]. The altitude of a graph $G$ is bounded above by its chromatic index $\chi'(G)$, as first noted in [10]; the difference can be arbitrary, as shown by the upper bounds for $\alpha(K_n)$ in [3, 8].

As mentioned in the introduction, every vertex ordering of a graph $G$ corresponds to an orientation $D$ of $G$. Not all orientations of $G$ correspond in this way to a vertex ordering of $G$; in fact, it is well known (see for example [2, Exercise 10.1.3]) that vertex orderings and orientations of a graph are related as follows.

**Remark 1** *The following statements are equivalent.*

- *(i) An orientation $D$ of a graph $G$ corresponds to a vertex ordering of $G$.*
- *(ii) Every induced subdigraph of $D$ has a vertex $v$ with* $\operatorname{id} v = 0$.
- *(iii) Every induced subdigraph of $D$ has a vertex $v$ with* $\operatorname{od} v = 0$.
- *(iv) $D$ has no directed cycles.*

We henceforth mainly consider acyclic orientations of a graph rather than vertex orderings and denote the set of all such orientations of $G$ by

$\mathcal{D}(G)$. We denote a directed path $u_1, u_2, ..., u_k$ by $u_1 \to u_2 \to ... \to u_k$ and the arc $(u, v)$ by $u \to v$ or $v \leftarrow u$.

Define the parameter $a(G)$ of a graph $G$ to be the largest integer $k$ such that, for any acyclic orientation of $G$, there is a directed path with $k$ vertices. Thus $a(G)$ for acyclic orientations (vertex orderings) corresponds to $\alpha(G)$ for edge orderings. The following well-known result by Roy (1967) and Gallai (1968), as cited in [2, p. 174], shows that $a(G)$ is even more closely connected to the chromatic number $\chi(G)$ than $\alpha(G)$ is to the chromatic index.

**Remark 2** *For any graph $G$, $a(G) = \chi(G)$.*

Note that the compression $\sigma(G)$ differs from $a(G)$ in that we only consider maximal directed paths and look for such paths of length at most $k$. These concepts are defined more formally in the next section.

# 3 Compression of a graph

A *maximal directed path* in a digraph is a path $a_1 \to a_2 \to ... \to a_k$ such that whenever $s \to a_1$, we have $s = a_i$ for some $i = 2, ..., k$, and whenever $t \leftarrow a_k$, we have $t = a_i$ for some $i = 1, ..., k-1$. Thus a maximal directed path in an acyclic orientation of a graph is a path $a_1 \to a_2 \to ... \to a_k$ such that id $a_1 =$ od $a_k = 0$. Consider the maximal directed paths with respect to a given $D \in \mathcal{D}(G)$, let the *compression* $\sigma(D)$ of $D$ be the minimum length amongst all such paths, and define the *compression* $\sigma(G)$ of $G$ by

$$\sigma(G) = \max_{D \in \mathcal{D}(G)} \{\sigma(D)\}.$$

To show that $\sigma(G) = k$, we must therefore show that

(a) each acyclic orientation of $G$ has a maximal directed path of length at most $k$ – this shows that $\sigma(G) \leq k$,

(b) there exists an acyclic orientation of $G$ with no maximal directed path of length less than $k$, i.e. for which each directed path of length $l$, where $l < k$, can be extended to a directed path of length $k$ – this shows that $\sigma(G) \geq k$.

The compression of some graphs is easy to determine. As mentioned in the introduction, $\sigma(P_n) = n - 1$ and $\sigma(K_n) = 1$. Further, consider any acyclic orientation $D$ of $C_n$ and a vertex $v$ with id $v = 0$. Such a vertex exists by Remark 1. Let $v \to v_1 \to \cdots \to v_k$ and $v \to u_1 \to \cdots \to u_l$ be the two maximal directed paths originating at $v$. Then $\min\{k, l\} \leq \lfloor n/2 \rfloor$, hence $\sigma(C_n) \leq \lfloor n/2 \rfloor$. Moreover, if $k = \lfloor n/2 \rfloor$ and $v_k = u_l$, then $\sigma(D) = \lfloor n/2 \rfloor$. It follows that $\sigma(C_n) = \lfloor n/2 \rfloor$.

Let $\tau(G)$ denote the *detour length* of $G$, that is, the length of a longest path in $G$. Then $\sigma(G) \leq \tau(G)$. The difference between $\sigma(G)$ and $\tau(G)$ can be arbitrary, as shown by the cycles $C_n$. This upper bound can be refined as follows. For each edge $e$ of $G$, define $\tau(e)$ to be the length of a longest path in $G$ containing $e$, and

$$\tau'(G) = \min_{e \in E(G)} \{\tau(e)\}.$$

Obviously $\tau'(G) \leq \tau(G)$ and the difference can be arbitrary: if $T$ is the tree obtained by joining the central vertex $u$ of $P_{2n+1}$ to a new vertex $v$, then $\tau(T) = 2n$ and $\tau'(G) = \tau(uv) = n + 1$.

**Proposition 3** *For any graph $G$, $\sigma(G) \leq \tau'(G)$.*

*Proof.* Consider any acyclic orientation $D$ of $G$. Any $e \in E(G)$ is contained in a maximal directed path $P$ of length at most $\tau(e)$, implying that $\sigma(D) \leq \tau(e)$. Therefore $\sigma(D) \leq \tau'(G)$ for each $D \in \mathcal{D}(G)$ and the result follows. ■

Our next result concerns graphs with compression 1 and 2. For a set $S \subseteq V(G)$ and $u \in V(G) - S$, define $d(u, S) = \min_{v \in S} d(u, v)$.

**Theorem 4** *Let $G$ be connected. Then*

(*i*) *$\sigma(G) = 1$ if and only if $G = K_n$, $n \geq 2$;*
(*ii*) *$\sigma(G) = 2$ if and only if* diam $G = 2$.

*Proof.* (*i*) Suppose $\sigma(G) = 1$ and $G \neq K_n$. Then $k =$ diam $G \geq 2$. Let $v$ be a peripheral vertex (i.e. $v$ has eccentricity equal to diam $G$) of $G$. Define $V_0 = \{v\}$ and, for $i = 1, ..., k$, let $V_i$ denote the set of vertices at distance $i$ from $v$. Then $V_i \neq \phi$ for each $i$, and each vertex in $V_i$, $i \geq 1$, is adjacent to a vertex in $V_{i-1}$ (and to no vertices in $V_j$, $j \leq i - 2$). Let $x \in V_k$. Partition $V_1$ into the sets $U$ and $W$, where

$$U = \{u \in V_1 : N(u) \cap V_2 = \phi\} \text{ and } W = \{u \in V_1 : N(u) \cap V_2 \neq \phi\};$$

possibly $U = \phi$, but certainly $W \neq \phi$. Any $u \in U$ is adjacent to a vertex $w \in W$, otherwise $d(u, x) > k$, which is impossible.

Consider any vertex ordering $f$ of $G$ such that $f(v) = 1$, $f(u) < f(w)$ whenever $u \in U$, $w \in W$, and for any $i < j$ and $v_i \in V_i$, $v_j \in V_j$, $f(v_i) < f(v_j)$. Let $D$ be the acyclic orientation of $G$ corresponding to $f$.

To show that $\sigma(D) \geq 2$ we must show that every arc $e$ of $D$ can be extended to a directed path of length two. This will be the case if and only if for every arc $e = (z, z')$ with id $z = 0$, there exists a vertex $z''$ with $(z', z'') \in E(D)$. But by the definition of $D$, $\{z \in V(D) : \text{id } z = 0\} = \{v\}$, hence we only need to consider arcs $(v, z)$.

If $e = (v, u)$, $u \in U$, then there exists $w \in W$ such that $v \to u \to w$. If $e = (v, w)$, $w \in W$, then there exists $z \in V_2$ such that $v \to w \to z$. As these are the only arcs of $D$ incident from $v$, it follows that $\sigma(D) \geq 2$. Hence $\sigma(G) \geq 2$.

(*ii*) Let diam $G = 2$. By (*i*), $\sigma(G) \geq 2$. By Remark 1 any acyclic orientation of $G$ has a vertex $u$ with id $u = 0$ and a vertex $v$ with od $v = 0$. If $u$ and $v$ are adjacent, then $u \to v$ is a maximal directed path. If not, then $d(u, v) = 2$ and there is a vertex $w$ such that $u \to w \to v$. Hence $\sigma(G) \leq 2$ and so $\sigma(G) = 2$.

Assume diam $G = k \geq 3$. Let $v$ be a peripheral vertex of $G$ and partition $V(G)$ into the sets $V_0, V_1, ..., V_k$, where $V_i$ consists of the set of all vertices at distance $i$ from $v$. Then $V_i \neq \phi$ for each $i$, and each vertex in $V_i$, $i \geq 1$, is adjacent to a vertex in $V_{i-1}$ (and to no vertices in $V_j$, $j \leq i-2$). Let $\mathcal{P}$ be the set of all paths $v, w_1, w_2, w_3$ of length 3 from $v$ to $V_3$, i.e. $w_i \in V_i$ for each $i = 1, 2, 3$. For $i = 1, 2$, define

$$W_i = \{w_i \in V_i : w_i \text{ lies on a path in } \mathcal{P}\}.$$

Thus if $u \in V_2$ is adjacent to some vertex in $V_3$, then $u \in W_2$. Let $H_1, ..., H_m$ be the components of $\langle V_2 \rangle$, where we may choose the subscripts of the $H_i$ so that for some $l$ with $1 \leq l \leq m$, each component $H_i$ with $1 \leq i \leq l$ contains a vertex in $W_2$, while $V(H_i) \cap W_2 = \phi$ for each $i = l+1, ..., m$. Define

$$\begin{aligned} U_2 &= \bigcup_{i=1}^{l} V(H_i) - W_2, \\ U_{2,i} &= \{u \in U_2 : d_{\langle V_2 \rangle}(u, W_2) = i\}, \\ Y_2 &= \bigcup_{i=l+1}^{m} V(H_i), \text{ and} \\ U_1 &= \{u \in V_1 : N(u) \cap V_2 \subseteq Y_2\}. \end{aligned}$$

See Figure 1, in which

$$W_1 = \{w_1\},\ W_2 = \{w_2\},\ Y_2 = \{y\},\ U_{2,1} = \{u_{21}\},$$
$$U_{2,2} = \{u_{22}\} \text{ and } U_1 = \{u_1, \underline{u}_1\}.$$

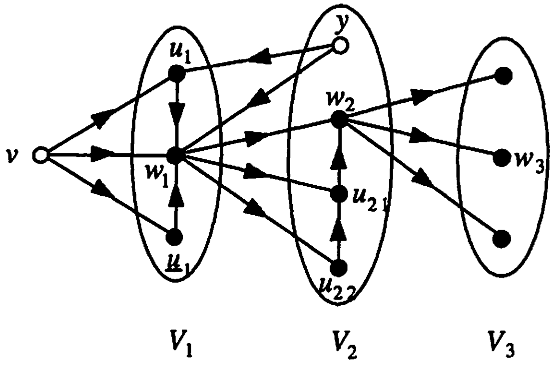


Figure 1: An illustration of the proof of Theorem 4(*ii*).

(Here, $N(\underline{u}_1) \cap V_2 = \phi \subseteq Y_2$.) By definition of $W_2$ and $U_2$, no vertex in $U_2$ is adjacent to a vertex in $V_3$. If some vertex $y \in Y_2$ is nonadjacent to all vertices in $W_1$, then there is no path from $y$ to $V_k$ of length $k$ or less, which is impossible since diam $G = k$. Therefore each vertex in $Y_2$ is adjacent to some vertex in $W_1$. Similarly, each vertex in $U_1$ is adjacent to some vertex in $W_1$. Form an orientation $D$ of $G$ as follows.

For each $i = 2, ..., k-1$, if $v_i \in V_i$ and $v_{i+1} \in N(v_i) \cap V_{i+1}$, then $v_i \to v_{i+1}$.
For each $x \in V_0 \cup Y_2$ and $v_1 \in N(x) \cap V_1$, $x \to v_1$.
For each $v_1 \in V_1$ and $v_2 \in N(v_1) \cap (V_2 - Y_2)$, $v_i \to v_{i+1}$.

This takes care of all edges between the $V_i$.

For each $u_1 \in U_1$ and $w_1 \in N(u_1) \cap (V_1 - U_1)$, $u_1 \to w_1$.
For each $z_1 \in V_1 - U_1 - W_1$ and $w_1 \in N(z_1) \cap W_1$, $z_1 \to w_1$.
For each $u_{21} \in U_{2,1}$ and $w_2 \in N(u_{21}) \cap W_2$, $u_{21} \to w_2$.
For each $i = 2, ...,$ if $u_{2,i} \in U_{2,i}$ and $u_{2,i-1} \in N(u_{2,i}) \cap U_{2,i-1}$, then $u_{2,i} \to u_{2,i-1}$.

Orient all other edges of $\langle U_1 \rangle$, $\langle W_1 \rangle$, $\langle V_1 - U_1 - W_1 \rangle$, $\langle W_2 \rangle$, $\langle Y_2 \rangle$, each $\langle U_{2,i} \rangle$ and each $\langle V_i \rangle$, $i \geq 3$, arbitrarily acyclically. This is possible by Remark 1. It is easy to verify that the resulting orientation $D$ of $G$ is acyclic.

To show that $\sigma(D) \geq 3$ we must show that every directed path of length one or two can be extended to a directed path of length three. This will be the case if and only if for every path $z_1 \to z_2$ with id $z_1 = 0$, there exists a vertex $z_3$ such that $z_1 \to z_2 \to z_3$, and for every path $z_1 \to z_2 \to z_3$ with id $z_1 = 0$, there exists a vertex $z_4$ with $(z_3, z_4) \in E(D)$. The only vertices with indegree 0 are $v$ and some vertices in $Y_2$.

As in the proof of $(i)$, each arc $(v, z)$, $z \in V_1$, can be extended to a directed path of length two. Consider the path $\vec{P} : v \to z \to z'$.

(a) If $z \in U_1$, then $z' \in V_1$ because $N(z) \cap V_2 \subseteq Y_2$ and no vertex in $Y_2$ is adjacent from a vertex in $V_1$. If $z' \in U_1$, there exists $w_1 \in N(z') \cap W_1$ and $\vec{P} \to w_1$, and if $z' \in V_1 - U_1$, there exists $v_2 \in N(z') \cap V_2$ such that $\vec{P} \to v_2$.

(b) If $z \in W_1$, then $z' \in W_1 \cup W_2 \cup U_2$. If $z' \in W_1$, then there exists $w_2 \in N(z') \cap W_2$ and $\vec{P} \to w_2$, and if $z' \in W_2$, then there exists $w_3 \in N(z') \cap V_3$ and $\vec{P} \to w_3$. Suppose $z' \in U_2$. If $z' \in U_{21}$, then there exists $z'' \in N(z') \cap W_2$ and if $z' \in U_{2,i}$, $i \geq 2$, then there exists $z'' \in N(z') \cap U_{2,i-1}$. In either case $\vec{P} \to z''$.

(c) If $z \in V_1 - W_1 - U_1$, then $z' \in (V_1 - U_1) \cup U_2$. If $z' \in W_1 \cup U_2$, then $\vec{P}$ can be extended to a path of length three as in the above two cases, and if $z' \in V_1 - W_1 - U_1$, then there exists $v_2 \in N(z') \cap U_2$ and $\vec{P} \to v_2$.

This exhausts all cases of paths of length two beginning with $v$. Next, let $y \in Y_2$ and first consider the arcs $(y, z)$. Then $z \in V_1 \cup Y_2$.

(d) If $z \in V_1$, then by the above results (a) – (c) for $v$, there exists a vertex $z'$ such that $v \to z \to z'$, and for each such vertex $z'$ there exists a vertex $z''$ such that $v \to z \to z' \to z''$; note that $z', z'' \notin Y_2$. By replacing $v$ by $y$ we have a path $y \to z \to z' \to z''$ as required.

(e) If $z \in Y_2$, then there exists a vertex $z' \in N(z) \cap W_1$ and $y \to z \to z'$.

Now consider the directed paths $y \to z \to z'$, where by (d) we may assume that $z \in Y_2$. Then $z' \in V_1 \cup Y_2$ and $y \to z \to z'$ can be extended as shown in (d) – (e).

This exhausts all possibilities and it follows that $\sigma(D) \geq 3$; consequently, $\sigma(G) \geq 3$. ■

**Corollary 5** *For all $m, n$ except $m = n = 1$, $\sigma(K_{m,n}) = 2$.*

Although Theorem 4 states that for $i = 1, 2$, $G$ has compression $i$ if and only if diam $G = i$, in general the diameter of a graph is neither an upper nor a lower bound for its compression. For example, if $T$ is the tree

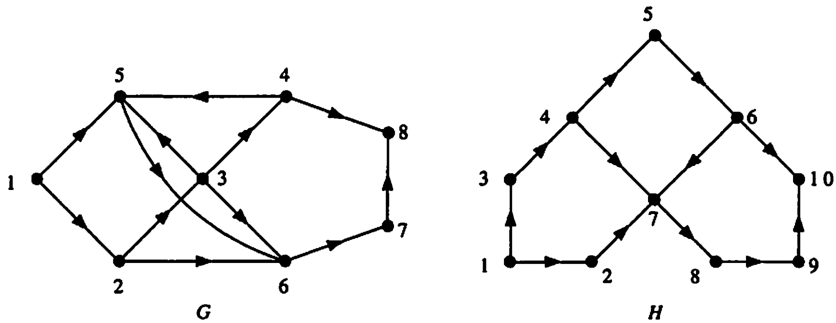


Figure 2: diam $G = 3$ and $\sigma(G) \geq 4$, while diam $H = 4$ and $\sigma(H) \geq 5$

obtained by joining the central vertex $u$ of $P_{2n+1}$ to a new vertex $v$, then diam $T = 2n$ and by Proposition 3, $\sigma(T) \leq \tau'(T) = \tau(uv) = n + 1$. On the other hand, the graph $G$ in Figure 2 has diam $G = 3$ and $\sigma(G) \geq 4$, while diam $H = 4$ and $\sigma(H) \geq 5$.

# 4 Compression of trees

For any tree $T$, diam $T = \tau(T)$ and thus, unlike the case for cyclic graphs, the compression of a tree is bounded above by its diameter – a weaker bound, however, than that of Proposition 3. The latter bound is not exact for all trees, as $\tau'(T) = 4$ ($<$ diam $T = 5$) for the tree $T$ in Figure 3, while it follows from our next result that $\sigma(T) = 3$.

Let $B(T)$ and $L(T)$ denote the set of all *branch vertices*, i.e. vertices of degree at least three, and the set of all leaves, respectively, of the tree $T$. An *internal* vertex is a vertex that is not a leaf. For $v \in V(T)$, a $v - L$ *path* is a path from $v$ to a leaf of $T$. If $l \in L(T)$, a $v - l$ *endpath*, or $v$-*endpath* if the specific leaf is unimportant, is a $v - l$ path $P$ such that each internal vertex of $P$ has degree two in $T$. For each vertex $v$, let $e_1, ..., e_{\deg v}$ be the edges incident with $v$ and define $\ell_i(v)$ to be the minimum length amongst all $v - L$ paths containing $e_i$. We shall assume throughout that the edges incident with $v$ have been labelled so that $\ell_1(v) \leq \cdots \leq \ell_{\deg v}(v)$. In Figure 3, $\ell_1(u) = 1$ and $\ell_2(u) = \ell_3(u) = 2$.

As mentioned in Section 3, $\sigma(P_n) = n - 1$. Note that for any internal vertex $v$ of $P_n$, $\ell_1(v) + \ell_{\deg v}(v) = n - 1 = \sigma(P_n)$, and for either leaf $v$ of $P_n$, $\ell_{\deg v}(v) = n - 1$. This observation provides the key to determining the compression of all other trees, except that it is sufficient to consider only branch vertices.

**Theorem 6** *For any tree $T \neq P_n$ and each vertex $v$ of $T$, let $e_1, ..., e_{\deg v}$ be the edges incident with $v$ and let $\ell_i(v)$ be the minimum length amongst all $v - L$ paths containing $e_i$. Assume that the $e_i$ have been labelled so that $\ell_1(v) \leq \cdots \leq \ell_{\deg v}(v)$. Then*

$$\sigma(T) = \max_{v \in B(T)} \{\ell_1(v) + \ell_{\deg v}(v)\}.$$

*Proof.* Let $u$ be a branch vertex of $T$ of degree (say) $r$ such that

$$\ell_1(u) + \ell_r(u) = \max_{v \in B(T)} \{\ell_1(v) + \ell_{\deg v}(v)\}.$$

For $i = 1, ..., r$, let $u_i$ be the neighbour of $u$ incident with $e_i$, and $T_i$ the subtree of $T - u$ containing $u_i$. For $i = 1, ..., r-1$, direct $e_i$ and the edges of $T_i$ away from $u$ so that all $u - L$ paths containing $e_i$ are directed paths from $u$ to the leaves of $T_i$. Direct $e_r$ and the edges of $T_r$ towards $u$ so that all $u - L$ paths containing $e_r$ are directed paths from the leaves of $T_r$ to $u$. (See Figure 3.) In the resulting orientation $D$ of $T$, the maximal directed paths are precisely the paths from the leaves of $T_r$ to the leaves of $T_i$, $i = 1, ..., r-1$. The minimum length amongst these paths is $\ell_1(u) + \ell_r(u)$, hence $\sigma(T) \geq \ell_1(u) + \ell_r(u)$.

To prove the upper bound, we must show that each acyclic orientation of each tree $T \neq P_n$ has a maximal directed path of length at most $k$, where

$$k = \max_{v \in B(T)} \{\ell_1(v) + \ell_{\deg v}(v)\}. \tag{1}$$

Suppose this is not true. Amongst all trees for which this bound does not hold, let $T$ be one such that $|B(T)|$ is minimum. Let $D$ be an acyclic orientation of $T$ such that each maximal directed path of $D$ has length greater than $k$. We first prove a lemma.

**Lemma 6.1** If $w$ is an internal vertex such that $T$ contains a $w$-endpath, then $\text{id}_D\ w \neq 0$ and $\text{od}_D\ w \neq 0$.

*Proof of Lemma* 6.1. Suppose id $w = 0$ and let $P$ be a nontrivial $w - l$ endpath. Since id $w = 0$ and each internal vertex of $P$ has degree two in $T$, $P$ contains a maximal directed path $Q$ of $D$; say $Q$ has length $s$. Let $w'$ be a branch vertex at minimum distance from $w$, where $w' = w$ if $w \in B(T)$. Then $w'$ is not an internal vertex of $P$ and the $w' - l$ path in $T$ is an endpath containing $P$ and hence $Q$. But then

$$s \leq \ell_{\deg w'}(w') < k,$$

a contraction. Hence id $w \neq 0$. Similarly od $w \neq 0$. □

We continue with the proof of Theorem 6. Suppose firstly that $T$ has exactly one branch vertex $w$; say $\deg w = t$. Then all $w - L$ paths are endpaths. For $i = 1, ..., t$, let $e_i = ww_i$ be the edges incident with $w$ (labelled so that $\ell_i(w) \leq \ell_j(w)$ for $i < j$) and denote the unique $w - l_i$ endpath containing $e_i$ by $P_i$ (so $P_i$ has length $\ell_i(w)$). Without loss of generality assume that id $l_1 = 0$. By Lemma 6.1, od $x \neq 0$ for each internal vertex $x$ of $P_1$. Thus $P_1$ is a directed path from $l_1$ to $w$. Similarly $P_i$ is a directed path for $i = 2, ..., t$. Since Lemma 6.1 implies that od $w \neq 0$, at least one path $P_i$, $i \geq 2$, is directed from $w$ to $l_i$. But then $P_1$ followed by $P_i$ is a maximal directed path in $D$ of length

$$\ell_1(w) + \ell_i(w) \leq \ell_1(w) + \ell_t(w) = k,$$

a contradiction. Hence $|B(T)| \geq 2$.

Now let $w$ be a branch vertex with incident edges $e_i = ww_i$, $i = 1, ..., t = \deg w$, such that all $w - L$ paths except those containing $e_r$, for an $r \in \{1, ..., t\}$, are endpaths. Denote these unique $(w, l_i)$-endpaths, $i = 1, ..., t$, $i \neq r$, by $P_i$. Define $\varepsilon = 1$ if $r \neq 1$ and $\varepsilon = 2$ if $r = 1$. Thus

$$\ell_\varepsilon(w) = \min_{i=1}^{t}\{\ell_i(w) : P_i \text{ is a } w\text{-endpath of length } \ell_i(w)\}. \qquad (2)$$

Let $T'$ be the tree obtained by deleting all the internal vertices and leaves of the endpaths $P_i$, $i \neq \varepsilon$, and $D'$ the orientation of $T'$ induced by $D$. Then $|B(T')| = |B(T)| - 1 \geq 1$. Let $L' = L(T')$ and for each branch vertex $w'$ of $T'$ and each edge $e'_j$ incident with $w'$, define $\ell'_j(w')$ to be the minimum length amongst all $w' - L'$ paths in $T'$ containing $e'_j$.

**Lemma 6.2** *For each branch vertex $w'$ of $T'$ and each $j = 1, ..., \deg w'$, $\ell'_j(w') = \ell_j(w')$.*

*Proof of Lemma* 6.2. Note that $\deg_{T'} w' = \deg_T w'$ and any $w' - L$ path in $T$ that does not contain $w$ is a $w' - L'$ path. Suppose $Q$ is a $w' - L$ path in $T$ of length $\ell_j(w')$ for some $j = 1, ..., \deg w'$, containing $w$ but not $l_\varepsilon$. Then $Q$ contains the leaf $l_i$ of $P_i$, $i \neq \varepsilon, r$. By (2) and the minimality of $Q$, $\ell_i(w) = \ell_\varepsilon(w)$. Hence $(Q - P_i) \cup P_\varepsilon$ is a $w' - L'$ path in $T'$ with the same length as $Q$. It follows that $\ell'_j(w') = \ell_j(w')$. □

Let

$$k' = \max_{v \in B(T')} \{\ell'_1(v) + \ell'_{\deg v}(v)\}.$$

By Lemma 6.2 and (1), $k' \leq k$. By the choice of $T$ as a tree with the minimum number of branch vertices such that (1) is not an upper bound for $\sigma(T)$, $D'$ has a maximal directed path $P'$ of length at most $k'$. But $P'$ is a maximal directed path in $D$ unless $P'$ starts or ends at $w$, in which case $\mathrm{id}_{D'}w = 0$ or $\mathrm{od}_{D'}w = 0$. Since $D$ does not contain a maximal directed path of length at most $k$ we may assume without loss of generality that $\mathrm{od}_{D'}w = 0$; i.e. $w_r \to w$ and $w_\varepsilon \to w$. By Lemma 6.1, $\mathrm{od}_D w \neq 0$. Thus for some edge $e_x = ww_x$, $x \in \{1, ..., t\} - \{\varepsilon, r\}$, $w \to w_x$ in $D$. Lemma 6.1 implies that $P_x$ is a $w - l_x$ directed path in $D$, while $P_\varepsilon$ is an $l_\varepsilon - w$ directed path. Hence $Q = P_\varepsilon \cup P_x$ is a maximal $l_\varepsilon - l_x$ directed path in $D$. Therefore, if $\varepsilon = 1$, then $Q$ has length at most

$$\ell_1(w) + \ell_t(w) \leq k,$$

contrary to assumption. Hence $\varepsilon = 2$, $r = 1$ and $\ell_1(w) < \ell_2(w)$. See Figure 4.

Let $P_1$ be any $w - L$ path in $T$ of length $\ell = \ell_1(w)$ that contains $ww_1$, say $P_1 : u_0, u_1, ..., u_\ell$, where $u_0 = w$, $u_1 = w_1$ and $u_\ell \in L(T)$. Then $P_1$ is a $w - L$ path of minimum length. This fact will help us to establish the following lemma. For each $j \in \{1, ..., \ell - 1\}$, let $\overline{\ell}(u_j)$ denote the length of a $u_j - L$ path in $T$ of minimum length containing $u_{j-1}$.

**Lemma 6.3** *For each $j \in \{1, ..., \ell - 1\}$, the $u_j - u_\ell$ subpath $P_{1j}$ of $P_1$ is a $u_j - L$ path in $T$ of minimum length $\ell_1(u_j)$ and $\ell_1(u_j) < \overline{\ell}(u_j)$.*

*Proof of Lemma* 6.3. The path $P_{1j}$ has length $\ell - j$. Suppose $Q$ is a $u_j - L$ path not containing $u_{j-1}$ of length less than $\ell - j$. Then $u_0, ..., u_j$ followed by $Q$ gives a $w - L$ path in $T$ of length less than $\ell_1(w)$, which is impossible. Now suppose $R$ is a $u_j - L$ path of length $\overline{\ell}(u_j) \leq \ell - j$ containing $u_{j-1}$. Let $j'$ be the smallest index such that $u_{j'} \in V(R) \cap V(P_1)$, and $R'$ the $u_{j'} - L$ subpath of $R$ with length (say) $r'$. (Possibly $j' = 0$ and $R' = P_2$.) Then

$$j' \leq j - 1 \tag{3}$$

and, by definition of $\ell_1(w)$,

$$\ell = \ell_1(w) \leq j' + r'. \tag{4}$$

But then the length of $R$ is

$$\overline{\ell}(u_j) = j - j' + r' \leq \ell - j,$$

which after substituting (4) contradicts (3). The result follows. □

Now, $P_1$ is not a directed path in $D'$, for otherwise $P_1 \cup P_x$ is a directed path in $D$ of length at most $k$, a contradiction. Thus there exists a smallest

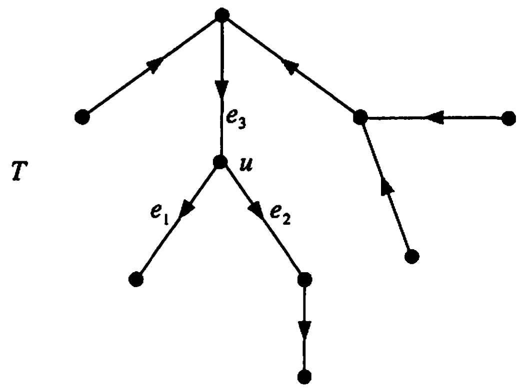


Figure 3: An orientation of $T$ that shows that $\sigma(T) \geq 3$

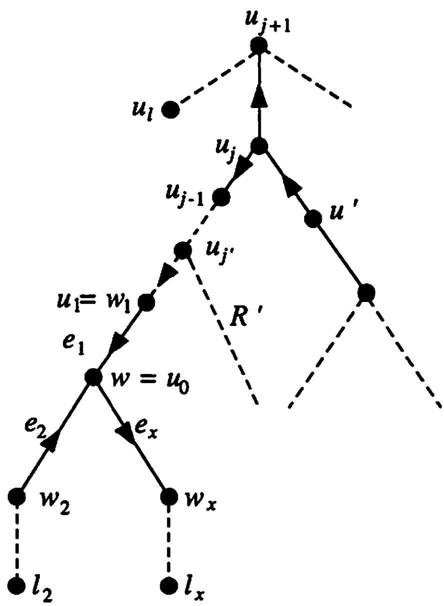


Figure 4: An illustration of the proof of Theorem 6.

index $j \geq 1$ such that $u_j \to u_{j+1}$. (Note that $u_j \to u_{j-1} \to \cdots \to u_1 = w_1 \to w$.) Then $\mathrm{id}_{D'} u_j \neq 0$, otherwise $\mathrm{id}_D u_j = 0$ and $u_j, \ldots, u_0 = w, \ldots, l_x$ is a maximal directed path in $D$ of length less than $k$. In particular, $u_j \in B(T)$ and

$$\text{there exists a vertex } u' \in N(u_j) - \{u_{j-1}, u_{j+1}\} \text{ such that } u' \to u_j. \quad (5)$$

Let $T^*$ be the subtree of $T - u_{j-1}$ that contains $u_j$, and $D^*$ the orientation of $T^*$ induced by $D$. Then $|B(T^*)| \leq |B(T)| - 1$. Let $L^* = L(T^*)$ and for each branch vertex $b$ of $T^*$ and each edge $bb_i$ incident with $b$, define $\ell_i^*(b)$ to be the minimum length amongst all $b - L^*$ paths in $T^*$ containing $bb_i$. We need one more lemma.

**Lemma 6.4** *If*

$$k^* = \max_{v \in B(T^*)} \left\{ \ell_1^*(v) + \ell^*_{\deg_{T^*} v}(v) \right\},$$

*then* $k^* \leq k$.

*Proof of Lemma* 6.4. We must show that for each vertex $v \in B(T^*)$ there exists a vertex $\overline{v} \in B(T)$ such that

$$\ell_1^*(v) + \ell^*_{\deg_{T^*} v}(v) \leq \ell_1(\overline{v}) + \ell_{\deg_T \overline{v}}(\overline{v}). \quad (6)$$

(a) Let $\deg_T u_j = z \geq 3$ (so $\deg_{T^*} u_j = z - 1$). Then $\ell^*_{z-1}(u_j) \leq \ell_z(u_j)$ and by Lemma 6.3, $\ell_1^*(u_j) = \ell_1(u_j)$. Thus (6) holds for $u_j$ with $\overline{u_j} = u_j$.

Let $b \in B(T^*) - \{u_j\}$, $bb_i$ an edge incident with $b$ for some arbitrary $i = 1, \ldots, \deg_T b$, and $P^*$ a $b - L$ path in $T$ of length $\ell_i(b)$ containing $bb_i$. By definition of $T^*$, if $u_{j-1} \in V(P^*)$ then $u_{j-1}$ is preceded on $P^*$ by $u_j$.

(b) If $u_j \notin V(P^*)$, then $P^*$ is a $b - L^*$ path in $T^*$. Suppose $u_j \in V(P^*)$. If $P^*$ contains the subpath $b, \ldots, u^* u_j$, where $u^* \in N(u_j) - \{u_{j-1}, u_{j+1}\}$, then by Lemma 6.3, the $u_j - L$ subpath of $P^*$ has length $\ell_1(u_j) = \ell_1^*(u_j) < \overline{\ell}(u_j)$, hence $P^*$ does not contain $u_{j-1}$ and thus is a $b - L^*$ path in $T^*$. In each of the above cases, no $b - L^*$ path in $T^*$ containing $bb_i$ has shorter length than $P^*$. Thus $\ell_i^*(b) = \ell_i(b)$ for each $i$, and if $i = \deg b$ it follows that, in particular, (6) holds with $\overline{b} = b$.

(c) Now suppose $P^*$ contains the subpath $b, \ldots, u_{j+1} u_j u_{j-1}$. In this case it is possible that $\ell_i(b) < \ell_i^*(b)$, so we find $\overline{b} \neq b$. Let $y$ be the largest index such that $u_y \in V(P^*) \cap V(P_1)$. Then $j + 1 \leq y < l$, the $u_y - L$ subpath $R$ of $P^*$ contains $u_{y-1}$ and thus has length $\overline{\ell}(u_y)$. By Lemma 6.3, $\ell_1(u_y) < \overline{\ell}(u_y)$ and $P_{1y}$ has length $\ell_1(u_y)$. If $b \neq u_y$, then $bb_i \in E(P^*) - E(R)$ and

$(P^*-R) \cup P_{1y}$ is a shorter $b-L$ path containing $bb_i$ than $P^*$, a contradiction. Hence $b = u_y$, $R = P^*$ (thus $\ell_i(b) = \bar{\ell}(b)$), and $P^*$ and $P_{1y}$ are internally disjoint. Since $\bar{\ell}(b) > \ell_1(b)$ it follows that $i > 1$. We now determine an upper bound for $\ell_i^*(b)$.

Let $Q$ be the $u_j - L$ subpath of $P^*$ (so $Q$ has length $\ell_i(b) - y + j$) and $S$ a $u_j - L$ path not containing $u_{j-1}$ or $u_{j+1}$ of minimum length $\lambda$. By definition of $\ell_i(b)$ and $\ell_{\deg u_j}(u_j)$,

$$\ell_i(b) - y + j \le \lambda \le \ell_{\deg u_j}(u_j).$$

Now $(P^* - Q) \cup S$ is a $b - L^*$ path in $T^*$ containing $bb_i$ of length $\Lambda \ge \ell_i^*(b)$, hence

$$\ell_i^*(b) \le \Lambda = \ell_i(b) - (\ell_i(b) - y + j) + \lambda = y - j + \lambda \le y - j + \ell_{\deg u_j}(u_j).$$

But then

$$\ell_1^*(b) + \ell_i^*(b) = \ell_1(b) + \ell_i^*(b) \le \ell_1(b) + y - j + \ell_{\deg u_j}(u_j) = \ell_1(u_j) + \ell_{\deg u_j}(u_j).$$

In particular, $\ell_1^*(b) + \ell_{\deg b}^*(b) \le \ell_1(u_j) + \ell_{\deg u_j}(u_j)$ and so (6) holds with $\bar{b} = u_j$.

It now follows from (a), (b) and (c) that $k^* \le k$. □

By the choice of $T$, $D^*$ has a maximal directed path $R^*$ of length at most $k^*$. Now $R^*$ is a maximal directed path in $T$ unless $R^*$ starts or ends at $u_j$, that is, $\mathrm{id}_{D^*} u_j = 0$ or $\mathrm{od}_{D^*} u_j = 0$. But by the choice of $j$, $u_j \to u_{j+1}$ and by (5), $u' \to u_j$, so neither of these equations holds. Therefore $R^*$ is a maximal directed path in $T$ of length at most $k^* \le k$. This final contradiction establishes the result. ■

# 5 Problems

We conclude with ideas for further investigation.

**Problem 1** Characterize graphs with compression three.

**Problem 2** Characterize (or find classes of) graphs $G$ with $\sigma(G) = \mathrm{diam}\, G \ge 3$, or $\sigma(G) = \tau(G)$, or $\sigma(G) = \tau'(G)$.

**Problem 3** It seems easier to find graphs with $\sigma(G) < \mathrm{diam}\, G$ than graphs with $\sigma(G) > \mathrm{diam}\, G$. What is the maximum ratio $\sigma(G)/\mathrm{diam}\, G$?

**Problem 4** Determine the compression of other classes of graphs.

**Problem 5** Study classes of graphs with the property that the addition/deletion of any edge or the deletion of any vertex changes the compression of the graph.

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