# Neighborhood Connected Domination in Graphs

S. ARUMUGAM
Core Group Research Facility (CGRF)
National Centre for Advanced Research in Discrete Mathematics (n-CARDMATH)
Kalasalingam University
Anand Nagar, Krishnankoil-626190, INDIA.
e-mail: *s_arumugam_akce@yahoo.com*
and
C. SIVAGNANAM
Department of Mathematics
St. Joseph's College of Engineering
Chennai-600119, INDIA.
e-mail: *c_sivagnanam71@yahoo.co.uk*

**Abstract**

Let $G = (V, E)$ be a connected graph. A dominating set $S$ of $G$ is called a *neighborhood connected dominating set* (*ncd-set*) if the induced subgraph $\langle N(S) \rangle$ is connected. The minimum cardinality of a *ncd-set* of $G$ is called the *neighborhood connected domination number* of $G$ and is denoted by $\gamma_{nc}(G)$. In this paper we initiate a study of this parameter.

**Keywords :** Neighborhood connected domination, total domination, connected domination, paired domination.
**2000 Mathematics Subject Classification:** 05C69

# 1 Introduction

By a graph $G = (V, E)$ we mean a finite, undirected and connected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [1].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of $v$ are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If

$S \subseteq V$ and $u \in S$, then the private neighbor set of $u$ with respect to $S$ is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A subset $S$ of $V$ is called a dominating set of $G$ if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$ ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [4]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [5].

Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the Appendix of Haynes et al. [4]. Sampathkumar and Walikar [6] introduced the concept of connected domination in graphs. A dominating set $S$ of $G$ is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. Cockayne et al. [2] introduced the concept of total domination in graphs. A dominating set $S$ of $G$ is called a total dominating set of $G$ if $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. Haynes and Slater [3] introduced the concept of paired domination in graphs. A dominating set $S$ of $G$ is called a paired dominating set if $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set of $G$ is called the paired domination number of $G$ and is denoted by $\gamma_{pr}(G)$.

For a dominating set $S$ of $G$ it is natural to look at how $N(S)$ behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_4\}$ are dominating sets, $\langle N(S_1) \rangle$ is not connected and $\langle N(S_2) \rangle$ is connected. Motivated by this example, in this paper we introduce the concept of neighborhood connected domination and initiate a study of the corresponding parameter.

# 2 Main Results

**Definition 2.1.** *A dominating set $S$ of a connected graph $G$ is called a neighborhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. A ncd-set $S$ is said to be minimal if no proper subset of $S$ is a ncd-set. The minimum cardinality of a ncd-set of $G$ is called the neighborhood connected domination number of $G$ and is denoted by $\gamma_{nc}(G)$.*

**Remark 2.2.**

(i) *Clearly $\gamma_{nc} \geq \gamma$. Further if $S$ is a total dominating set or a paired dominating set or a connected dominating set with $|S| > 1$, then $N(S) = V$ and hence $\gamma_{nc} \leq \gamma_t, \gamma_{nc} \leq \gamma_{pr}$ and $\gamma_{nc} \leq \gamma_c$ if $\gamma_c \neq 1$.*

(ii) *For any connected graph $G, \gamma_{nc} = 1$ if and only if there exists a non-cut vertex $v$ such that deg $v = n-1$. Thus $\gamma_{nc}(G) = 1$ if and only if $G = H + K_1$ for some connected graph $H$.*

**Theorem 2.3.** *For any graph $G$, $\gamma(G) \leq \gamma_{nc}(G) \leq 2\gamma(G)$. Further given two positive integers $a$ and $b$ with $a \leq b \leq 2a$, there exists a graph $G$ with $\gamma(G) = a$ and $\gamma_{nc}(G) = b$.*

*Proof.* We have $\gamma \leq \gamma_{nc}(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$. Now, let $a$ and $b$ be positive integers with $a \leq b \leq 2a$. Let $b = a + k, 0 \leq k \leq a$. If $k = 0$, let $G = K_a \circ K_1$. Suppose $1 \leq k \leq a$. Consider the complete graph $K_a$ with $V(K_a) = \{v_1, v_2, \ldots, v_a\}$. Let $H$ be the graph obtained from $K_a$ by attaching a path of length two to each vertex of $K_a$. Let $u_1, u_2, \ldots, u_a$ be the support vertices of $H$ with $u_i$ adjacent to $v_i$. Let $G$ be the graph obtained from $H$ by attaching an edge to each $u_i$ with $1 \leq i \leq k$. Then $\gamma(G) = a$ and $\gamma_{nc}(G) = a + k = b$. □

**Theorem 2.4.** *For the path $P_n$, $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$.*

*Proof.* Let $P_n = (v_1, v_2, \ldots, v_n)$. If $n$ is even, then $S = \{v_i : i = 2k, 2k+1$ and $k$ is odd$\}$ is a *ncd-set* of $P_n$ and if $n$ is odd, then $S_1 = S \cup \{v_n\}$ is a *ncd-set* of $P_n$. Hence $\gamma_{nc}(P_n) \leq \lceil \frac{n}{2} \rceil$. Further if $S$ is any $\gamma_{nc}$-set of $P_n$, then $N(S)$ contains all the internal vertices of $P_n$ and hence $|S| \geq \lceil \frac{n}{2} \rceil$. Thus $\gamma_{nc}(P_n) = \lceil \frac{n}{2} \rceil$. □

**Corollary 2.5.** *For any nontrivial path $P_n$,*

(i) *$\gamma_{nc}(P_n) = \gamma(P_n)$ if and only if $n = 2$ or $4$.*

(ii) *$\gamma_{nc}(P_n) = \gamma_c(P_n)$ if and only if $n = 4$ or $5$.*

(iii) *$\gamma_{nc}(P_n) = \gamma_t(P_n)$ if and only if $n \geq 3$ and $n \not\equiv 2 (mod\ 4)$.*

*Proof.* Since $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma_c(P_n) = n - 2$ and

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (mod\ 4) \\ \lfloor \frac{n}{2} \rfloor + 1 & \text{otherwise} \end{cases}$$

the corollary follows. □

**Theorem 2.6.** $\gamma_{nc}(C_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 3 (mod\ 4) \\ \lfloor \frac{n}{2} \rfloor & n \equiv 3 (mod\ 4). \end{cases}$

*Proof.* Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 4k + r$, where $0 \leq r \leq 3$. Let $S = \{v_i : i = 2j, 2j+1, j$ is odd and $1 \leq j \leq 2k - 1\}$.

$$\text{Let } S_1 = \begin{cases} S & \text{if } n \equiv 0 (mod\ 4) \\ S \cup \{v_n\} & n \equiv 1 \text{ or } 2 (mod\ 4) \\ S \cup \{v_{n-1}\} & \text{if } n \equiv 3 (mod\ 4). \end{cases}$$

Clearly $S_1$ is a *ncd-set* of $C_n$ and hence

$$\gamma_{nc}(C_n) \le \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 3(mod\ 4) \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 3(mod\ 4). \end{cases}$$

Now, let $S$ be any $\gamma_{nc}$-set of $C_n$. Then $\langle S \rangle$ contains at most one isolated vertex and

$$\langle N(S) \rangle = \begin{cases} P_{n-1} & \text{if } n \not\equiv 0(mod\ 4) \\ C_n & \text{if } n \equiv 0(mod\ 4). \end{cases}$$

Hence $|S| \ge \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \not\equiv 3(mod\ 4) \\ \lfloor \frac{n}{2} \rfloor & \text{if } n \equiv 3(mod\ 4) \end{cases}$

and the result follows. □

**Corollary 2.7.**

(i) $\gamma_{nc}(C_n) = \gamma(C_n)$ *if and only if* $n = 3, 4$ *or* $7$

(ii) $\gamma_{nc}(C_n) = \gamma_c(C_n)$ *if and only if* $n = 3, 4$ *or* $5$

(iii) $\gamma_{nc}(C_n) = \gamma_t(C_n)$ *if and only if* $n \equiv 0$ *or* $1(mod\ 4)$.

*Proof.* Since $\gamma(C_n) = \lceil \frac{n}{3} \rceil, \gamma_c(C_n) \equiv n - 2$ and

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2(mod\ 4) \\ \lceil \frac{n}{2} \rceil & \text{otherwise,} \end{cases}$$

the result follows. □

We now proceed to obtain a characterization of minimal ncd-sets.

**Lemma 2.8.** *A superset of a ncd-set is a ncd-set.*

*Proof.* Let $S$ be a ncd-set of a graph $G$ and let $S_1 = S \cup \{v\}$, where $v \in V - S$. Clearly, $v \in N(S)$ and $S_1$ is a dominating set of $G$. Now, let $x, y \in N(S_1)$. If $x, y \in N(S)$, then any $x$-$y$ path in $N(S)$ is a $x$-$y$ path in $N(S_1)$. If $x \in N(S)$ and $y \notin N(S)$, then $y \in N(v)$ and any $x$-$v$ path in $N(S)$ followed by the edge $vy$ is a $x$-$y$ path in $N(S_1)$. Also if $x, y \notin N(S)$, then $(x, v, y)$ is a $x$-$y$ path in $N(S_1)$. Thus $\langle N(S_1) \rangle$ is connected, so that $S_1$ is a ncd-set of $G$. □

**Theorem 2.9.** *A ncd-set $S$ of a graph $G$ is a minimal ncd-set if and only if for every $u \in S$, one of the following holds.*

(i) $pn[u, S] \neq \emptyset$.

(ii) *There exist two vertices $x, y \in N(S)$ such that every $x$-$y$ path in $\langle N(S) \rangle$ contains at least one vertex of $N(S) - N(S - \{u\})$.*

*Proof.* Let $S$ be a minimal ncd-set of $G$. Let $u \in S$ and let $S_1 = S - \{u\}$. Then either $S_1$ is not a dominating set of $G$ or $\langle N(S_1)\rangle$ is disconnected. If $S_1$ is not a dominating set of $G$, then $pn[u, S] \neq \emptyset$. If $\langle N(S_1)\rangle$ is disconnected then there exist two vertices $x, y \in N(S_1)$ such that there is no $x$-$y$ path in $\langle N(S_1)\rangle$. Since $\langle N(S)\rangle$ is connected, it follows that every $x$-$y$ path in $\langle N(S)\rangle$ contains at least one vertex of $N(S) - N(S - \{u\})$. Conversely, if $S$ is a ncd-set of $G$ satisfying the conditions of the theorem, then $S$ is 1-minimal and hence the result follows from Lemma 2.8. □

**Theorem 2.10.** *Let $G$ be a graph with $\Delta = n - 1$. Then $\gamma_{nc}(G) = 1$ or 2. Further $\gamma_{nc}(G) = 2$ if and only if $G$ has exactly one vertex $v$ with $deg\ v = n - 1$ and $v$ is a cut vertex of $G$.*

*Proof.* Let $v \in V(G)$ and $deg\ v = n - 1$. Then $\{u, v\}$, where $u \in V - \{v\}$, is a ncd-set of $G$ so that $\gamma_{nc}(G) \leq 2$. Now suppose $\gamma_{nc}(G) = 2$. Then $\langle N(v)\rangle = G - v$ is disconnected and hence $v$ is a cut vertex of $G$. Hence it follows that $v$ is the only vertex of $G$ with $deg\ v = n - 1$. The converse is obvious. □

In the following theorems we obtain a bound for $\gamma_{nc}$ and characterize trees and unicyclic graphs attaining the bound.

**Theorem 2.11.** *Let $G$ be a graph with $\Delta < n - 1$. Then $\gamma_{nc}(G) \leq n - \Delta$.*

*Proof.* Let $v \in V(G)$ and $deg\ v = \Delta$. Since $G$ is connected and $\Delta < n-1$, there exist two adjacent vertices $u$ and $w$ such that $u \in N(v)$ and $w \notin N[v]$. Now, let $S = (N(v) - \{u\}) \cup \{w\}$. Clearly $V - S$ is a ncd-set of $G$ and hence $\gamma_{nc}(G) \leq n - \Delta$. □

**Theorem 2.12.** *Let $T$ be any tree with $n > 2$. Then $\gamma_{nc}(T) = n - \Delta$ if and only if $T$ can be obtained from a star by subdividing $k$ of its edges, $k \geq 1$, once or by subdividing exactly one edge twice.*

*Proof.* Let $T$ be a tree with $\gamma_{nc}(T) = n - \Delta$. Let $v \in V(T)$ and $deg\ v = \Delta$. Clearly, $T$ is not a star and hence $\Delta < n - 1$. Let $N(v) = \{v_1, v_2, \ldots, v_\Delta\}, A = V(T) - N[v] = \{w_1, w_2, \ldots, w_k\}$ and $T_1 = \langle A\rangle$.

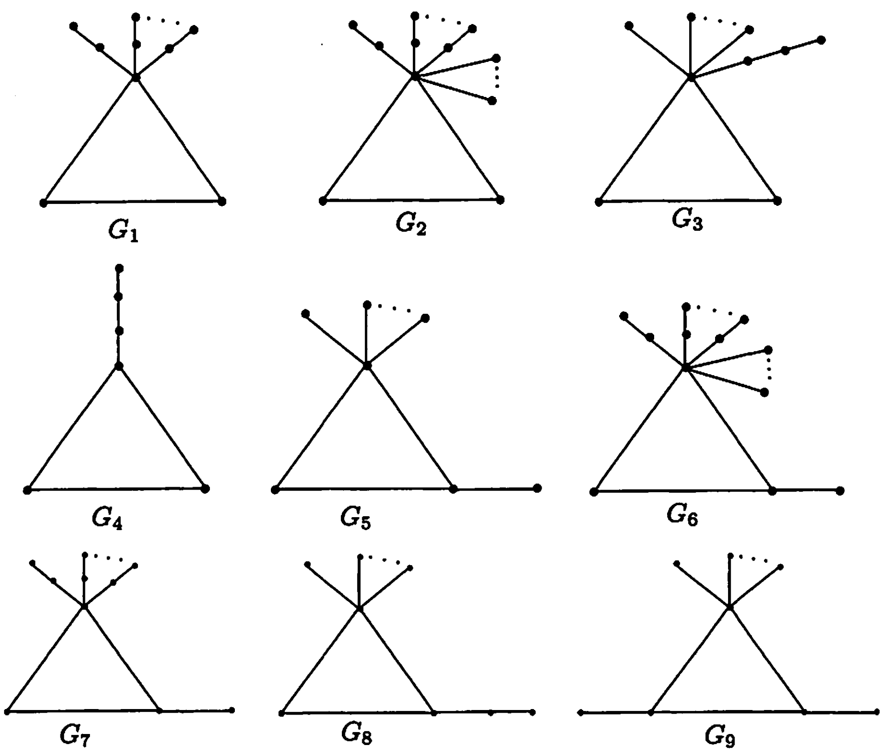
**Case i.** $E(T_1) = \emptyset$.

If $deg\ v_i \geq 3$ for some $v_i \in N(v)$, then $S = \{v, v_i\} \cup (A - (N(v_i) \cap A))$ is a ncd-set of $T$ and $|S| \leq n - \Delta - 1$, which is a contradiction. Hence $deg\ v_i \leq 2$, so that $T$ can be obtained from $K_{1,\Delta}$ by subdividing $k$ of its edges once, $k \geq 1$.

**Case ii.** $E(T_1) \neq \emptyset$.

Let $G_1$ be any non-trivial component of $T_1$ and we may assume without loss of generality that $v_1 \in N(V(G_1))$. If $G_1$ contains more than one pendant vertex of $T$, then $S = V(T) - (\{v_2, \ldots, v_\Delta\} \cup V_1)$ where $V_1$ is the set of all pendant vertices of $T$ in $G_1$ is a ncd-set of $G$ with $|S| < n-\Delta$, which is a contradiction. Thus $G_1$ has exactly one pendant vertex of $T$ and hence $G_1$ is a path. Let $G_1 = (x_1, x_2, \ldots, x_r)$ with $v_1 \in N(x_1)$. If $r > 2$, then $S = V(T) - \{v_2, \ldots, v_\Delta, x_1, x_2\}$ is a ncd-set of $T$ with $|S| = n - \Delta - 1$, which is a contradiction. Thus $G_1 = P_2$. Now, if $T_1$ has two nontrivial components $G_1 = (x_1, x_2)$ and $G_2 = (y_1, y_2)$, then $S = V(T) - \{v_2, \ldots, v_\Delta, x_1, y_1\}$ is a ncd-set of $T$, which is again a contradiction. Thus $T_1$ has exactly one nontrivial component and $T$ is the tree obtained from $K_{1,\Delta}$ by subdividing exactly one edge twice. The converse is obvious. □

**Theorem 2.13.** *Let $G$ be a unicyclic graph with cycle $C = (v_1, v_2, \ldots, v_r, v_1)$. Then $\gamma_{nc} = n - \Delta$ if and only if $G$ is isomorphic to $C_3, C_4, C_5$ or one of the graphs $G_i, 1 \le i \le 15$, given in figure 1.*



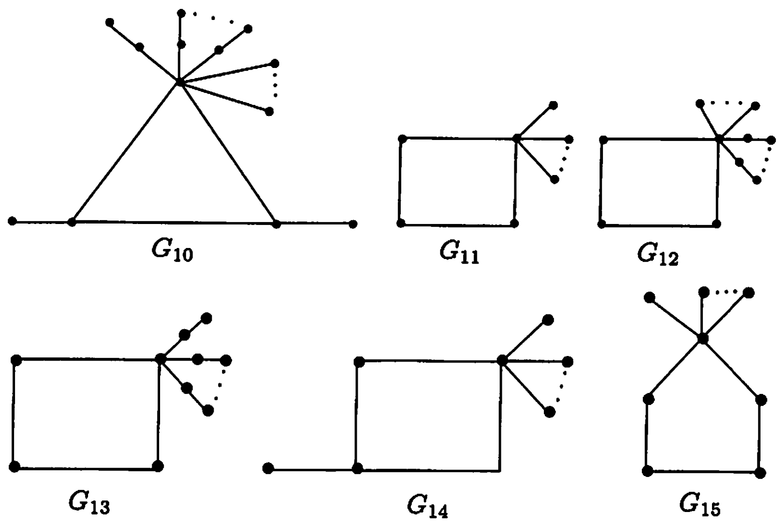


Figure 1

*Proof.* Let $G$ be a unicyclic graph with cycle $C$ and $\gamma_{nc} = n-\Delta$. If $G = C$, then it follows from Theorem 2.6 that $n \leq 5$ and hence $G$ is isomorphic to $C_3$ or $C_4$ or $C_5$.

Suppose $G \neq C$. Let $A$ denote the set of all pendant vertices in $G$ and let $|A| = k$. Clearly,

$$\Delta - 2 \leq k \leq \Delta. \tag{1}$$

**Claim 1.** If $v \in V(G)$ and $deg\ v = \Delta$ then $v$ lies on $C$.

Suppose $v$ is not on $C$. Then $k = \Delta - 1$ or $\Delta$. Now, $S = [V(G) - (V(C) \cup A)] \cup S_1$, where $S_1$ is a $\gamma_{nc}$-set of $C$, is a ncd-set of $G$ and $|S| < n - \Delta$, which is a contradiction. Hence $v \in V(C)$.

Let $C = (v_1, v_2, \ldots, v_r, v_1)$ and $deg\ v_1 = \Delta$.

**Claim 2.** $deg\ w = 1$ or $2$ for all $w \in V(G) - V(C)$.

Suppose there exists a vertex $w \in V(G) - V(C)$ with $deg\ w > 2$. Then $k = \Delta - 1$ or $\Delta$. If $k = \Delta - 1$, all the vertices of $V(C) - \{v_1\}$ have degree 2 and hence $S = V(G) - [A \cup \{v_2, v_3\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$.

If $k = \Delta$, then at least one vertex $v_i$ on $C$ has degree 2 and $S = V(G) - [A \cup \{v_i\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$. Hence it follows that $deg\ w = 1$ or $2$ for all $w \in V(G) - V(C)$.

**Claim 3.** Every vertex of $V(C) - \{v_1\}$ has degree 2 or 3.

It follows from (1) that $deg\ v_i \leq 4$ for all $i \neq 1$. If there exists a vertex $v_i \in V(C)$ with $deg\ v_i = 4$, then $k = \Delta$ and $deg\ v_j = 2$ for all $j \neq 1, i$. Now, $S = V(G) - [A \cup \{v_j\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$. This proves Claim 3.

**Claim 4.** $r \leq 5$.

Suppose $r \geq 6$.

If $k = \Delta$ then there exists a vertex $v_i$ such that $deg\ v_i = 2$ and $S = V(G) - [A \cup \{v_i\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$.

If $k = \Delta - 1$ then there exist two adjacent vertices $v_i$ and $v_j$ such that $d(v_i) = d(v_j) = 2$. Now, $S = V(G) - [A \cup \{v_i, v_j\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$.

If $k = \Delta - 2$, then every vertex of $V(C) - \{v_1\}$ has degree 2 and hence $S = V(G) - [A \cup \{v_2, v_3, v_5\}]$ is a ncd-set of $G$ with $|S| < n - \Delta$. Thus, $r \leq 5$.

**Claim 5.** $d(w, C) \leq 3$ for all $w \in A$.

Suppose there exists a pendant vertex $w_1$ such that $d(w_1, C) \geq 4$. Let $(w_1, w_2, \ldots, w_k, v_i)$, $k \geq 4$, be the unique $w_1 - v_i$ path. Then $S = (V(G) - [A \cup \{w_2, w_3, v_2, v_3\}]) \cup \{w_1\}$ is a ncd-set of $G$ with $|S| < n - \Delta$. Hence $d(w, C) \leq 3$ for all $w \in A$.

**Claim 6.** There exists at most one vertex $w \in A$ such that $d(w, C) = 3$.

Suppose $w_1, w_1' \in A$ and $d(w_1, C) = d(w_1', C) = 3$. Let $(w_1, w_2, w_3, v_i)$ and $(w_1', w_2', w_3', v_j)$ be the unique $w_1 - v_i$ and $w_1' - v_j$ paths respectively.

If $k = \Delta$ then $S = (V - [A \cup \{w_2, w_3, w_2', w_3'\}]) \cup \{w_1, w_1'\}$ is a ncd-set of $G$ with $|S| < n - \Delta$.

If $k < \Delta$ then there exists a vertex $v$ on $C$ with $d(v) = 2$. Hence $S = (V - [A \cup \{w_2, w_3, w_2', w_3', v\}]) \cup \{w_1, w_1'\}$ is a ncd-set of $G$ with $|S| < n - \Delta$.

This proves Claim 6.

Now, let $n_1, n_2$ and $n_3$ denote respectively the number of pendant vertices with distance 1,2 and 3 from $C$, so that $n_1, n_2 \geq 0, 0 \leq n_3 \leq 1$ and $n_1 + n_2 + n_3 = k$. We consider three cases.

**Case (1).** $k = \Delta - 2$.

In this case $deg\ x = 1$ or $2$ for all $x \in V(G) - \{v_1\}$.

Now, if $r = 5$ and if there exists a vertex $w \in V(G) - V(C)$ such that $deg\ w = 2$, then $S = V(G) - [A \cup \{v_2, v_4, v_5\}]$ is a ncd-set of $G$ with $|S| = n - \Delta - 1$. Hence $deg\ w = 1$ for all $w \in V(G) - V(C)$ and $G$ is isomorphic to $G_{15}$.

Now, suppose $r \leq 4$.

If $r = 4$ and $n_3 = 1$, let $w_1 \in A$ be such that $(w_1, w_2, w_3, v_1)$ is the unique $w_1 - v_1$ path. Now, $S = (V(G) - [A \cup \{v_2, v_3, w_2, w_3\}]) \cup \{w_1\}$ is a ncd-set of $G$ with $|S| < n - \Delta$. Hence if $r = 4$ then $n_3 = 0$. Hence $G$ is isomorphic to one of the graphs $G_1, G_2, G_3, G_4, G_{11}, G_{12}, G_{13}$.

**Case 2.** $k = \Delta - 1$.

In this case $deg\ v_i = 3$ for exactly one vertex $v_i \neq v_1$ on $C$ and $deg\ x = 1$ or 2 for all $x \in V - \{v_1, v_i\}$. Further $C$ has at most two vertices of degree 2 and hence $r = 3$ or 4. Now, if there exists a pendant vertex $w_1 \in A$ such that $d(w_1, C) = 3$, let $(w_1, w_2, w_3, v_i)$ be the unique $w - v_i$ path in $G$. Then $S = V(G) - [(A - \{w_1\}) \cup \{v_j, w_2, w_3\}]$ where $deg\ v_j = 2$ is a ncd-set of $G$

with $|S| < n - \Delta$. Hence $n_3 = 0$. When $r = 4$, by a similar argument we get $n_2 = 0$ and the two vertices of dgree 2 on $C$ are non-adjacent, so that $G$ is isomorphic to $G_{14}$. Now, suppose $r = 3$ and $deg\ v_2 = 3$. If there exists a pendant vertex $w_1$ such that $d(w_1, C) = d(w_1, v_2) = 2$, then all neighbours of $v_1$ not on $C$ are pendant vertices and $G$ is isomorphic to $G_8$. Otherwise $G$ is isomorphic to one of the graphs $G_5, G_6$ or $G_7$.

**Case 3.** $k = \Delta$.

In this case $C$ has no vertex of degree 2 and hence $r = 3$ and every vertex on $C$ is a support. Thus $G$ is isomorphic to $G_9$ or $G_{10}$. The converse is obvious. □

**Problem 2.14.** *Characterize the class of graphs for which $\gamma_{nc} = n - \Delta$.*

**Theorem 2.15.** *For any graph $G$, $\gamma_{nc}(G) \leq \lceil \frac{n}{2} \rceil$.*

*Proof.* Since $\gamma_{nc}(G) \leq \gamma_{nc}(T)$ for any spanning tree $T$ of $G$, it is enough to prove the result for trees, which we prove by induction on $n$. Obviously the result is true when $n = 2$ or 3. We now assume that the result is true for all trees of order less than $n$ and let $T$ be a tree of order $n, n \geq 4$. If $n$ is odd, let $T_1 = T - \{v\}$ where $v$ is a pendant vertex of $T$. Then $\gamma_{nc}(T_1) \leq \frac{n-1}{2}$, so that $\gamma_{nc}(T) \leq \gamma_{nc}(T_1) + 1 \leq \lceil \frac{n}{2} \rceil$. Now, suppose $n$ is even. Let $u$ be a support vertex with maximum eccentricity in $T$. If $deg\ u = 2$, let $T_1 = T - \{u, v\}$ where $v$ is the pendant vertex adjacent to $u$. If $deg\ u \geq 3$, let $T_1 = T - \{v_1, v_2\}$ where $v_1$ and $v_2$ are two pendant vertices adjacent to $u$. Then $\gamma_{nc}(T_1) \leq \frac{n-2}{2}$ and $\gamma_{nc}(T) \leq \gamma_{nc}(T_1) + 1 \leq \lceil \frac{n}{2} \rceil$. □

**Remark 2.16.** *The bound given in Theorem 2.15 is sharp. If $T_1$ is any non-trivial tree, then for the carona $T = T_1 \circ K_1$, we have $\gamma_{nc}(T) = \frac{n}{2}$.*

**Problem 2.17.** *Characterize graphs for which $\gamma_{nc}(G) = \lceil \frac{n}{2} \rceil$.*

**Theorem 2.18.** *Let $G$ be a graph with $diam\ G = 2$. Then $\gamma_{nc}(G) \leq \delta + 1$ and the bound is sharp.*

*Proof.* If $v \in V(G)$ and $deg\ v = \delta$, then $N[v]$ is an ncd-set of $G$ and hence the result follows. The bound is attained for $K_{1,n}$ and $C_5$. □

**Theorem 2.19.** *Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then*

$$\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) \leq \begin{cases} \lceil \frac{n}{2} \rceil + 2 & \text{if } diam\ G \geq 3 \\ \lceil \frac{n}{2} \rceil + 3 & \text{if } diam\ G = 2. \end{cases}$$

*Proof.* If $diam\ G \geq 3$, then $S = \{u, v\}$, where $d(u, v) = diam(G)$, is a ncd-set of $\overline{G}$ so that $\gamma_{nc}(\overline{G}) = 2$. If $diam\ G = 2$, then it follows from Theorem 2.18 that $\gamma_{nc}(G) = 3$. Hence the result follows from Theorem 2.15. □

**Remark 2.20.** *The bounds given in Theorem 2.19 are sharp. The graph* $G = C_5$ *has diameter 2,* $\gamma_{nc}(G) = \gamma_{nc}(\overline{G}) = 3$, *and* $\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) = 6 = \lceil \frac{n}{2} \rceil + 3$ . *For the graph* $G = C_k \circ K_1$, *where* $k \not\equiv 3(mod\ 4)$, *diam* $G \geq 3$ *and* $\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) = \lceil \frac{n}{2} \rceil + 2$.

**Problem 2.21.** *Characterize graphs which attain the bounds given in Theorem 2.18 and Theorem 2.19.*

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