Six-vertex graph packings and coverings of λK_v *

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Abstract. Let λK_v be the complete multigraph, G a finite simple graph. A G-design (G-packing, G-covering, resp.) of λK_v is denoted by $GD(v, G, \lambda)$ ($PD(v, G, \lambda)$, $CD(v, G, \lambda)$, resp.). In this paper, we will give some construction methods of graph packings and graph coverings, determine the existence spectrum for the G-designs of λK_v, and construct the maximum packings and the minimum coverings of λK_v with G for any positive integer λ. Where the graph G is $(K_4 - e) \bigcup P_1$ or $C_5 \bigodot P_1$. Therefore, the problems of G-coverings and G-packings of λK_v are solved completely when G is a graph with order 6 and $|E(G)| \le 6$.

Key words: *G-design; G-packing; G-covering*

1 Introduction

A graph is called (p,q)-graph if it has p vertices and q edges. Let v and λ be two positive integers, G a finite simple (p,q)-graph. We denote by λK_v the complete multigraph of order v and index λ. *A G-design (G-packing, G-covering, resp.) of λK_v, denoted by $GD(v, G, \lambda)$ ($PD(v, G, \lambda)$, $CD(v, G, \lambda)$, resp.), is a pair $(X, \mathcal{B})$ where X is the vertex set of K_v and $\mathcal{B}$ is a collection of subgraphs of K_v, called *blocks*, such that
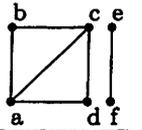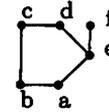
each block is isomorphic to G and any two distinct vertices in K_v are joined in exactly (at most, at least, resp.) λ blocks of $\mathcal{B}$. A G-packing (G-covering, resp.) is said to be *maximum* (*minimum*, resp.), denoted by $MPD(v, G, \lambda)$ ($MCD(v, G, \lambda)$, resp.), if no other such G-packing (G-covering, resp.) has more (fewer, resp.) blocks. The number of blocks in a maximum G-packing (minimum G-covering, resp.), denoted by $p(v, G, \lambda)$, ($c(v, G, \lambda)$, resp.), is called the *packing* (*covering*, resp.) *number*. A simple counting shows that $p(v, G, \lambda) \leq \lfloor \frac{\lambda v(v-1)}{2q} \rfloor \leq \lceil \frac{\lambda v(v-1)}{2q} \rceil \leq c(v, G, \lambda)$, where $\lfloor x \rfloor$ denotes the greatest integer y such that $y \leq x$, and $\lceil x \rceil$ denotes the least integer y such that $y \geq x$. A $PD(v, G, \lambda)$ ($CD(v, G, \lambda)$, resp.) is said to be *optimal* and is denoted by $OPD(v, G, \lambda)$ ($OCD(v, G, \lambda)$, resp.) if the left (right, resp.) equality holds. Obviously, there exists a $GD(v, G, \lambda)$ if and only if $p(v, G, \lambda) = c(v, G, \lambda)$. A $GD(v, G, \lambda)$ can be regarded as an $OPD(v, G, \lambda)$ or an $OCD(v, G, \lambda)$.

The leave edge graph $L_\lambda(\mathcal{B})$ of a packing $\mathcal{D}=(X,\mathcal{B})$ is a subgraph of λK_v whose edges are the complement of $\mathcal{B}$ in λK_v. The number of edges in $L_\lambda(\mathcal{B})$ is denoted by $|L_\lambda(\mathcal{B})|$. In particular, when $\mathcal{D}$ is maximum, $|L_\lambda(\mathcal{B})|$ is called the *leave edge number* and it is denoted by $l_\lambda(v)$. It is clear that $l_\lambda(v) = \lambda \binom{v}{2} - q|\mathcal{B}|$. Similarly, the *repeat edge graph* $R_\lambda(\mathcal{B})$ of a covering $\mathcal{D}=(X,\mathcal{B})$ is a subgraph of λK_v whose edges are the complement of λK_v in $\mathcal{B}$. When $\mathcal{D}$ is minimum, $|R_\lambda(\mathcal{B})|$ is called the *repeat edge number* and it is denoted by $r_\lambda(v)$. The number $r_\lambda(v) = q|\mathcal{B}| - \lambda \binom{v}{2}$. Generally, the symbols $L_\lambda(\mathcal{B})$, $l_\lambda(v)$, $R_\lambda(\mathcal{B})$ and $r_\lambda(v)$ can be denoted more briefly by L_λ, l_λ, R_λ and r_λ.

Remark: for the above notes, we drop the λ if $\lambda = 1$. For instance, the r_1 and $GD(v, G, 1)$ are denoted by r and $GD(v, G)$, respectively.

Graph decompositions have many important applications in communications, cryptography and networking (see [17,18]). Many researchers have been involved in graph designs, graph packings and graph coverings of λK_v with five vertices or less(see [1-10]). Yin[11] listed the spectrum of the following two graph designs when $\lambda = 1$.

Table A

note	$G_1=(K_4-e)\bigcup P_1$	$G_2=C_5\odot P_1$
graph		
spectrum	$v \equiv 0,1,4,9(mod\ 12)$	$v \equiv 0,1,4,9(mod\ 12)$

Throughout this paper, each graph in the Table A is denoted by $[a, b, c, d, e, f]$. In what follows, the notations$(a, b \in Z)$: $[a, b] = \{x \in Z | a \le x \le b\}$, $[a, b]_k = \{x \in Z | a \le x \le b, x \equiv a(mod\ k)\}$ for $a, b \in Z$, $[a, b, \cdots, c] + i = [a+i, b+i, \cdots, c+i]$ and $(Z_n)_m = \{i_m | i \in Z_n\}$ are used frequently. The edge set $\{(a_1, a_2), (a_2, a_3), \cdots, (a_{n-1}, a_n)\}$ is denoted by $(a_1, a_2, \cdots, a_n)$.

Let graph G has six vertices and its edge number not greater than 6. The G-designs, maximum G-packings and minimum G-coverings of λK_v except the graphs in *Table A* were solved by Yin and Gong [11], and Liang and Wang [13, 14, 16]. In this paper, we obtain some construction ways of graph packings and graph coverings, and give G-designs, maximum G-packings and minimum G-coverings of λK_v by the graph G in *Table A*. Therefore, we completely solve the problem of G-coverings and G-packings of λK_v, where G has no isolated vertices and $|V(G)|=6$, $|E(G)| \le 6$.

2 Recursive constructions

In the following results, if we replace MPD and MCD by OPD and OCD respectively, then they are also true.

By $K_{n_1, n_2, \cdots, n_h}$ we mean the complete multipartite graph with h parts of sizes $n_1, n_2, \cdots, n_h$. Let $X = \bigcup_{1 \le i \le h} X_i$ be the vertex set of $K_{n_1, n_2, \cdots, n_h}$ where $X_i (1 \le i \le h)$ are disjoint sets with $|X_i| = n_i$ and $v = \sum_{1 \le i \le h} n_i$. When $n_1 = n_2 = \cdots = n_h = n$, we denote $K_{n_1, n_2, \cdots, n_h}$ by $K_h(n)$. λ copies of $K_{n_1, n_2, \cdots, n_h}$ is denoted by $\lambda K_{n_1, n_2, \cdots, n_h}$. For any fixed graph G, if $\lambda K_{n_1, n_2, \cdots, n_h}$ can be decomposed into edge-disjoint subgraphs isomorphic to G, then we call $(X, \mathcal{G}, \mathcal{A})$ a *holey G-design*, where $\mathcal{G}=\{X_1, X_2, \cdots, X_h\}$, and $\mathcal{A}$ is the collection of all subgraphs called *G-blocks* (or simply *blocks*).

Each set $X_i(1 \le i \le h)$ is said to be a *hole* (or *group*) and the multiset $T=\{n_1, n_2, \cdots, n_h\}$ is called the type of the holey G-design. We denote the design by (G, λ)-$HGD(n_1^1 n_2^1 \cdots n_h^1)$ (or (G, λ)-$HGD(T)$) and use an "exponential" notation to describe its type in general: a type $1^i 2^j 3^k \cdots$, denotes i occurrences of 1, j occurrences of 2, etc.. A (G, λ)-$HGD(1^{v-w} w^1)$ (shortly $IGD(v, w, G, \lambda)$) is called an *incomplete G-design*. Obviously, a $GD(v, G, \lambda)$ is a (G, λ)-$HGD(1^v)$ which can be thought of as an $IGD(v, w, G)$ with $w = 0$ or 1. The symbols $(K_k, 1)$-$HGD(T)$, $(G, 1)$-$HGD(T)$ and $IGD(v, w, G, 1)$ can be briefly written by k-$HGD(T)$, G-$HGD(T)$ and $IGD(v, w, G)$ respectively. A transversal design $TD(k, n)$ is a k-$HGD(n^k)$.

Let $H = \{S_1, S_2, \cdots, S_n\}$ be a partition of finite set S. A *holey Latin square* having partition H is a $|S| \times |S|$ array L indexed by S, satisfying the following conditions: (1). for $1 \le i \le n$, the cells of $S_i \times S_i$ contain no symbols, and each of the other cells contains one symbol; and

(2). for $1 \le i \le n$, each row and column with indices in S_i contains each symbol in $S \setminus S_i$ exactly once.

The order of L is $|S|$, and the type of L is the multiset $T = \{|S_i| \mid i \in [1, n]\}$. A holey Latin square is called *symmetric* if the element in cell (i, j) is the element in cell (j, i) for all i and j. We simply write $HSL(T)$ for a holey symmetric Latin square of type T.

Theorem 2.1[12] There exists a $HSL(2^n)$ for any integer $n \ge 3$.

Theorem 2.2 Let (p, q)-graph $G=(V, E)$ be 3-colorable. If there exists an injection f from V to Z_q such that when each edge uv is assigned the label $|f(u) - f(v)|$ the resulting edge labels are distinct, then there exists a G-$HGD((2q)^n)$ for any integer $n \ge 3$.

Construction: Let $S=[1, 2n]$, $H=\{S_t \mid S_t = \{2t-1, 2t\}, t \in [1, n]\}$, vertex set $X=Z_q \times S$, hole set $\mathcal{G} = \{Z_q \times S_t \mid t \in [1, n]\}$. If $V=\{v_1, v_2, \cdots, v_p\}$ is the vertex set of G, then G is denoted by $[v_1, v_2, \cdots, v_p]$. By Theorem 2.1, we know that there exists a $HSL(2^n)$, written $L=(a_{ij})$. Let $A=\{\{i, j, a_{ij}\} \mid 1 \le i, j \le 2n, (i, j) \notin S_t \times S_t, t \in [1, n], a_{ij} \in L\}$. We construct $\mathcal{B}$ on the set X as follows: for every $\{i, j, a_{ij}\} \in A$, we regard i, j, a_{ij} as 3 colors of G. We get

$$[(f(v_1), x_1), (f(v_2), x_2), \cdots, (f(v_p), x_p)] \bmod(q, \text{-}).$$

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Where $x_k \in \{i, j, a_{ij}\}$, $k \in [1, p]$, and x_k is the color corresponding to v_k. Then $(X, \mathcal{G}, \mathcal{B})$ is a $G\text{-}HGD((2q)^n)$.

Proof It is easy to verify that $(f(v_1), x_1), (f(v_2), x_2), \cdots, (f(v_p), x_p)$ for all $\{i, j, a_{ij}\} \in A$ are mutually different.

It is easy to see that any edge to join distinct vertices in same hole $Z_q \times S_t$ do not occurs in any block of $\mathcal{B}$. If $(f(u), i)$ and $(f(v), j)$ are two elements in distinct holes of $\mathcal{G}$, then there are $(i, j) \notin \cup_{t=1}^n S_t \times S_t$ and $a_{ij} \in S$, such that $\{i, j, a_{ij}\} \in A$. Therefore the edge $((f(u), i), (f(v), j))$ occurs in some block of $\mathcal{B}$.

On the other hand, since $|A| = (4n^2 - 4n)/2 = 2n(n-1)$, the set $\mathcal{B}$ contains $q|A| = 2n(n-1)q$ blocks. Thus the number of edges contained in all blocks of $\mathcal{B}$ is

$$2n(n-1)q^2 = 4q^2 \binom{n}{2} = |E(K_n(2q))|.$$

To sum up, the $(X, \mathcal{B})$ is a $G\text{-}HGD((2q)^n)$. □

Theorem 2.3 Let (p, q)-graph $G = (V, E)$ with vertex set $V = \{v_i \mid i \in [1, p]\}$ be 3-colorable. Suppose that there exists a mapping $f \colon V \to Z_q$ such that its induced mapping $f^* \colon E \to Z_q$ defined by $f^*(uv) = |f(u) - f(v)|$ form a multiset

$$\{|f^*(uv)| \mid uv \in E\} = \begin{cases} \{0, 1, 1, \cdots, (q-1)/2, (q-1)/2\}, & \text{if } q \text{ odd} \\ \{0, 1, 1, \cdots, (q-2)/2, (q-2)/2, q/2\}, & \text{if } q \text{ even} \end{cases}$$

If $[(f(v_1), x_1), (f(v_2), x_2), \cdots, (f(v_p), x_p)]$ is isomorphic to G, where $x_k \in \{i, j, a_{ij}\}$, $k \in [1, p]$, and x_k is the color corresponding to v_k, then there exists a $G\text{-}HGD((2q)^n)$ for any integer $n \geq 3$.

Construction: Let $S = [1, 2n]$, $H = \{S_t \mid S_t = \{2t-1, 2t\}, t \in [1, n]\}$, vertex set $X = Z_q \times S$, hole set $\mathcal{G} = \{Z_q \times S_t | t \in [1, n]\}$. If $V = \{v_1, v_2, \cdots, v_p\}$ is the vertex set of G, then G is denoted by $[v_1, v_2, \cdots, v_p]$. By Theorem 2.1, we know that there exists a $HSL(2^n)$, written $L = (a_{ij})$. Let $A = \{\{i, j, a_{ij}\} \mid 1 \leq i, j \leq 2n, (i, j) \notin S_t \times S_t, t \in [1, n], a_{ij} \in L\}$. Next we construct $\mathcal{B}$. For every 3-coloring of G $\{i, j, a_{ij}\} \in A$, we obtain the block $[(f(v_1), x_1), (f(v_2), x_2), \cdots, (f(v_p), x_p)]$,
where $x_k \in \{i, j, a_{ij}\}$, $k \in [1, p]$, and x_k is the color corresponding to v_k. Let $\mathcal{A}$ be the set consist of these blocks. Then $\mathcal{B} = \{B + i \mid i \in Z_q, B \in \mathcal{A}\}$ is the collection of $G\text{-}HGD((2q)^n)$. □

Theorem 2.4 Let G be a graph in Table A. Then there exists a G-$HGD(12^n)$ for $n \geq 3$.

Proof Let $S = [1, 2n]$, $H = \{S_t | \ S_t = \{2t - 1, 2t\}, t \in [1, n]\}$, vertex set $X = Z_6 \times S$, hole set $\mathcal{G} = \{Z_6 \times S_t | \ t \in [1, n]\}$. By Theorem 2.1, we know that $HSL(2^n)$ exists, denoted by $L = (a_{ij})$. We construct $\mathcal{B}$ as follows:

for G_1 $[(1, i), (3, a_{ij}), (1, j), (0, a_{ij}), (0, i), (3, j)]$ (mod 6,-).

for G_2 $[(0, j), (4, a_{ij}), (3, j), (2, i), (0, a_{ij}), (3, i)]$ (mod 6,-).

Where (i, j, a_{ij}) over all elements in $\{\{i, j, a_{ij}\} \ | \ 1 \leq i, j \leq 2n, (i, j) \notin S_t \times S_t, t \in [1, n], a_{ij} \in L\}$. It is easy to verify that $(X, \mathcal{G}, \mathcal{B})$ is a G-$HGD(12^n)$ for $G \in \{G_i \ | \ i \in [1, 2]\}$. □

Theorem 2.5 Let G be a graph in Table A. If both $IGD(12+w, w, G, \lambda)$ and $MPD(12+w, G, \lambda)$ $(MCD(12+w, G, \lambda)$, resp.) exist, then an $MPD(12n + w, G, \lambda)$ $(MCD(12n + w, G, \lambda)$, resp.) exists for any integer $n \geq 3$.

Proof By Theorem 2.4, there exists a (G, λ)-$HGD(12^n) = (X, \mathcal{G}, \mathcal{A})$ for $G \in \{G_1, G_2\}$. Let $Y = (Z_n \times Z_{12}) \bigcup \{\infty_1, \infty_2, \cdots, \infty_w\}$, $Y_j = (\{j\} \times Z_{12}) \bigcup \{\infty_1, \infty_2, \cdots, \infty_w\}$, for $j \in Z_n$. Since $IGD(12 + w, w, G, \lambda)$ and $MPD(12 + w, G, \lambda)$ exist, $IGD(12 + w, w, G, \lambda)$ on Y_j $(j \in Z_n^*)$ exists, denoted by $(Y_j, \mathcal{A}_j)$, and $MPD(12 + w, G, \lambda)$ exists on the set Y_0, denoted by $(Y_0, \mathcal{A}_0)$. We have $|\mathcal{A}| = 12\lambda n(n-1)$, $|\bigcup_{1 \leq j \leq n-1} \mathcal{A}_j| = \lambda(n-1)(12 + 2w - 1)$ and $|\mathcal{A}_0| = \lambda(12 + 2w - 1) + \lfloor \lambda \frac{w(w-1)}{12} \rfloor$.

Thus $|\mathcal{A}| + |\bigcup_{1 \leq j \leq n-1} \mathcal{A}_j| + |\mathcal{A}_0| = \lambda(12n^2 + 2nw - n) + \lfloor \lambda \frac{w(w-1)}{12} \rfloor$

$= \lfloor \frac{\lambda(12n+w)(12n+w-1)}{12} \rfloor$.

This implies that $(Y, \mathcal{A} \bigcup (\bigcup_{0 \leq j \leq n-1} \mathcal{A}_j))$ is an $MPD(12n + w, G, \lambda)$.

In the same way we can prove that an $MCD(12n + w, G, \lambda)$ exists. □

Theorem 2.6 Let G be a (p, q)-graph. If both (G, λ)-$HGD((hm)^u)$ and (G, λ)-$HGD(h^m)$ exist, then (G, λ)-$HGD(h^{mu})$ exists.

Proof Let $(X, \mathcal{G}, \mathcal{A}_0)$ be a (G, λ)-$HGD((hm)^u)$. By hypothesis condition of the theorem, we obtain that there exists a (G, λ)-$HGD(h^m)$. Suppose that $(X_i, \mathcal{G}_i, \mathcal{A}_i)$, $i \in [1, u]$ are u (G, λ)-$HGD(h^m)$. Next we verify that the $(X, \bigcup_{1 \leq j \leq u} \mathcal{G}_j, \bigcup_{0 \leq j \leq u} \mathcal{A}_j)$ is a (G, λ)-$HGD(h^{mu})$.

It is not difficult to see that every pair of distinct element of X occurs in exactly λ blocks of $\bigcup_{0 \leq j \leq u} \mathcal{A}_j$ or one group of $\bigcup_{1 \leq j \leq u} \mathcal{G}_j$, but not both.

Again $\sum_{i=0}^{u} |\mathcal{A}_i| = |\mathcal{A}_0| + \sum_{i=1}^{u} |\mathcal{A}_i|$

$= \lambda[\binom{u}{2}(hm)^2/q + u\binom{m}{2}h^2/q] = \lambda h^2[u(u-1)m^2 + um(m-1)]/2q$

$=\lambda h^2 mu(mu-1)/2q=\lambda h^2\binom{mu}{2}/q$.

Thus we have the required number, $\lambda h^2\binom{mu}{2}/q$, of blocks as the total number of blocks of (G,λ)-$HGD(h^{mu})$. Therefore, we claim that the $(X,\bigcup_{1\leq j\leq u}\mathcal{G}_j,\bigcup_{0\leq j\leq u}\mathcal{A}_j)$ is the required (G,λ)-$HGD(h^{mu})$. □

Theorem 2.7 Let $(X,\mathcal{G},\mathcal{B})$ be a k-HGD, and let $w\colon X\to Z^+\bigcup\{0\}$ be a weight function on X. Suppose that there exists a (G,λ)-HGD of type $\{w(x)|\ x\in B\}$ for each block $B\in\mathcal{B}$. Then there is a (G,λ)-HGD of type $\{\sum_{x\in G}w(x)|\ G\in\mathcal{G}\}$.

Proof Let $\mathcal{D}=\{\bigcup_{x\in g}(\{x\}\times Z_{w(x)})|\ g\in\mathcal{G}\}$, $Y=\bigcup_{x\in X}(\{x\}\times Z_{w(x)})$. By hypothesis condition of the theorem, we obtain that there exists a (G,λ)-HGD of type $\{w(x)|\ x\in B\}$ $(\bigcup_{x\in B}(\{x\}\times Z_{w(x)}),\mathcal{G}_B,\mathcal{A}_B)$ for any $B\in\mathcal{B}$. Then $(Y,\mathcal{D},\bigcup_{B\in\mathcal{B}}\mathcal{A}_B)$ is a (G,λ)-HGD of type $\{\sum_{x\in g}w(x)|\ g\in\mathcal{G}\}$. □

Theorem 2.8 If there exist a G-$HGD(n_1^1 n_2^1\cdots n_h^1)$ and a $TD(h,n)$, then there exists a G-$HGD((nn_1)^1(nn_2)^1\cdots(nn_h)^1)$ for any positive integer n.

Proof Let $(X,\mathcal{G},\mathcal{B})$ be a $TD(h,n)$. For each $B=\{b_1,b_2,...,b_h\}\in\mathcal{B}$, we construct weight function $w(b_i)=n_i$, $i\in[1,h]$. By Theorem 2.7, we obtain the result. □

Theorem 2.9 If there exist a G-$HGD(n_1^1 n_2^1\cdots n_m^1)$, an $IGD(n_i+h,h,G)$, $i\in[1,m-1]$ and an $MPD(n_m+h,G)$ $(MCD(n_m+h,G)$, $GD(n_m+h,G)$, resp.), then there exists an $MPD(v,G)$ $(MCD(v,G)$, $GD(v,G)$, resp.), where $v=h+\sum_{i=1}^{m}n_i$.

Theorem 2.10[13] Let l be the leave edge number of the $OPD(n,G,1)$ and $\bar{\lambda}=e(G)/gcd(e(G),l)$. If there exist an $OPD(n,G,\lambda)$ and an $OCD(n,G,\lambda)$ for $1\leq\lambda\leq\bar{\lambda}$, then there exist an $OPD(n,G,\lambda)$ and an $OCD(n,G,\lambda)$ for any positive integer λ.

Theorem 2.11[13] Given positive integers v, λ and μ. Let X be a v-set.

(1) Suppose that there exists an $MPD(v,\ G,\ \lambda)=(X,\ \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$ and an $MPD(v,\ G,\ \mu)=(X,\ \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $|L_\lambda(\mathcal{D})|+|L_\mu(\mathcal{E})|=l_{\lambda+\mu}(v)<e(G)$, then there exists an $MPD(v,\ G,\ \lambda+\mu)$ with leave edge graph $L_\lambda(\mathcal{D})\bigcup L_\mu(\mathcal{E})$.

(2) Suppose that there exists an $MCD(v,\ G,\ \lambda)=(X,\ \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and an $MCD(v,\ G,\ \mu)=(X,\ \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $|R_\lambda(\mathcal{D})|+|R_\mu(\mathcal{E})|=r_{\lambda+\mu}(v)<e(G)$, then there exists an

$MCD(v,\ G,\ \lambda+\mu)$ with repeat edge graph $R_\lambda(\mathcal{D})\bigcup R_\mu(\mathcal{E})$.

(3) Suppose that there exists an $MPD(v,\ G,\ \lambda)=(X,\ \mathcal{D})$ with leave edge graph $L_\lambda(\mathcal{D})$ and an $MCD(v,\ G,\ \mu)=(X,\ \mathcal{E})$ with repeat edge graph $R_\mu(\mathcal{E})$. If $R_\mu(\mathcal{E})\subset L_\lambda(\mathcal{D})$ and $|L_\lambda(\mathcal{D})|-|R_\mu(\mathcal{E})|=l_{\lambda+\mu}(v)<e(G)$, then there exists an $MPD(v,\ G,\ \lambda+\mu)$ with leave edge graph $L_\lambda(\mathcal{D})\backslash R_\mu(\mathcal{E})$.

(4) Suppose that there exists an $MCD(v,\ G,\ \lambda)=(X,\ \mathcal{D})$ with repeat edge graph $R_\lambda(\mathcal{D})$ and an $MPD(v,\ G,\ \mu)=(X,\ \mathcal{E})$ with leave edge graph $L_\mu(\mathcal{E})$. If $L_\mu(\mathcal{E})\subset R_\lambda(\mathcal{D})$ and $|R_\lambda(\mathcal{D})|-|L_\mu(\mathcal{E})|=r_{\lambda+\mu}(v)<e(G)$, then there exists an $MCD(v,\ G,\ \lambda+\mu)$ with repeat edge graph $R_\lambda(\mathcal{D})\backslash L_\mu(\mathcal{E})$.

Corollary 2.12[15] If both $GD(v,G,\lambda_1)$ and $GD(v,G,\lambda_2)$ exist, then $GD(v,G,\lambda_1+\lambda_2)$ exists.

Theorem 2.13[13] If there exist a $GD(v,G,\lambda_1)$ and an $MPD(v,G,\lambda_2)$ $(MCD(v,G,\lambda_2),$ resp.), then there exists an $MPD(v,G,\lambda_1+\lambda_2)(MCD(v,$ $G,\lambda_1+\lambda_2),$ resp.).

Theorem 2.14[14] If there exist an $IGD(v,w,G)$ and a $GD(w,G)$ $(MPD(w,G), MCD(w,G),$ resp.), then there exists a $GD(v,G)$ $(MPD(v,$ $G), MCD(v,G),$ resp.).

3   Direct constructions

Theorem 3.1[14] Let G be a (p,q)-graph. If positive integer n satisfies $n(n-1)<2q$, then there exists an $OPD(v,G)$ with the leave edge graph K_n if and only if there exists an $IGD(v,n,G)$.

Let $G=(V,E)$, and let $D_i=(V,E_i)$, $i=1,$ 2 be two subgraphs of G. We say that D_1 and D_2 are mutual complement on G (denoted by $D_2=C_G D_1$) if $\{E_1,E_2\}$ is a partition of E. D_1 is called a self-complement graph on G if D_1 is isomorphic to $C_G D_1$. Let P_n be a path of length n. It is easy to see that there are two self-complement graphs on G_2 in Table A (i.e. P_3 and $P_2\bigcup P_1$ (disjoint union)), and there is no self-complement graph on G_1 in Table A.

Theorem 3.2 If there exists an $MPD(v,G)$ with leave edge graph L which is a self-complement graph on G, then there exist an $MPD(v,G,\lambda)$ and an $MCD(v,G,\lambda)$ for any positive integer λ.

Proof Let L be the leave edge graph of $MPD(v,G)=(X,\mathcal{A})$. Since

L is a self-complement graph on G, L can conduct a graph G' that is isomorphic to G, and $V(L) \subset V(G') \subset X$. Let $\mathcal{B}=\mathcal{A}\bigcup\{G'\}$. Then $(X,\mathcal{B})$ is an $MCD(v,G)$ with repeat edge graph $R \cong L$. It follows from Theorems 2.10 and 2.11 that there exist an $MPD(v,G,\lambda)$ and an $MCD(v,G,\lambda)$ for any positive integer λ. □

Corollary 3.3 For graph G_2 in Table A, if $MPD(v,G_2)$ has leave edge graph P_3 or $P_2\bigcup P_1$ (disjoint union), then there exist an $MPD(v,G_2,\lambda)$ and an $MCD(v,G_2,\lambda)$ for any positive integer λ.

Theorem 3.4 Let G be a (p,q)-graph.

(1) Let the leave edge graph L of $MPD(v,G)$ has $q/2$ edges, and L be a graph isomorphic to some subgraph of G. If there exist an $MPD(v,G)$ and a $GD(v,G,2)$, then there exist an $MPD(v,G,\lambda)$ and an $MCD(v,G,\lambda)$ for any positive integer λ.

(2) Let L_λ be the leave edge graph of $MPD(v,G,\lambda)$. If L_λ is isomorphic to some subgraph of G, then there exists an $MCD(v,G,\lambda)$.

Proof Let $MPD(v,G)=(X,\mathcal{A})$. We span G' by L, where G' is isomorphic to G. Then $(X,\mathcal{A}\bigcup\{G'\})$ is an $MCD(v,G)$. When $\lambda=2n$, take n graph designs $GD(v,G,2)$ form a $GD(v,G,\lambda)$. When $\lambda=2n+1$, take n graph designs $GD(v,G,2)$ and one $MPD(v,G)$ ($MCD(v,G)$, resp.) form an $MPD(v,G,\lambda)$ ($MCD(v,G,\lambda)$, resp.). The result (2) is obvious. □

We write the disjoint union of k copies of G for kG. A subgraph B of G is said to be a *block* of G if either it is a bridge or else it is a maximal 2-connected subgraph of G.

Theorem 3.5 Let G be a (p,q)-graph, L a leave edge graph of $MPD(v,G)$ and R a repeat edge graph of $MCD(v,G)$. (i) If every block of G and R are isomorphic to L, then there exist an $MPD(v,G,\lambda)$ and an $MCD(v,G,\lambda)$ for any positive integer λ.

(ii) Let s and t be two positive integers. If G is isomorphic to tL, and R is isomorphic to sL, then there exist an $MPD(v,G,\lambda)$ and an $MCD(v,G,\lambda)$ for any positive integer λ.

Proof Since every vertex belonging to at least two blocks is a cut-vertex of G and two distinct blocks in G have at most one vertex in common, the conclusion follows from Theorem 2.11. The result (ii) is obvious. □

Corollary 3.6 If the leave edge number of $MPD(v,G)$ is 1, then there

exist an $MPD(v, G, \lambda)$ and an $MCD(v, G, \lambda)$ for any positive integer λ.

Lemma 3.7 Let G be a graph in Table A. When $w \in \{2, 3, 5, 6, 7, 8, 10, 11\}$, there exists an $IGD(12 + w, w, G)$.

Proof see Appendix. □

Lemma 3.8 When $v \in \{7, 8, 10, 11, 14, 15, 17, 18, 19, 20, 22, 23, 26, 27, 29, 30, 31, 32, 34, 35\}$ and $G \in \{G_1, G_2\}$, there exist an $OPD(v, G, \lambda)$ and an $OCD(v, G, \lambda)$ for any positive integer λ.

Proof See Appendix. □

4 Case v=6

Lemma 4.1 There exist an $OPD(6, G_2, \lambda)$ and an $OCD(6, G_2, \lambda)$ for any positive integer λ.

Proof On the set Z_6, the blocks of $OPD(6, G_2)$ are given by $[0, 1, 2, 3, 4, 5]$, $[2, 4, 1, 3, 5, 0]$. Leave edges: $(3, 0, 2)$, $(1, 5)$.

Therefore, there exist an $OPD(6, G_2, \lambda)$ and an $OCD(v, G_2, \lambda)$ for any positive integer λ from Corollary 3.3. □

Lemma 4.2 $p(6, G_1)=1$, $c(6, G_1)=3$.

Proof On the set Z_6, the blocks of $OCD(6, G_1)$ are given by $[0, 1, 2, 3, 4, 5]$, $[1, 4, 3, 5, 0, 2]$, $[4, 2, 5, 0, 1, 3]$. Repeat edges: $(3, 1)$, $(2, 0)$, $(4, 5)$. therefore, an $OCD(6, G_1)$ exists and $c(6, G_1)=3$.

Since $K_6 - G_1$ does not contains graph G_1, there does not exists $OPD(6, G_1, 1)$. This implies that $p(6, G_1)=1$. □

Lemma 4.3 There does not exists $GD(6, G_1, 2)$, and $p(6, G_1, 2)=4$, $c(6, G_1, 2)=6$.

Proof Suppose that there exists a $GD(6, G_1, 2)$ with five blocks. Then the type of each vertex on $2K_6$ is one of 2^5, $3^1 2^3 1^1$, $3^2 2^1 1^2$. Let there are x vertices with type 2^5, y vertices with type $3^1 2^3 1^1$, z vertices with type $3^2 2^1 1^2$. Then we have the system of equations

$$\begin{cases} x + y + z & = 6 \\ 5x + 3y + z & = 10 \\ y + 2z & = 10 \end{cases}$$

Its nonnegative integer solutions are

$$(I) \quad \begin{cases} x = 0 \\ y = 2 \\ z = 4 \end{cases} \qquad (II) \quad \begin{cases} x = 1 \\ y = 0 \\ z = 5 \end{cases}$$

Let two vertices with type $3^1 2^3 1^1$ of $2K_6$ be 0, 1. We distinguish two cases.

Case 1: if the edge 01 occurs in one of subgraphs $K_4 - e$ in the five blocks, then the edge 01 can not occur in other blocks. Else, the vertex 0 or 1 will occurs two time in same block.

Case 2: if the edge 01 occurs in two subgraphs $K_4 - e$ of the five blocks, then after five $K_4 - e$ was obtained, there is at least one block with repeat vertex. This is a contradiction. Therefore, the solution (I) is not feasible.

For the solution (II), if one vertex of $2K_6$, say 0, occur in one vertex with degree 2 of the five blocks, other one vertex with degree 2 in the five blocks are labelled 1, 2, 3, 4, 5, respectively. However, after five $K_4 - e$ was obtained, there exist at least one block with repeat vertex. Therefore, the solution (II) is not feasible also.

Let Z_6 be the vertex set of K_6. $\mathcal{A}$: [0, 1, 2, 3, 4, 5], [1, 5, 3, 4, 0, 2], [5, 2, 4, 0, 1, 3], [4, 2, 3, 0, 1, 5]. Thus $(Z_6, \mathcal{A})$ is an $MPD(6, G_1, 2)$, i.e. $p(6, G_1, 2) = 4$. Let $\mathcal{B} = \mathcal{A} \bigcup \{[1, 0, 5, 2, 3, 4], [1, 4, 3, 5, 0, 2]\}$.

Then $(Z_6, \mathcal{B})$ is an $MCD(6, G_1, 2)$, i.e. $c(6, G_1, 2) = 6$. $\qquad \square$

Lemma 4.4 For integer λ, $3 \le \lambda \le 6$, there exist an $OPD(6, G_1, \lambda)$ and an $OCD(6, G_1, \lambda)$.

Proof Let Z_6 be the vertex set of K_6. Define

$\mathcal{A}$: [4, 2, 3, 0, 1, 5], [3, 0, 5, 2, 1, 4], [1, 4, 5, 3, 0, 2];

$\mathcal{B}$: [3, 0, 1, 2, 4, 5], [5, 2, 4, 0, 1, 3], [2, 1, 0, 5, 3, 4], [1, 0, 4, 2, 3, 5];

$\mathcal{C}$: [0, 1, 2, 3, 4, 5], [0, 5, 2, 4, 1, 3], [5, 2, 1, 0, 3, 4], [2, 1, 0, 4, 3, 5], [1, 5, 3, 4, 0, 2], [0, 1, 5, 4, 2, 3], [1, 2, 4, 3, 0, 5], [5, 4, 2, 3, 0, 1], [4, 1, 3, 0, 2, 5].

We obtain an $OPD(6, G_1, 3) = (Z_6, \mathcal{A} \bigcup \mathcal{B})$ with the leave edge set $\{02, 15, 34\}$, and an $OPD(6, G_1, 5) = (Z_6, \mathcal{A} \bigcup \mathcal{C})$ with the leave edge set $\{03, 12, 45\}$. By Theorem 3.4, there exists an $OCD(6, G_1, \lambda)$ for $\lambda = 3$, 5. By Theorem 2.12 and Lemma 4.2, there exists a $GD(6, G_1, \lambda)$ for $\lambda = 4$, 6. $\qquad \square$

Lemma 4.5 (1) Any odd number $s \ge 7$ can be written by $3x + 4y$ or $4y + 5z$.

(2) Any even number $s \geq 8$ can be written by $4x + 6y$. Where x, y, z are three nonnegative integers.

Proof (1) When $s=4n+1$, taking $y=n-1$, $z=1$, we have $s=4y+5z$. When $s=4n+3$, taking $y=n$, $x=1$, we have $s=3x+4y$.

(2) When $s=4n$, taking $y=0$, $x=n$; When $s=4n+2$, taking $y=1$, $x=n-1$. □

Lemma 4.6 There exist an $OPD(v, G_1, \lambda)$ and an $OCD(v, G_1, \lambda)$ for any integer $\lambda \geq 3$.

Proof The result is obtained from Theorem 2.11, Lemmas 4.4 and 4.5. □

5 Conclusion

Theorem 5.1 There exist an $OPD(v, G_i, \lambda)$ and an $OCD(v, G_i, \lambda)$ for $i = 1$, 2 and $v \geq 6$, exception of $(v, i, \lambda)=(6, 1, 1)$, $(6, 1, 2)$ for packing, and $(v, i, \lambda)=(6, 1, 2)$ for covering.

Proof By Lemmas 3.7 and 3.8, Theorem 2.5 and Lemmas 4.1-4.6, we obtain the result. □

Lemma 5.2[15] The necessary conditions for a $GD(v, G, \lambda)$ to exists are (1) $\lambda v(v-1) \equiv 0 \pmod{2e(G)}$; (2) $\lambda(v-1) \equiv 0 \pmod{n}$, where $n = gcd(\{d(u)|u \in V(G)\})$.

Theorem 5.3 The necessary conditions for the existence of a $GD(v, G_i, \lambda)$ are also sufficient for $i \in [1, 2]$ and $\lambda \geq 1$, except for $(v, i, \lambda)=(6, 1, 2)$.

Proof Since an $OPD(v, G, \lambda)$ is also a $GD(v, G, \lambda)$ when $p(v, G, \lambda) = c(v, G, \lambda)$, the result holds by Theorem 5.1. □

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Appendix

Proof of Lemma 3.7

Table B $IGD(12 + w, w, G)$

w	2	3	5	6	7	8	10	11
12+w	14	15	17	18	19	20	22	23
number of blocks	15	17	21	23	25	27	31	33

Let the incomplete graph design $IGD(12 + w, w, G) = (X, \mathcal{B})$ for $G \in \{G_1, G_2\}$. The number of blocks in the $IGD(12 + w, w, G)$ see Table B. We construct $\mathcal{B}$ as follows:

<u>$w = 2$</u> On the set $X = Z_{12} \bigcup \{a, b\}$.

For G_1: $[0, 1, 3, 7, 6, 9]$, $[1, 2, 4, 8, 5, 11]$, $[2, 3, 5, 9, 1, 11]$, $[3, 4, 6, 10, 5, 0]$,

$[4, 5, 7, 11, b, 9]$, $[7, 8, 10, 2, 5, 1]$, $[8, 9, 11, 3, 5, 6]$, $[9, 10, 0, 4, b, 3]$,

$[11, 0, 2, 6, 7, 1]$, $[a, 6, 8, 0, 3, 9]$, $[a, 7, 9, 1, b, 11]$, $[a, 11, 10, 5, 6, 1]$,

$[b, 0, 6, 7, a, 2]$, $[b, 2, 8, 5, a, 4]$, $[b, 4, 10, 1, a, 3]$.

For G_2: $[0, a, 6, 7, 9, 3] + i$, $i \in Z_3$, $[b, 1, 2, 4, 7, 11] + i$, $i \in Z_5$,

$[9, 2, 7, 1, 5, 10] + i$, $i \in Z_2$, $[a, 9, 10, 0, 3, 7]$, $[a, 10, 11, 1, 4, 9]$,

$[0, 2, 5, a, 11, 4]$, $[1, 3, 6, b, 0, 5]$, $[4, 8, 1, 6, 0, 7]$.

<u>$w = 3$</u> On the set $X = Z_{12} \bigcup \{a, b, c\}$

For G_1: $[0, 1, 3, 7, c, 4]$, $[1, 2, 4, 8, c, 6]$, $[3, 4, 6, 10, c, 7]$, $[4, 5, 7, 11, b, 9]$,

$[7, 8, 10, 2, c, 1]$, $[8, 9, 11, 3, c, 10]$, $[9, 10, 0, 4, 2, 5]$, $[11, 0, 2, 6, 7, 1]$,

$[a, 6, 8, 0, 3, 5]$, $[a, 7, 9, 1, b, 11]$, $[a, 11, 10, 5, 3, 9]$, $[b, 0, 6, 7, a, 2]$,

$[b, 2, 8, 5, a, 4]$, $[b, 4, 10, 1, a, 3]$, $[6, 1, 5, 9, c, 8]$, $[c, 0, 5, 11, b, 3]$,

$[c, 3, 2, 9, 1, 11]$.

For G_2: $[a, 6, 7, 9, 0, c] + i, i \in Z_6$, $[3, 6, b, 0, 1, 9] + i, i \in Z_3$,

$[6, 9, b, 3, 4, 10] + i, i \in Z_2$, $[0, 4, 8, 1, 6, 11] + i, i \in Z_2$,

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$[10, 3, 8, 2, 6, c] + i, i \in Z_2$, $[b, 5, 6, 8, 11, c]$, $[8, 0, 5, 10, c, 9]$.

<u>$w = 5$</u> On the set $X = Z_{12} \bigcup \{a, b, c, d, e\}$

For G_1: $[2, 3, 5, 9, 0, 1]$, $[3, 4, 6, 10, 1, 2]$, $[4, 5, 7, 11, 0, 3]$, $[5, 6, 8, 0, 1, 4]$,
$[6, 7, 9, 1, e, 5]$, $[7, 8, 10, 2, 4, 0]$, $[8, 9, 11, 3, d, 6]$, $[10, 11, 1, 5, d, 9]$,
$[11, 0, 2, 6, e, 9]$, $[0, a, 6, b, c, 11]$, $[1, a, 7, b, d, 11]$, $[2, a, 8, b, e, 11]$,
$[3, a, 9, b, e, 2]$, $[4, a, 10, b, e, 3]$, $[5, a, 11, b, e, 6]$, $[c, 1, 3, 7, d, 2]$,
$[c, 2, 4, 8, d, 3]$, $[9, 10, 0, c, d, 5]$, $[0, d, 7, e, c, 5]$, $[4, d, 9, e, c, 10]$,
$[1, d, 8, e, c, 6]$.

For G_2: $[a, 7, 8, 10, 1, d] + i, i \in Z_3$, $[7, b, 1, 2, 4, 10] + i, i \in Z_2$,
$[b, 3, 4, 6, 9, e] + i, i \in Z_3$, $[4, 8, 1, 6, 0, c] + i, i \in Z_4$,
$[d, 9, 1, e, 4, c] + i, i \in Z_3$, $[9, 0, a, 6, 7, d]$, $[1, 4, a, 10, 11, c]$,
$[2, 5, a, 11, 0, e]$, $[b, 0, 1, 3, 6, 11]$, $[8, 0, 5, 10, c, 9]$, $[e, 8, d, 0, 7, c]$.

<u>$w = 6$</u> On the set $X = Z_{12} \bigcup \{a, b, c, d, e, f\}$

For G_1: $[1, 2, 4, 8, 3, 7]$, $[3, 4, 6, 10, 0, 7]$, $[5, 6, 8, 0, 2, 9]$, $[7, 8, 10, 2, 5, 9]$,
$[9, 10, 0, 4, 1, 3]$, $[11, 0, 2, 6, 3, 5]$, $[0, a, 6, b, 5, 7]$, $[1, a, 7, b, 4, 11]$,
$[2, a, 8, b, 7, 11]$, $[3, a, 9, b, 6, 1]$, $[4, a, 10, b, 7, 9]$, $[5, a, 11, b, 9, 1]$,
$[0, c, 1, d, 9, 11]$, $[2, c, 3, d, e, 1]$, $[4, c, 5, d, f, 1]$, $[6, c, 7, d, e, 10]$,
$[8, c, 9, d, f, 10]$, $[10, c, 11, d, 8, 3]$, $[0, e, 3, f, 10, 1]$, $[2, e, 5, f, 11, 1]$,
$[4, e, 7, f, 10, 5]$, $[6, e, 9, f, 11, 3]$, $[8, e, 11, f, 1, 5]$.

For G_2: $[1, a, 7, 9, 4, f] + i, i \in Z_5$, $[6, b, 0, 2, 9, f] + i, i \in Z_6$,
$[d, 1, 0, c, 7, 6] + i, i \in Z_4$, $[11, 3, e, 9, 5, c] + i, i \in Z_2$,
$[8, 3, 0, a, 6, d]$, $[d, 5, 4, c, 11, 0]$, $[e, 6, 2, 8, 0, d]$, $[9, 1, e, 7, 3, f]$,
$[2, e, 8, 4, 10, 11]$, $[e, 11, 7, 1, 5, 6]$.

<u>$w = 7$</u> On the set $X = Z_{12} \bigcup \{a, b, c, d, e, f, g\}$

For G_1: $[1, 2, 4, 8, g, 0]$, $[3, 4, 6, 10, 0, 7]$, $[5, 6, 8, 0, 2, 9]$, $[7, 8, 10, 2, 5, 9]$,
$[9, 10, 0, 4, g, 8]$, $[11, 0, 2, 6, 3, 5]$, $[0, a, 6, b, 5, 7]$, $[1, a, 7, b, 4, 11]$,
$[2, a, 8, b, 7, 11]$, $[3, a, 9, b, 6, 1]$, $[4, a, 10, b, 7, 9]$, $[5, a, 11, b, 9, 1]$,
$[0, c, 1, d, 9, 11]$, $[2, c, 3, d, g, 4]$, $[4, c, 5, d, g, 2]$, $[6, c, 7, d, g, 5]$,
$[8, c, 9, d, g, 6]$, $[10, c, 11, d, 8, 3]$, $[0, e, 3, f, g, 9]$, $[2, e, 5, f, 11, 1]$,
$[4, e, 7, f, g, 10]$, $[6, e, 9, f, 11, 3]$, $[8, e, 11, f, 1, 5]$, $[10, e, 1, f, g, 11]$,
$[g, 1, 3, 7, 10, 5]$.

For G_2: $[a, 6, 7, 9, 0, c] + i, i \in Z_4$, $[b, 1, 2, 4, 7, c] + i, i \in Z_5$,
$[4, 8, 1, 6, 0, d] + i, i \in Z_4$, $[d, 9, 1, e, 4, c] + i, i \in Z_3$,
$[g, 1, f, 5, 10, e] + i, i \in Z_2$, $[f, 10, 4, g, 8, e] + i, i \in Z_2$,

$[a, 10, 11, 1, 4, f]$, $[2, 5, a, 11, 0, f]$, $[b, 0, 1, 3, 6, g]$, $[d, 8, 0, e, 7, g]$,
$[g, 3, f, 7, 0, 5]$.

$\underline{w = 8}$ On the set $X = Z_{12} \bigcup \{a, b, c, d, e, f, g, h\}$

For G_1: $[1, 2, 4, 8, g, 0]$, $[3, 4, 6, 10, 0, 7]$, $[5, 6, 8, 0, h, 1]$, $[7, 8, 10, 2, h, 0]$,
$[9, 10, 0, 4, g, 8]$, $[11, 0, 2, 6, 3, 5]$, $[0, a, 6, b, 5, 7]$, $[1, a, 7, b, h, 3]$,
$[2, a, 8, b, h, 6]$, $[3, a, 9, b, h, 8]$, $[4, a, 10, b, 7, 9]$, $[5, a, 11, b, h, 10]$,
$[0, c, 1, d, 9, 11]$, $[2, c, 3, d, g, 4]$, $[4, c, 5, d, g, 2]$, $[6, c, 7, d, g, 5]$,
$[8, c, 9, d, g, 6]$, $[10, c, 11, d, 8, 3]$, $[0, e, 3, f, g, 9]$, $[2, e, 5, f, 11, 1]$,
$[4, e, 7, f, g, 10]$, $[6, e, 9, f, 11, 3]$, $[8, e, 11, f, 1, 5]$, $[10, e, 1, f, g, 11]$,
$[g, 1, 3, 7, 10, 5]$, $[h, 2, 9, 5, 6, 1]$, $[h, 4, 11, 7, 9, 1]$.

For G_2: $[a, 6, 7, 9, 0, d] + i, i \in Z_4$, $[b, 1, 2, 4, 7, e] + i, i \in Z_3$,
$[0, 4, 8, 1, 6, c] + i, i \in Z_3$, $[d, 9, 1, e, 4, h] + i, i \in Z_3$,
$[g, 1, f, 5, 10, e] + i, i \in Z_2$, $[h, 9, c, 11, 7, d] + i, i \in Z_2$,
$[a, 10, 11, 1, 4, f]$, $[2, 5, a, 11, 0, f]$, $[6, b, 0, 1, 3, h]$, $[10, b, 4, 5, 7, 3]$,
$[8, 11, b, 5, 6, g]$, $[f, 7, 0, g, 3, c]$, $[8, f, 10, 4, g, 7]$, $[f, 11, 5, g, 9, 3]$,
$[11, h, 1, c, 4, 9]$, $[h, 2, c, 5, 0, e]$.

$\underline{w = 10}$ On the set $X = Z_{12} \bigcup \{a, b, c, d, e, f, g, h, i, j\}$

For G_1: $[1, 2, 4, 8, g, 0]$, $[3, 4, 6, 10, j, 0]$, $[5, 6, 8, 0, h, 1]$, $[7, 8, 10, 2, h, 0]$,
$[9, 10, 0, 4, g, 8]$, $[11, 0, 2, 6, i, 4]$, $[0, a, 6, b, i, 9]$, $[1, a, 7, b, h, 3]$,
$[2, a, 8, b, h, 6]$, $[3, a, 9, b, h, 8]$, $[4, a, 10, b, 7, 9]$, $[5, a, 11, b, h, 10]$,
$[0, c, 1, d, 9, 11]$, $[2, c, 3, d, g, 4]$, $[4, c, 5, d, g, 2]$, $[6, c, 7, d, g, 5]$,
$[8, c, 9, d, g, 6]$, $[10, c, 11, d, i, 6]$, $[0, e, 3, f, g, 9]$, $[2, e, 5, f, j, 6]$,
$[4, e, 7, f, g, 10]$, $[6, e, 9, f, j, 7]$, $[8, e, 11, f, 1, 5]$, $[10, e, 1, f, g, 11]$,
$[g, 1, 3, 7, i, 10]$, $[h, 2, 9, 5, 6, 1]$, $[h, 4, 11, 7, j, 8]$, $[i, 0, 7, 5, j, 2]$,
$[i, 8, 3, 11, j, 4]$, $[j, 3, 5, 10, i, 1]$, $[j, 9, 1, 11, i, 2]$.

For G_2: $[7, 9, 0, a, 6, c] + x, x \in Z_3$, $[d, 9, 1, e, 4, h] + x, x \in Z_3$,
$[g, 1, f, 5, 10, e] + x, x \in Z_2$, $[h, 1, c, 4, 11, i] + x, x \in Z_2$,
$[c, 11, 7, h, 9, i] + x, x \in Z_2$, $[0, 4, 8, 1, 6, g] + x, x \in Z_2$,
$[a, 10, 11, 1, 4, f]$, $[0, 2, 5, a, 11, j]$, $[6, b, 0, 1, 3, i]$, $[5, 7, 10, b, 4, i]$,
$[11, b, 5, 6, 8, d]$, $[d, 2, 8, e, 0, f]$, $[e, 7, d, 3, 9, 4]$, $[i, 2, 6, j, 1, d]$,
$[j, 5, i, 6, 10, 3]$, $[4, 7, b, 1, 2, j]$, $[6, 9, b, 3, 4, j]$, $[5, 8, b, 2, 3, f]$,
$[3, a, 9, 10, 0, j]$, $[0, g, 3, f, 7, j]$, $[f, 10, 4, g, 8, 3]$, $[f, 11, 5, g, 9, j]$,
$[j, 8, i, 7, 3, c]$.

$\underline{w = 11}$ On the set $X = Z_{12} \bigcup \{a, b, c, d, e, f, g, h, i, j, k\}$

For G_1: $[1, 2, 4, 8, g, 0]$, $[3, 4, 6, 10, j, 0]$, $[5, 6, 8, 0, h, 1]$, $[7, 8, 10, 2, h, 0]$,

$\quad$ $[9, 10, 0, 4, g, 8]$, $[11, 0, 2, 6, i, 4]$, $[0, a, 6, b, i, 9]$, $[1, a, 7, b, h, 3]$,

$\quad$ $[2, a, 8, b, h, 6]$, $[3, a, 9, b, k, 8]$, $[4, a, 10, b, k, 0]$, $[5, a, 11, b, k, 10]$,

$\quad$ $[0, c, 1, d, k, 2]$, $[2, c, 3, d, g, 4]$, $[4, c, 5, d, g, 2]$, $[6, c, 7, d, g, 5]$,

$\quad$ $[8, c, 9, d, g, 6]$, $[10, c, 11, d, i, 6]$, $[0, e, 3, f, g, 9]$, $[2, e, 5, f, j, 6]$,

$\quad$ $[4, e, 7, f, g, 10]$, $[6, e, 9, f, j, 7]$, $[8, e, 11, f, k, 3]$, $[10, e, 1, f, g, 11]$,

$\quad$ $[g, 1, 3, 7, i, 10]$, $[h, 2, 9, 5, k, 4]$, $[h, 4, 11, 7, j, 8]$, $[i, 0, 7, 5, j, 2]$,

$\quad$ $[i, 8, 3, 11, j, 4]$, $[j, 3, 5, 10, i, 1]$, $[j, 9, 1, 11, i, 2]$, $[k, 7, 9, 11, h, 8]$,

$\quad$ $[k, 6, 1, 5, h, 10]$.

For G_2: $[7, 9, 0, a, 6, c] + x, x \in Z_3$, $[4, a, 10, 11, 1, k] + x, x \in Z_2$,

$\quad$ $[9, b, 3, 4, 6, k] + x, x \in Z_2$, $[d, 9, 1, e, 4, h] + x, x \in Z_3$,

$\quad$ $[g, 1, f, 5, 10, k] + x, x \in Z_3$, $[4, g, 8, f, 10, e] + x, x \in Z_2$,

$\quad$ $[h, 1, c, 4, 11, i] + x, x \in Z_2$, $[0, 4, 8, 1, 6, g] + x, x \in Z_2$,

$\quad$ $[10, 0, 3, a, 9, k]$, $[5, 8, b, 2, 3, c]$, $[8, e, 0, d, 2, j]$, $[9, e, 7, d, 3, h]$,

$\quad$ $[7, h, 9, c, 11, j]$, $[8, h, 10, c, 0, f]$, $[i, 2, 6, j, 1, d]$, $[i, 6, 10, j, 5, k]$,

$\quad$ $[i, 3, 8, j, 4, f]$, $[i, 10, 3, j, 9, 4]$, $[1, 3, 6, b, 0, j]$, $[11, b, 5, 6, 8, d]$,

$\quad$ $[3, k, 8, i, 7, j]$, $[7, b, 1, 2, 4, k]$.

Proof of Lemma 3.8

Table C

v	7	8	10	11	14	15	17	18	19	20
l	3	4	3	1	1	3	4	3	3	4
r	3	2	3	5	5	3	2	3	3	2
p	3	4	7	9	15	17	22	25	28	31
c	4	5	8	10	16	18	23	26	29	32

v	22	23	26	27	29	30	31	32	34	35
l	3	1	1	3	4	3	3	4	3	1
r	3	5	5	3	2	3	3	2	3	5
p	38	42	54	58	67	72	77	82	93	99
c	39	43	55	59	68	73	78	83	94	100

Let $OPD(v, G)=(X, \mathcal{A})$, and $OCD(v, G)=(X, \mathcal{B})$, for $G \in \{G_1, G_2\}$. the l, r, p and c see Table C. We construct $\mathcal{A}$ and $\mathcal{B}$ as follows:

$\underline{v = 7}$ For G_1, on the set $X=(Z_3 \times Z_2) \bigcup \{\infty\}$, $\mathcal{A}$: $[0_0, \infty, 1_1, 2_1, 1_0, 2_0] + i$, $i \in Z_3$. Leave edges: $(0_0, 0_1)$, $(1_0, 1_1)$, $(2_0, 2_1)$.

On the set $X=Z_7$, $GD(7, G_1, 2)$: $[0, 1, 3, 6, 2, 4] + i, i \in Z_7$. From Theorem 3.4, we obtain that there exist an $OPD(v, G_1, \lambda)$ and an $OCD(v, G_1, \lambda)$ for any positive integer λ.

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For G_2, on the set $X=(Z_3 \times Z_2) \bigcup \{\infty\}$, $\mathcal{A}$: $[1_1, 0_1, \infty, 0_0, 1_0, 2_1]$, $[1_1, \infty,$
$1_0, 2_0, 2_1, 0_0] + i$, $i \in Z_2$. Leave edges: $(0_0, 1_1, 2_0, 0_1)$. Therefore, there exist
an $OPD(v, G_2, \lambda)$ and an $OCD(v, G_2, \lambda)$ for any positive integer λ from
Corollary 3.3.

$\underline{v = 8}$ For G_1, on the set $X=Z_4 \bigcup \{a, b, c, d\}$, $\mathcal{A}$: $[a, b, c, d, 3, 0]$, $[1, a, 3, b, 0, 2]$,
$[0, c, 1, d, 2, b]$, $[2, c, 3, d, 0, b]$. Leave edges: $(1, 2, a, 0)$, (b, d). $\mathcal{B}=\mathcal{A}\bigcup\{[a, 1, 2,$
$0, b, d]\}$. Repeat edges: $(a, 1)$, $(2, 0)$.

<div align="center">

Table D

λ	1	2
L_λ	$P_3 \bigcup P_1$	$2P_1$
R_λ	$2P_1$	$P_3 \bigcup P_1$

</div>

For G_2, on the set $X=Z_8$, $\mathcal{A}$: $[0, 1, 4, 2, 6, 5] + i$, $i \in Z_3$, $[5, 0, 3, 4, 7, 2]$.
Leave edges: $(4, 5, 1)$, $(6, 1, 3)$. $\mathcal{B}=\mathcal{A}\bigcup\{[5, 4, 0, 6, 1, 3]\}$. Repeat edges:
$(6, 0, 4)$.

<div align="center">

Table E

λ	1	2
L_λ		P_2
R_λ	P_2	

</div>

Therefore, there exist an $OPD(8, G_i, \lambda)$ and an $OCD(8, G_i, \lambda)$ for any
positive integer λ and $i=1, 2$ from Theorem 2.11 and Tables D-E.

$\underline{v = 10}$ For G_1, On the set $X=Z_7 \bigcup \{\infty_1, \infty_2, \infty_3\}$, $\mathcal{A}$: $[6, \infty_3, 2, 0, \infty_2, \infty_1]$,
$[0, \infty_1, 3, 1, \infty_2, 2]$, $[1, \infty_1, 4, 2, \infty_3, 3]$, $[2, \infty_1, 5, 3, \infty_3, 4]$, $[3, \infty_2, 6, 4, \infty_3, 5]$,
$[4, \infty_2, 0, 5, \infty_1, 6]$, $[5, \infty_2, 1, 6, \infty_3, 0]$. Leave edges: $(1, \infty_3, \infty_1)$, (∞_2, ∞_3).

On the set $X=Z_5 \times Z_2$, blocks of $GD(10, G_1, 2)$ are as follows:
$[0_0, 1_0, 1_1, 2_1, 2_0, 4_0]$ mod $(5, -)$, $[0_0, 1_0, 2_1, 3_0, 1_1, 3_1]$ mod $(5, -)$,
$[0_1, 1_1, 2_0, 2_1, 0_0, 4_1]$ mod $(5, -)$.

From Theorem 3.4, there exist an $OPD(v, G_1, \lambda)$ and an $OCD(v, G_1, \lambda)$
for any positive integer λ.

For G_2, on the set $X=Z_7 \bigcup \{\infty_1, \infty_2, \infty_3\}$, $\mathcal{A}$: $[0, x, 3, 6, 1, \infty_3] + i$, $i \in$
Z_6, when $i \leq 2$, $x=\infty_1$, when $i > 2$, $x=\infty_2$, $[0, 5, 2, \infty_2, \infty_3, \infty_1]$. Leave
edges: $(\infty_2, \infty_1, 6, 0)$.

It follows from Corollary 3.3 that there exist an $OPD(v, G_2, \lambda)$ and an
$OCD(v, G_2, \lambda)$ for any positive integer λ.

<div align="center">

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</div>

$\underline{v=11}$ On the set $X=(Z_3 \times Z_3)\bigcup\{\infty_1, \infty_2\}$

For G_1, $\mathcal{A}$: $[\infty_1, 0_1, 0_0, 1_2, 1_0, 2_0]$ mod $(3, -)$, $[\infty_2, 0_0, 0_2, 1_1, 0_1, 2_1]$ mod $(3, -)$, $[0_0, 1_1, 2_2, 2_1, 0_2, 1_2]$ mod $(3, -)$. Leave edge: (∞_1, ∞_2).

For G_2, $\mathcal{A}$: $[1_0, 2_1, 0_2, 1_2, \infty_1, 1_1]$ mod $(3, -)$, $[2_0, 2_2, 0_0, 0_1, \infty_2, 0_2]$ mod $(3, -)$, $[0_0, 1_0, 0_1, 1_1, 1_2, 2_1]$ mod $(3, -)$. Leave edge: (∞_1, ∞_2).

By Corollary 3.6, there exist an $OPD(v, G_i, \lambda)$ and an $OCD(v, G_i, \lambda)$ for any positive integer λ and $i=1, 2$.

$\underline{v=14}$ Apply Lemma 3.7, Theorem 3.1 and Corollary 3.6, we obtain the result.

$\underline{v=15}$ For G_1, on the set $X=Z_{15}$, $\mathcal{A}$: $[1, 2, 4, 9, 3, 7]+i$, $i \in Z_6$, $[7, 8, 10, 0, 12, 1]+i$, $i \in Z_4$, $[11, 12, 14, 4, 3, 9]+i$, $i \in Z_2$, $[13, 7, 1, 6, 12, 3]+i$, $i \in Z_2$, $[0, 1, 3, 8, 5, 11]$, $[1, 5, 14, 10, 6, 12]$, $[2, 6, 0, 11, 13, 14]$. Leave edges: $(14, 0, 9, 13)$.

On the set $X=Z_5 \times Z_3$, $GD(15, G_1, 2)$: $[0_0, 1_0, 1_1, 2_1, 0_2, 2_2]$ mod $(5, -)$, $[0_0, 1_0, 2_1, 3_0, 2_0, 0_2]$ mod $(5, -)$, $[0_1, 1_1, 2_0, 2_1, 0_0, 4_2]$ mod $(5, -)$, $[0_0, 1_2, 2_2, 2_0, 3_1, 3_2]$ mod $(5, -)$, $[0_1, 1_2, 2_2, 2_1, 0_0, 4_2]$ mod $(5, -)$, $[0_0, 1_2, 3_1, 2_2, 0_1, 3_2]$ mod $(5, -)$, $[0_2, 2_0, 2_2, 3_1, 0_1, 1_2]$ mod $(5, -)$.

The result is obtained by Theorem 3.4.

For G_2, on the set $X=Z_{15}$, $\mathcal{A}$: $[0, 1, 4, 9, 7, 3]+i$, $i \in Z_{15}$ and $i \neq$ 3, 4, 7, $[12, 3, 9, 0, 6, 10]+i$, $i \in Z_2$, $[3, 4, 7, 12, 10, 14]$, $[4, 5, 8, 13, 11, 2]$, $[7, 8, 11, 1, 14, 5]$. Leave edges: $(2, 8, 14)$, $(5, 11)$.

The result is obtained by Corollary 3.3.

$\underline{v=17}$ On the set $X=Z_{17}$

For G_1, $\mathcal{A}$: $[1, 9, 7, 6, 8, 12]+i$, $i \in Z_9$, $[10, 1, 16, 15, 5, 12]+i$, $i \in Z_5$, $[15, 6, 4, 3, 14, 0]+i$, $i \in Z_3$, $[2, 5, 9, 12, 0, 4]+i$, $i \in Z_3$, $[0, 3, 7, 10, 6, 9]$, $[1, 4, 8, 11, 13, 16]$. Leave edges: $(8, 5, 15, 12)$, $(6, 16)$.

$\mathcal{B}=\mathcal{A}\bigcup\{[15, 8, 5, 12, 6, 16]\}$. Repeat edges: $(5, 12)$, $(8, 15)$.

The result is obtained by Table D and Theorem 2.11.

For G_2, $\mathcal{A}$: $[16, 5, 1, 3, 4, 7]+i$, $i \in Z_9$ and $i \neq 1, 2, 4$, $[8, 14, 10, 12, 13, 6]+i$, $i \in Z_7$ and $i \neq 2, 5$, $[0, 8, 16, 13, 3, 12]+i$, $i \in Z_4$, $[0, 6, 2, 4, 5, 13]+i$, $i \in Z_2$, $[13, 2, 15, 0, 1, 8]+i$, $i=0, 2$, $[3, 9, 5, 7, 8, 15]$, $[10, 16, 12, 14, 15, 7]$, $[4, 12, 5, 8, 11, 1]$. Leave edges: $(2, 9, 6)$, $(0, 7, 16)$.

$\mathcal{B}=\mathcal{A}\bigcup\{[6, 9, 2, 0, 7, 16]\}$. Repeat edges: $(2, 0)$, $(6, 7)$.

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λ	1	2
L_λ	$2P_2$	$2P_1$
R_λ	$2P_1$	$2P_2$

Therefore, there exist an $OPD(v, G_2, \lambda)$ and an $OCD(v, G_2, \lambda)$ for any positive integer λ from Theorem 2.11.

$\underline{v = 18}$ On the set $X = Z_{18}$, for G_1, $\mathcal{A}$: $[0, 8, 6, 5, 7, 10] + i$, $i \in Z_{11}$, $[0, 3, 7, 14, 2, 11] + i$, $i \in Z_7$, $[11, 1, 17, 16, 14, 3] + i$, $i \in Z_4$, $[15, 5, 3, 2, 10, 14] + i$, $i \in Z_3$. Leave edges: $(0, 9)$, $(1, 10)$, $(13, 17)$.

On the set $X = Z_{17} \bigcup \{\infty\}$, $GD(18, G_1, 2)$: $[1, 9, 7, 6, 2, 5] \bmod (17, -)$, $[1, 8, 10, 7, 2, 6] \bmod (17, -)$, $[1, \infty, 5, 6, 2, 9] \bmod (17, -)$. The result is obtained by Theorem 3.4.

For G_2, the result is obtained by Lemma 3.7 and Theorem 2.14.

$\underline{v = 19, 20, 22, 23}$, taking $w = 7, 8, 10, 11$ in Theorem 2.14, we obtain the Lemma.

$\underline{v = 26}$ On the set $X = Z_{26}$

For G_1, $\mathcal{A}$: $[0, 9, 11, 16, 2, 8] + i$, $i \in Z_{18}$, $[18, 1, 3, 8, 22, 2] + i$, $i \in Z_4$, $[22, 5, 7, 12, 8, 15] + i$, $i \in Z_4$, $[0, 12, 4, 3, 15, 22] + i$, $i \in Z_{17}$, $[17, 3, 21, 20, 2, 15] + i$, $i \in Z_9$, $[0, 6, 13, 20, 11, 24] + i$, $i \in Z_2$. Leave edge: $(12, 19)$.

For G_2, $\mathcal{A}$: $[1, 4, 8, 15, 0, 13] + i$, $i \in Z_{13}$, $[13, 14, 17, 21, 2, 8] + i$, $i \in Z_3$, $[24, 5, 16, 17, 20, 14] + i$, $i \in Z_4$, $[21, 24, 2, 9, 20, 0] + i$, $i \in Z_6$, $[5, 13, 22, 12, 0, 2] + i$, $i \in Z_{22}$, $[22, 1, 9, 18, 8, 14] + i$, $i \in Z_4$, $[0, 6, 12, 18, 24, 22] + i$, $i \in Z_2$. Leave edge: $(5, 11)$.

The result is obtained by Corollary 3.6.

$\underline{v = 27}$ On the set $X = Z_{27}$

For G_1, $\mathcal{A}$: $[0, 3, 1, 12, 4, 10] + i$, $i \in Z_{23}$, $[23, 26, 24, 8, 2, 9] + i$, $i \in Z_4$, $[0, 4, 9, 19, 1, 8]$, $[1, 5, 10, 20, 4, 17] + i$, $i \in Z_{23}$, $[24, 1, 6, 16, 10, 17] + i$, $i \in Z_3$, $[0, 6, 13, 20, 24, 4] + i$, $i \in Z_4$. Leave edges: $(17, 24)$, $(18, 25)$, $(19, 26)$.

$GD(27, G_1, 2)$: $[0, 3, 1, 12, 4, 10] + i, i \in Z_{23}$, $[23, 26, 24, 8, 2, 9] + i, i \in Z_4$, $[0, 4, 9, 19, 1, 8]$, $[1, 5, 10, 20, 4, 17] + i, i \in Z_{23}$, $[24, 1, 6, 16, 10, 17] + i, i \in Z_3$, $[0, 6, 13, 20, 24, 4] + i, i \in Z_4$, $[0, 3, 1, 12, 17, 24]$, $[1, 4, 2, 13, 5, 11] + i, i \in Z_{22}$, $[23, 26, 24, 8, 2, 9] + i, i \in Z_4$, $[0, 4, 9, 19, 18, 25]$, $[1, 5, 10, 20, 19, 26]$, $[2, 6, 11, 21, 5, 18] + i, i \in Z_{22}$, $[24, 1, 6, 16, 18, 25]$, $[25, 2, 7, 17, 11, 18] + i, i \in$

Z_2, $[0,6,13,20,19,26]$, $[1,7,14,21,25,5]+i, i \in Z_4$. The result is obtained by Theorem 3.4.

For G_2, $\mathcal{A}$: $[3,7,14,0,1,4]+i$, $i \in Z_{23}$, $[26,1,5,12,25,4]+i$, $i \in Z_2$,
$[1,6,14,23,13,2]+i$, $i \in Z_{26}$, $[7,13,19,25,1,22]+i$, $i \in Z_3$,
$[12,0,5,13,22,16]$, $[18,24,0,6,12,1]$, $[23,24,26,3,10,16]$, $[11,24,25,0,4,10]$.
Leave edges: $(5,11,17,23)$. The result is obtained by Corollary 3.3.

$\underline{v=29}$ On the set $X=Z_{29}$
for G_1, $\mathcal{A}$: $[1,5,2,13,9,0]+i$, $i \in Z_{20}$, $[21,25,22,4,19,0]+i$, $i \in Z_9$,
$[0,13,6,8,9,24]+i$, $i \in Z_{20}$, $[20,4,26,28,0,24]+i$, $i \in Z_9$,
$[0,10,20,15,4,9]+i$, $i \in Z_9$. Leave edges: $(14,19,9,28)$, $(13,18)$.

$\mathcal{B}=\mathcal{A}\bigcup\{[9,14,19,28,13,18]\}$. Repeat edges: $(19,28)$, $(9,14)$.

The result is obtained by Table D and Theorem 2.11.

For G_2, $\mathcal{A}$: $[0,1,3,6,11,23]+i$, $i \in Z_{29}$, $[0,4,10,17,9,22]+i$, $i \in Z_{29}$,
$[20,1,15,0,10,24]+i$, $i \in Z_4$, $[5,19,4,14,24,9]+i$, $i \in Z_5$. Leave edges: $(9,19,0,14,28)$.

$\mathcal{B}=\mathcal{A}\bigcup\{[9,19,0,14,28,1]\}$. Repeat edges: $(1,28,9)$.

Table G

λ	1	2
L_λ	P_4	P_2
R_λ	P_2	P_4

Therefore, there exist an $OPD(v, G_2, \lambda)$ and an $OCD(v, G_2, \lambda)$ for any positive integer λ from Theorem 2.11.

$\underline{v=30}$ On the set $X=Z_{30}$
For G_1, $\mathcal{A}$: $[0,4,1,12,2,11]+i$, $i \in Z_{30}$, $[0,13,6,8,1,15]+i$, $i \in Z_{30}$,
$[0,5,15,20,2,12]+i$, $i \in Z_3$, $[3,8,18,23,17,27]+i$, $i \in Z_3$,
$[6,11,21,26,12,17]+i$, $i \in Z_3$, $[9,14,24,29,27,2]+i$, $i \in Z_3$.
Leave edges: $(12,27)$, $(13,28)$, $(14,29)$.

$GD(30, G_1, 2)$: $[0,4,1,12,2,11]+i, i \in Z_{30}$, $[0,13,6,8,1,15]+i, i \in Z_{30}$,
$[0,5,15,20,2,12]+i, i \in Z_3$, $[3,8,18,23,17,27]+i, i \in Z_3$,
$[6,11,21,26,12,17]+i, i \in Z_3$, $[9,14,24,29,27,2]+i, i \in Z_3$,
$[0,4,1,12,2,11]+i, i \in Z_{30}$, $[0,13,6,8,1,15]+i, i \in Z_{30}$,
$[7,12,22,27,13,18]+i, i \in Z_2$, $[1,6,16,21,3,13]+i, i \in Z_2$,
$[4,9,19,24,18,28]+i, i \in Z_2$, $[10,15,25,0,28,3]+i, i \in Z_2$,

$[6, 11, 21, 26, 12, 27]$, $[0, 5, 15, 20, 14, 29]$, $[3, 8, 18, 23, 13, 28]$,

$[9, 14, 24, 29, 13, 28]$, $[12, 17, 27, 2, 14, 29]$.

The result is obtained by Theorem 3.4.

For G_2, $\mathcal{A}$: $[0, 1, 11, 23, 2, 17] + i$, $i \in Z_{15}$, $[17, 15, 16, 26, 8, 14] + i$, $i \in Z_{15}$,

$[7, 23, 26, 1, 24, 5] + i$, $i \in Z_{29}$, $[10, 18, 22, 0, 6, 28] + i$, $i \in Z_6$,

$[2, 6, 14, 18, 26, 0] + i$, $i \in Z_4$, $[22, 25, 0, 23, 6, 12]$, $[0, 4, 12, 16, 24, 28]$,

$[17, 25, 1, 5, 13, 7]$. Leave edges: $(25, 29, 23, 4)$.

The result is obtained by Corollary 3.3.

$\underline{v = 31}$ Since $G\text{-}HGD(4^2 7^1)$(see [15]) and $TD(3, 2)$ exist, the result is true by Theorem 2.8 and Table A.

$\underline{v=32}$ On the set $X=Z_{32}$

For G_1, $\mathcal{A}$: $[1, 10, 16, 6, 22, 23] + i$, $i \in Z_{13}$, $[14, 23, 29, 19, 18, 21] + i$, $i \in Z_{14}$,

$[28, 5, 11, 1, 19, 30] + i$, $i \in Z_5$, $[1, 15, 13, 20, 19, 23] + i$, $i \in Z_{13}$,

$[14, 28, 26, 1, 22, 30] + i$, $i \in Z_{13}$, $[27, 9, 7, 14, 24, 3] + i$, $i \in Z_6$,

$[0, 4, 3, 11, 1, 17] + i$, $i \in Z_{16}$, $[16, 20, 19, 27, 30, 9] + i$, $i \in Z_2$.

Leave edges : $(29, 21, 22, 18, 29)$.

Table H

λ	1	2
L_λ	C_4	$2P_1$
R_λ	$2P_1$	C_4

Therefore, there exist an $OPD(v, G_2, \lambda)$ and an $OCD(v, G_2, \lambda)$ for any positive integer λ from Theorem 2.11 and Table H.

For G_2, $\mathcal{A}$: $[1, 4, 2, 6, 0, 18] + i$, $i \in Z_{14}$, $[16, 20, 14, 15, 18, 31] + i$, $i \in Z_{13}$,

$[31, 29, 1, 27, 28, 8] + i$, $i \in Z_5$, $[7, 17, 2, 11, 0, 24] + i$, $i \in Z_{13}$,

$[13, 20, 30, 15, 24, 19] + i$, $i \in Z_{13}$, $[11, 28, 5, 26, 1, 21] + i$, $i \in Z_6$,

$[15, 3, 8, 16, 2, 18] + i$, $i \in Z_{16}$, $[13, 1, 6, 14, 0, 20]$, $[15, 1, 14, 2, 7, 27]$.

Leave edges: $(0, 5, 13)$, $(12, 31, 19)$. The same as $v=17$, we can obtain the result.

$\underline{v=34}$ On the set $X=Z_{34}$

For G_1, $\mathcal{A}$: $[0, 7, 20, 15, 1, 10] + i$, $i \in Z_{34}$, $[0, 4, 3, 11, 1, 18] + i$, $i \in Z_{17}$,

$[17, 21, 20, 28, 1, 25] + i$, $i \in Z_9$, $[26, 30, 29, 3, 1, 7] + i$, $i \in Z_8$,

$[0, 12, 10, 16, 7, 25] + i$, $i \in Z_9$, $[9, 21, 19, 25, 1, 3] + i$, $i \in Z_9$,

$[18, 30, 28, 0, 3, 25] + i$, $i \in Z_7$. Leave edges : $(11, 33)$, $(10, 32)$, $(9, 15)$.

$(34, G_1, 2)\text{-}GD=(X, \mathcal{C})$, $\mathcal{C}$: $[0, 7, 20, 15, 1, 10] + i$, $i \in Z_{34}$,

$[0, 4, 3, 11, 1, 18] + i$, $i \in Z_{17}$, $[17, 21, 20, 28, 1, 25] + i$, $i \in Z_9$,
$[26, 30, 29, 3, 1, 7] + i$, $i \in Z_8$, $[0, 12, 10, 16, 7, 25] + i$, $i \in Z_9$,
$[9, 21, 19, 25, 1, 3] + i$, $i \in Z_9$, $[18, 30, 28, 0, 3, 25] + i$, $i \in Z_7$,
$[0, 7, 20, 15, 1, 10] + i$, $i \in Z_{34}$, $[0, 4, 3, 11, 1, 18] + i$, $i \in Z_{17}$,
$[1, 13, 11, 17, 8, 26] + i$, $i \in Z_8$, $[18, 22, 21, 29, 2, 26] + i$, $i \in Z_8$,
$[27, 31, 30, 4, 2, 8] + i$, $i \in Z_7$, $[10, 22, 20, 26, 2, 4] + i$, $i \in Z_8$,
$[19, 31, 29, 1, 4, 26] + i$, $i \in Z_7$, $[0, 12, 10, 16, 9, 15]$, $[17, 21, 20, 28, 9, 15]$,
$[26, 30, 29, 3, 10, 32]$, $[9, 21, 19, 25, 11, 33]$, $[18, 30, 28, 0, 11, 33]$.

The result is obtained by Theorem 3.4.

For G_2, $\mathcal{A}$: $[0, 1, 3, 6, 10, 26] + i$, $i \in Z_{34}$, $[11, 22, 14, 0, 5, 20] + i$, $i \in Z_8$,
$[30, 22, 8, 13, 19, 7] + i$, $i \in Z_9$, $[31, 17, 22, 28, 5, 26] + i$, $i \in Z_8$,
$[30, 2, 13, 5, 25, 32] + i$, $i \in Z_9$, $[0, 7, 16, 28, 13, 30] + i$, $i \in Z_{17}$,
$[11, 30, 17, 24, 33, 8] + i$, $i \in Z_8$. Leave edges: $(19, 4, 25)$, $(32, 7)$.

The result is obtained by Corollary 3.3.

<u>v=35</u> On the set $X = Z_{35}$

For G_1, $\mathcal{A}$: $[0, 4, 3, 11, 1, 15] + i$, $i \in Z_{35}$, $[0, 20, 13, 30, 1, 10] + i$, $i \in Z_{35}$,
$[0, 12, 10, 16, 4, 6] + i$, $i \in Z_6$, $[6, 18, 16, 22, 4, 10] + i$, $i \in Z_6$,
$[12, 24, 22, 28, 4, 29] + i$, $i \in Z_6$, $[18, 30, 28, 34, 6, 29] + i$, $i \in Z_6$,
$[24, 1, 34, 5, 10, 29] + i$, $i \in Z_5$. Leave edges :$(15, 34)$.

For G_2, $\mathcal{A}$: $[0, 1, 3, 6, 10, 26] + i$, $i \in Z_{35}$, $[14, 21, 30, 7, 27, 10] + i$, $i \in Z_{29}$,
$[15, 24, 1, 21, 8, 29] + i$, $i \in Z_6$, $[11, 22, 14, 0, 5, 16] + i$, $i \in Z_5$,
$[10, 16, 27, 19, 5, 34] + i$, $i \in Z_6$, $[22, 33, 25, 11, 16, 8] + i$, $i \in Z_6$,
$[31, 17, 22, 28, 4, 21] + i$, $i \in Z_6$, $[10, 2, 23, 28, 34, 29] + i$, $i \in Z_6$.
Leave edges: $(10, 21)$.

The same as $v=11$, we can obtain the result.