

On the p -domination, the total domination and the connected domination numbers of graphs

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Abstract

Let G be a simple graph, and let p be a positive integer. A subset $D \subseteq V(G)$ is a p -dominating set of the graph G , if every vertex $v \in V(G) - D$ is adjacent with at least p vertices of D . The p -domination number $\gamma_p(G)$ is the minimum cardinality among the p -dominating sets of G . Note that the 1-domination number $\gamma_1(G)$ is the usual domination number $\gamma(G)$. A subset $S \subseteq V(G)$ is said to be a total dominating set if every vertex in $V(G)$ has at least one neighbor in S and it is a connected dominating set if the graph induced by S is connected. The total domination number $\gamma_t(G)$ represents the cardinality of a minimum total dominating set of G and the connected domination number $\gamma_c(G)$ the cardinality of a minimum connected dominating set.

Fink and Jacobson showed in 1985 that if G is a graph with $\Delta(G) \geq p \geq 2$, then $\gamma_p(G) \geq \gamma(G) + p - 2$. In this paper we will give some sufficient conditions for a graph G such that $\gamma_p(G) \geq \gamma(G) + p - 1$. We will show that for block graphs G the inequality $\gamma_p(G) \geq \gamma_t(G) + p - 2$ is valid and that for trees T the inequality $\gamma_p(T) \geq \gamma_c(T) + p - 1$ holds. Further, we characterize the trees T with $\gamma_p(T) = \gamma_c(T) + p - 1$, $\gamma_p(T) = \gamma_t(T) + p - 2$, $\gamma_p(T) = \gamma_t(T) + p - 1$ and $\gamma_p(T) = \gamma(T) + p - 1$.

Keywords: domination, total domination, connected domination, p -domination, tree

AMS subject classification: 05C69

1 Terminology

We consider finite, undirected, and simple graphs G with vertex set $V(G)$ and edge set $E(G)$. The number of vertices $|V(G)|$ of a graph G is called the *order* of G and is denoted by $n = n(G)$. The number of edges $|E(G)|$ is denoted with $m(G)$.

The *open neighborhood* $N(v) = N_G(v)$ of a vertex v consists of the vertices adjacent to v and $d(v) = d_G(v) = |N(v)|$ is the *degree* of v . The *closed neighborhood* of a vertex v is defined by $N[v] = N_G[v] = N(v) \cup \{v\}$. With $\Delta(G)$ and $\delta(G)$ we denote the maximum degree and respectively the minimum degree of all vertices in $V(G)$. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. An edge incident with a leaf is called a *pendant edge*. We denote with $L(G)$ the set of leaves of a graph G . For a subset $S \subseteq V(G)$, we define $N(S) = N_G(S) = \bigcup_{v \in S} N(v)$, $N[S] = N_G[S] = N(S) \cup S$, and $G[S]$ is the subgraph induced by S . The *complement* \bar{G} of a graph G is the graph with vertex set $V(\bar{G}) = V(G)$ such that an edge $e \in E(\bar{G})$ if and only if $e \notin E(G)$.

A *block* of a graph G is a maximal subgraph of G without a cut vertex. If every block of a graph is complete, then we speak of a *block graph*. We write K_n for the *complete graph* of order n , and $K_{p,q}$ for the *complete bipartite graph* with bipartition X, Y such that $|X| = p$ and $|Y| = q$. With $K_p - e$ we denote the complete graph K_p without an arbitrary edge e . A *triangulated graph* G is a graph where every cycle of length at least 4 contains at least one diagonal. A graph G is called *cubic* if every vertex in $V(G)$ has degree three.

The *subdivision graph* $S(G)$ of a graph G is that graph obtained from G by replacing each edge uv of G by a vertex w and edges uw and vw . In the case that G is the trivial graph, we define $S(G) = G$. Let SS_t be the subdivision graph of the *star* $K_{1,t}$. A tree is a *double star* if it contains exactly two vertices of degree at least two. A double star with respectively s and t leaves attached at each support vertex is denoted by $S_{s,t}$. Instead of $S(S_{s,t})$ we write $SS_{s,t}$. A *generalized star* is a tree that results from a star $K_{1,t}$ by subdividing its edges arbitrary many times.

The *corona graph* $G \circ K_1$ of a graph G is the graph constructed from a copy of G , where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.

Let p be a positive integer. A subset $D \subseteq V(G)$ is a *p -dominating set* of the graph G , if $|N_G(v) \cap D| \geq p$ for every $v \in V(G) - D$. The *p -domination number* $\gamma_p(G)$ is the minimum cardinality among the p -dominating sets of G . Note that the 1-domination number $\gamma_1(G)$ is the usual *domination number* $\gamma(G)$. A p -dominating set of minimum cardinality of a graph G is called a *$\gamma_p(G)$ -set*. A subset $D \subseteq V(G)$ is a *p -star-forming set* of G if for each vertex u in $V - D$ with less than p neighbors in D , $N(u) \cap D \neq \emptyset$ and at

least one vertex of $N(u) \cap D$ has at least $p - 1$ neighbors in D . This means that every vertex u of $V - D$ is contained in a (not necessarily induced) p -star $K_{1,p}$ of the subgraph induced by $D \cup \{u\}$. The minimum cardinality of a p -star-forming set of G is denoted $sf_p(G)$. As every p -dominating set is a p -star-forming set, $\gamma_p(G) \geq sf_p(G)$ for every graph G and every positive integer p . Note that $sf_1(G) = \gamma_1(G) = \gamma(G)$ for every graph G . A subset $S \subseteq V(G)$ is said to be a *total dominating set* if every vertex in $V(G)$ has at least one neighbor in S and it is a *connected dominating set*, if it is a dominating set and the graph induced by S is connected. The *total domination number* $\gamma_t(G)$ and the *connected domination number* $\gamma_c(G)$ represent the cardinality of a *minimum total dominating set* and, respectively, of a *minimum connected dominating set* of G .

The concepts of p -domination and p -star-forming sets were respectively introduced by Fink and Jacobson in [2], [3] and by Chellali and Favaron in [1]. For a comprehensive treatment of domination in graphs, see the monographs by Haynes, Hedetniemi, and Slater [4], [5].

2 Preliminary Results

Theorem 1 (Fink, Jacobson [2] 1985) *If G is a graph with $\Delta(G) \geq p \geq 2$, then*

$$\gamma_p(G) \geq \gamma(G) + p - 2.$$

Theorem 2 (Reed [6] 1996) *If G is a graph with $\delta(G) \geq 3$, then*

$$\gamma(G) \leq \frac{3}{8}n(G).$$

Theorem 3 (Fink, Jacobson [2] 1985) *If G is a graph of order n and maximum degree Δ , then*

$$\gamma_p(G) \geq \frac{p}{\Delta + p}n$$

for every integer $p \in \mathbb{N}$.

Theorem 4 (Chellali, Favaron [1] 2006) *Every nontrivial connected triangulated graph G satisfies $sf_2(G) = \gamma_t(G)$.*

3 Main Results

Lemma 1 *Let G be a nontrivial graph of order n such that $m(\bar{G}) \leq n - 2$ and $\Delta(G) \leq n - 2$. Then $m(\bar{G}) \geq \lceil \frac{n}{2} \rceil$ and G contains at least $m(\bar{G}) - 1$ induced cycles of length 4.*

Proof. Let $k = m(\bar{G})$. If $d_G(v) \leq n - 2$ for all $v \in V(G)$, then $n^2 - n - 2k = n(n - 1) - 2k = 2m(G) \leq n(n - 2) = n^2 - 2n$ and thus $k \geq \lceil \frac{n}{2} \rceil$.

Let l be the number of vertices in G with degree $n - 2$. Then $2m(G) = n^2 - n - 2k \leq l(n - 2) + (n - l)(n - 3) = n^2 - 3n + l$ and so it follows that $l \geq 2(n - k) \geq 4$.

We will now prove by induction on n that there are at least $k - 1$ induced cycles of length 4 in G . For $n \in \{2, 3\}$ the property $d_G(v) \leq n - 2$ is not satisfied for any $k \leq n - 2$. If $n = 4$, then the only possibility for having $\Delta(G) \leq n - 2$ is when $k = 2$ and $G \cong C_4$. Now suppose that $n \geq 5$.

Case 1. Suppose that there is a vertex x of degree $n - 2$ in G such that the only vertex that is not adjacent with x , say y , has degree less than $n - 2$. Then all vertices in $G' := G - x$ have degree at most $n - 3$ and $m(\bar{G}') \leq k - 1$ and so from the induction hypothesis follows that G' contains at least $k - 2$ induced cycles of length 4. If all neighbors of y would be adjacent to each other, that is, if $G[N_G(y)] \cong K_{d_G(y)+1}$, then, since $\Delta(G) \leq n - 2$, for each $z \in N_G(y)$ would exist a vertex $z' \in N_{\bar{G}}(y)$ such that $zz' \in E(\bar{G})$. But this would imply that $k \geq d_G(y) + d_{\bar{G}}(y) = n - 1$, which is a contradiction. So, there have to be two vertices $z, w \in N_G(y)$ such that $zw \in E(\bar{G})$. It follows that $xzywx$ is a new induced cycle of length 4 in G and thus G has at least $k - 1$ induced C_4 .

Case 2. Suppose that every vertex of degree $n - 2$ in G is exactly not adjacent to another vertex of degree $n - 2$. Then the vertices of degree $n - 2$ in G induce a matching M in \bar{G} and thus l is even and $|M| = l/2 \geq n - k$. This implies that if $xy \in M$ and $x'y' \in E(\bar{G}) - \{xy\}$, then the vertices x, x', y, y' induce a cycle of length 4 in G . It follows that there are $|M|(|M| - 1)/2$ induced C_4 with vertices in $V(M)$ and $|M|(k - |M|)$ induced C_4 with vertices in both $V(M)$ and $V(G) - V(M)$. Since $n/2 \geq |M| = l/2 \geq n - k$, we obtain that there are at least

$$\frac{(n - k - 1)(n - k)}{2} + \frac{(n - k)(2k - n)}{2} = \frac{(n - k)(k - 1)}{2} \geq k - 1$$

cycles of length 4. \square

Theorem 5 *If G is a graph with at most $p - 2$ induced cycles of length 4 for an integer p with $\Delta(G) \geq p \geq 2$, then $\gamma_p(G) \geq \gamma(G) + p - 1$.*

Proof. Let S be a $\gamma_p(G)$ -set. Since $p \leq \Delta(G)$, we conclude that $V(G) - S \neq \emptyset$. Let x be any vertex of $V(G) - S$ and A_x any subset of $N(x) \cap S$ of size p . If $\{x\} \cup (S - A_x)$ is a dominating set of G , then $\gamma(G) \leq |S| - p + 1$ and we are done. Thus let B be the set of vertices of $V(G) - S$ such that for every $y \in B$, $N(y) \cap S = A_x$ and y is not adjacent to x . Since G has at most $p - 2$ induced C_4 -cycles, the subgraph $G[A_x]$ has to be a clique minus at most $p - 2$ edges. Then $k := m(\overline{G[A_x]}) \leq p - 2 = n(G[A_x]) - 2$. If $G[A_x]$ contains no vertex of degree $p - 1$, then from Lemma 1 follows that $G[A_x]$ has at least $k - 1$ induced cycles of length 4. But this implies that G contains $k - 1 + k = 2k - 1 \geq 2 \lfloor \frac{p}{2} \rfloor - 1 \geq p - 1$ induced C_4 -cycles, which is a contradiction. Therefore, there has to be a vertex $u \in A_x$ of degree $p - 1$ in $G[A_x]$. It follows that $S - (A_x - \{u\})$ is a dominating set and hence $\gamma(G) \leq \gamma_p(G) - p + 1$. \square

Let G be a graph which is isomorphic to K_{p+1} with one subdivided edge, say the edge $xy \in E(K_{p+1})$ is subdivided into xv and vy . Then $V(G) - \{x, y\}$ is a $\gamma_p(G)$ -set, $\{x, y\}$ is a $\gamma(G)$ -set, and G contains $p - 1$ induced C_4 and fulfills $\gamma_p(G) = \gamma(G) + p - 2$. Thus Theorem 5 is best possible.

Corollary 1 *If G is a graph without an induced cycle C_4 , then $\gamma_p(G) \geq \gamma(G) + p - 1$ for every positive integer p with $\Delta(G) \geq p \geq 2$.*

Since triangulated graphs do not contain induced C_4 , we also obtain the following corollary.

Corollary 2 *If G is a triangulated graph, then $\gamma_p(G) \geq \gamma(G) + p - 1$ for every positive integer p with $\Delta(G) \geq p \geq 2$.*

In particular, every nontrivial block graph G has the property $\gamma_p(G) \geq \gamma(G) + p - 1$ for every positive integer p with $\Delta(G) \geq p \geq 2$. If we regard the graph $G = K_n \circ K_1$, where n is an integer with $n \geq p$, it is evident that G fulfills $\gamma_p(G) = n + p - 1$ and $\gamma(G) = n$. This shows that Corollary 2 is best possible.

Theorems 2 and 3 lead directly to the next observation.

Observation 1 *If G is a graph of order n , maximum degree Δ and $\delta(G) \geq 3$, then*

$$\gamma_p(G) - \gamma(G) \geq \frac{5p - 3\Delta}{8(\Delta + p)} n$$

for every integer $p \in \mathbb{N}$.

Remark 1 If G is a graph of order n with $\delta(G) \geq 3$ and $5p > 3\Delta(G)$, then previous observation yields $\gamma_p(G) \geq \gamma(G) + \frac{5p-3\Delta}{8(\Delta+p)} n$. This shows that if $c \geq p - 2$ is an arbitrary constant, then there exist only a finite number of graphs G such that $\delta(G) \geq 3$, $5p > 3\Delta(G)$ and $\gamma_p(G) \leq \gamma(G) + c$.

Remark 2 Let G be a cubic graph. Observation 1 implies that $\gamma_2(G) \geq \frac{n}{40} + \gamma(G)$ and $\gamma_3(G) \geq \frac{n}{8} + \gamma(G)$. Hence the cubic graphs with $\gamma_2(G) = \gamma(G) + 1$ have at most 40 vertices and those with $\gamma_3(G) = \gamma(G) + 1$ at most 8. Analyzing all cubic graphs with at most 8 vertices, it is a simple matter to verify that the only cubic graph that fulfills $\gamma_3(G) = \gamma(G) + 1$ is $G \cong K_{3,3}$.

If we regard the total domination number γ_t , we can give a similar result for block graphs to the one of Fink and Jacobson in Theorem 1. This will follow from the next theorem.

Theorem 6 Let G be a connected $\{C_4, K_4 - e\}$ -free graph and p, q two integers such that $2 \leq q \leq p \leq \Delta(G)$. Then $\gamma_p(G) \geq sf_q(G) + p - q$.

Proof. If $p = q$, we already know that $\gamma_p(G) \geq sf_p(G)$. So assume $p \geq q + 1$ and consider a $\gamma_p(G)$ -set D . Since $p \leq \Delta(G)$, $V - D \neq \emptyset$. Let $v \in V - D$, A a subset of $N(v) \cap D$ of size $p - q + 1$, and $D' = (D - A) \cup \{v\}$. Suppose that D' is not a q -star-forming set. Then there exists a vertex u in $V - D' = A \cup (V - (D \cup \{v\}))$ having less than q neighbors in D' and such that every vertex of $N(u) \cap D'$, if any, has degree less than $q - 1$ in D' . This vertex u cannot be in A since every vertex of A is adjacent to $v \in D'$ and (as v is p -dominated by D) v has at least $q - 1$ neighbors in $D - A \subseteq D'$. Therefore $u \in V - (D \cup \{v\})$. The vertex u , which has at least p neighbors in D but less than q in $D' = (D - A) \cup \{v\}$, is adjacent to every vertex in A but not to v . Since $|A| = p - q + 1 \geq 2$, the subgraph induced by $A \cup \{u, v\}$ contains either an induced cycle C_4 or a K_4 minus an edge, contradicting the hypothesis on G . Thus D' is a q -star-forming set and $sf_q(G) \leq |D'| = \gamma_p - p + q$. \square

As, by Theorem 4, $sf_2(G) = \gamma_t(G)$ holds for every nontrivial triangulated graph G , we obtain from Theorem 6 with $p = 2$ following corollary for block graphs.

Corollary 3 Let G be a nontrivial block graph. If $\Delta(G) \geq p$ for an integer $p \geq 2$, then $\gamma_p(G) \geq \gamma_t(G) + p - 2$.

For the block graphs G where $\gamma_t(G) > \gamma(G)$, this is a stronger result than Corollary 2. Now we will concentrate in trees, which are a special kind of block graphs.

Theorem 7 Let T be a tree with $\Delta(T) \geq p \geq 3$ for an integer p . Then $\gamma_p(T) \geq \gamma_c(T) + p - 1$ with equality if and only if T is a generalized star with p leaves or, in the case $p = 3$, if T has maximum degree 3 and no two vertices of degree 3 are adjacent to each other.

Proof. Let τ_i be the number of vertices of degree i in T , $\Delta = \Delta(T)$ and let L be the set of leaves in T . Since T is a tree,

$$\tau_1 = 2 + \sum_{i=3}^{\Delta} (i-2)\tau_i \quad (1)$$

holds (see for example the book by Volkmann [7], p. 31). Because every p -dominating set contains all vertices of degree less than p and $V(T) - L$ is a $\gamma_c(T)$ -set of T , we deduce that

$$\begin{aligned} \gamma_p(T) &\geq \sum_{i=1}^{p-1} \tau_i = 2 + \sum_{i=3}^{\Delta} (i-2)\tau_i + \sum_{i=2}^{p-1} \tau_i \\ &= (n - \tau_1) + 2 + \sum_{i=3}^{p-1} (i-2)\tau_i + \sum_{i=p}^{\Delta} (i-3)\tau_i \\ &\geq \gamma_c(T) + 2 + (\Delta - 3)\tau_{\Delta} \\ &\geq \gamma_c(T) + 2 + (p - 3) = \gamma_c(T) + p - 1. \end{aligned}$$

Case 1. Assume that $p \geq 4$. It is a simple matter to verify that if T is a generalized star with p leaves, then $\gamma_p(T) = \gamma_c(T) + p - 1$ holds. Assume now that $\gamma_p(T) = \gamma_c(T) + p - 1$. Then we obtain equality in the whole inequality chain above, in particular $\Delta = p$, $\tau_{\Delta} = \tau_p = 1$ and $\tau_i = 0$ for $3 \leq i \leq p - 1$. We deduce that $\gamma_p(T) = n - 1$ and the inequality chain implies

$$\gamma_c(T) + p - 1 = \gamma_p(T) = (n - \tau_1) + 2 + (p - 3) = n - 1,$$

which leads to $\tau_1 = p$ and $\gamma_c(T) = n - p$. The tree T consists then of a vertex of degree p , p vertices of degree 1 and the remaining vertices are of degree 2. It follows that T is a generalized star with p leaves.

Case 2. Assume that $p = 3$. If T has maximum degree $\Delta(T) = 3$ and no two vertices of degree 3 are adjacent to each other, then it is easy to check that $V(T) - L$ is a $\gamma_c(T)$ -set and that $V(T) - L_3$ is a $\gamma_3(T)$ -set, where L_3 is the set of vertices of degree 3. It follows with (1) that $\gamma_3(T) = n - \tau_3 = n - \tau_1 + 2 = \gamma_c(T) + 2$.

Conversely, assume that $\gamma_3(T) = \gamma_c(T) + 2$. It follows from the inequality chain that $\Delta = 3$ and $\gamma_3(T) = \tau_1 + \tau_2$. So, if L_2 is the set of vertices with degree 2 in T , then $L \cup L_2$ is a $\gamma_3(T)$ -set and $V(T) - L$ is a $\gamma_c(T)$ -set. This implies that no two vertices of degree 3 can be adjacent to each other. \square

If $p \geq 3$, then every star $K_{1,p}$ satisfies $\gamma_p(G) = \gamma_t(G) + p - 2$. We will show in the next theorem that these are the only trees which satisfy this equality and for all other trees T the inequality $\gamma_p(T) \geq \gamma_t(T) + p - 1$ is valid. We also give a characterization of those with $\gamma_p(T) = \gamma_t(T) + p - 1$.

Let \mathcal{S}_t be the family of trees that are obtained from a star $K_{1,t}$ for $t \geq 3$ by subdividing one edge twice and the remaining edges at most twice but not all edges are subdivided twice. Let T_t be the tree that is obtained from the star $K_{1,t}$ by subdividing one edge exactly three times and let \mathcal{T} be the family of graphs that are obtained from every tree $T \in \mathcal{S}_3 \cup \{T_3\}$ by attaching a leaf to one or to both support vertices which have distance at least 2 to the unique vertex of degree 3 in T .

Theorem 8 *Let T be a tree different to a star such that $\Delta(T) \geq p \geq 3$ for an integer p . Then $\gamma_p(T) \geq \gamma_t(T) + p - 1$ with equality if and only if $T \in \mathcal{S}_p \cup \{T_p\}$ or T is isomorphic to a subdivided star SS_p minus k leaves for an integer $0 \leq k \leq p - 1$ or $T \in \mathcal{T}$ in the case $p = 3$.*

Proof. If T is different to a star, then it evident that $\gamma_c(T) \geq \gamma_t(T)$. With Theorem 7, we obtain then $\gamma_p(T) \geq \gamma_c(T) + p - 1 \geq \gamma_t(T) + p - 1$.

Case 1. Suppose that $p \geq 4$. If $T \in \mathcal{S}_p \cup \{T_p\}$ or T is isomorphic to a subdivided star SS_p minus k leaves for an integer $0 \leq k \leq p - 1$, then it is simple to verify that $\gamma_p(T) = \gamma_t(T) + p - 1$.

Now assume that $\gamma_p(T) = \gamma_t(T) + p - 1$. Then $\gamma_p(T) = \gamma_c(T) + p - 1$ holds, too, and T is a generalized star. Additionally, regarding the proof of Theorem 7, $\gamma_t(T) = \gamma_c(T) = n - \tau_1$ holds and hence $V(T) - L$ has to be a $\gamma_t(T)$ -set. If v is the unique vertex of degree p , then it follows that every path from v to a leaf has length at most 4, otherwise $V(T) - L$ would not be a minimum total dominating set. Further, if there is a path of length 4 from v to a leaf, then the other neighbors of v have to be leaves, since otherwise it would again contradict the minimality of $V(T) - L$. Thus T is isomorphic to T_p . Assume now that every path from v has length at most 3. Then T must contain at least one path of length one or two. It is now evident that, since T is not a star, T is either isomorphic to a subdivided star, SS_p with p leaves minus k leaves for an integer $0 \leq k \leq p - 1$ or $T \in \mathcal{S}_p$.

Case 2. Assume that $p = 3$. It is obvious that, if $T \in \mathcal{T}$, then $\gamma_3(T) = \gamma_t(T) + 2$.

Suppose now that $\gamma_3(T) = \gamma_t(T) + 2$. As above, $\gamma_3(T) = \gamma_c(T) + 2$ and $V(T) - L$ is both $\gamma_c(T)$ and $\gamma_t(T)$ -set. From Theorem 7, T is a tree with maximum degree $\Delta(T) = 3$ such that no two vertices of degree 3 are adjacent to each other.

If T has only one vertex of degree three, then by using a same argument to that used in Case 1, $T \in \mathcal{S}_3 \cup \{T_3\}$ or T is isomorphic to a subdivided star SS_3 minus k leaves for an integer $0 \leq k \leq 2$. Thus we assume that T has at least two vertices of maximum degree. Now let H be the graph induced by the $\gamma_t(T)$ -set $V(T) - L$. Assume for a contradiction that H contains two vertices x, y of maximum degree and all their neighbors, that is $d_H(x) = d_H(y) = 3$. Then, since H is connected, all vertices on the unique path between x and y have degree two in T and so $V(T) - L$ minus any vertex adjacent to x or y on this path is a total dominating set, which is a contradiction. Thus let us assume that H contains one vertex u such that $d_H(u) = 3$, and let $N_H(u) = \{u_1, u_2, u_3\}$. Then $d_T(u_i) = 2$ for each i . The minimality of H implies that one of u_1, u_2, u_3 , say u_1 , is a support vertex in T . Since T has at least two vertices of maximum degree, we can assume that u_2 has a second neighbor v in H . Again by minimality $d_H(v) = 1$ and v is a support vertex of two leaves (since it has to be the second vertex of maximum degree). Now if $d_H(u_3) = 1$, then u_3 is a support vertex of exactly one leaf and thus $T \in \mathcal{T}$. Let now $w \neq u$ be the neighbor of u_3 in H . It is clear that $d_H(w) = 1$ and w is a support vertex of one or two leaves in T . It follows that $T \in \mathcal{T}$.

Assume now that $d_H(x) \leq 2$ for every vertex in H . Since H is connected, H is a path. Assume that a vertex u of maximum degree has two neighbors in H , say u_1 and u_2 . Then u is a support vertex, $d_T(u_1) = d_T(u_2) = 2$ and so, without loss of generality, u_1 has another neighbor in H , say w . By the minimality of the $\gamma_t(T)$ -set $V(T) - L$, it follows that w is a support vertex of two leaves and $d_H(w) = 1$. Now if $d_H(u_2) = 1$, then u_2 is a support vertex of one leaf in T and $T \in \mathcal{T}$. Thus let $z \neq u$ be the neighbor of u_2 in H . Then $d_H(z) = 1$ and z is a support vertex of one or two leaves in T and thus $T \in \mathcal{T}$.

Now we may assume that every vertex of maximum degree is a leaf in H . Let u be a vertex of maximum degree, v its unique neighbor in H and $w \neq u$ a neighbor of v in H . If $d_H(w) = 1$, then w is a support vertex of two leaves in T and so $T \in \mathcal{T}$. Thus let $z \neq v$ be a neighbor of w in H . Then $d_T(w) = 2$ and, again by minimality, $d_H(z) = 1$ and z is a support vertex of two leaves in T , which implies that $T \in \mathcal{T}$.

Now we have achieved every tree contained in $\mathcal{T} \cup \{T_3\}$, too, and the proof is complete. \square

In 2007, Volkmann [8] showed that a nontrivial tree T satisfies $\gamma_2(T) = \gamma(T) + 1$ if and only if T is a subdivided star SS_t or a subdivided star SS_t minus a leaf or a subdivided double star $SS_{s,t}$. As a supplement to this result and as a consequence of the previous theorem, we characterize all trees T with $\gamma_p(T) = \gamma(T) + p - 1$ for $p \geq 3$.

Theorem 9 *Let T be a tree such that $\Delta(T) \geq p \geq 3$ for an integer p . Then $\gamma_p(T) \geq \gamma(T) + p - 1$ and $\gamma_p(T) = \gamma(T) + p - 1$ if and only if T is isomorphic to a subdivided star SS_p minus k leaves for an integer $1 \leq k \leq p$.*

Proof. Since $\gamma_c(T) \geq \gamma(T)$, from Theorem 7 follows $\gamma_p(T) \geq \gamma(T) + p - 1$.

If T is isomorphic to a subdivided star SS_p minus k leaves for an integer $1 \leq k \leq p$, then it is easy to see that $\gamma_p(T) = \gamma(T) + p - 1$.

Conversely, let T be a tree with $\gamma_p(T) = \gamma(T) + p - 1$. Since $\gamma_t(T) \geq \gamma(T)$, it follows, together with the former theorem, that either T is a star or $\gamma_p(T) = \gamma_t(T) + p - 1$. If T is a star $K_{1,r}$, then it is easy to see that $r = p$. Assume now that T is not a star. Then T has to be of the form of the graphs of Theorem 8 satisfying $\gamma_t(T) = \gamma(T)$. It is now straightforward to verify that T is isomorphic to a subdivided star SS_p minus k leaves for an integer $1 \leq k \leq p$. \square

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