A Note on Graphs with Prescribed Complete Coloring Numbers

Gary Chartrand
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

Futaba Okamoto
Mathematics Department
University of Wisconsin - La Crosse
La Crosse, WI 54601, USA

Zsolt Tuza[1]
Computer and Automation Institute
Hungarian Academy of Sciences
Budapest, HUNGARY

Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008, USA

ABSTRACT

A complete coloring of a graph G is a proper vertex coloring of G having the property that for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j. The maximum positive integer k for which G has a complete k-coloring is the achromatic number $\psi(G)$ of G. A Grundy coloring of a graph G is a proper vertex coloring of G having the property that for every two colors (positive integers) i and j with $i < j$, every vertex colored j has a neighbor colored i. The maximum positive integer k for which a graph G has a Grundy k-coloring is the Grundy number $\Gamma(G)$. Thus $2 \leq \chi(G) \leq \Gamma(G) \leq \psi(G)$ for every nonempty graph G. It is shown that if a, b, and c are integers with $2 \leq a \leq b \leq c$, then there exists a connected graph G with $\chi(G) = a$, $\Gamma(G) = b$, and $\psi(G) = c$ if and only if $a = b = c = 2$ or $b \geq 3$.

Key Words: proper coloring, complete coloring, Grundy coloring.
AMS Subject Classification: 05C15.

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[1]Also affiliated with the Department of Computer Science, University of Pannonia, Veszprém, Hungary. Research supported by the Hungarian Scientific Research Fund, OTKA grant T-049613.

1 Introduction

A (proper) *coloring* of a graph G is a function $c : V(G) \to \mathbb{N}$ having the property that $c(u) \neq c(v)$ for every pair u, v of adjacent vertices of G. A *k-coloring* of G uses k colors. The *chromatic number* $\chi(G)$ of G is the minimum integer k for which G has a k-coloring.

A *complete coloring* of a graph G is a proper vertex coloring of G having the property that for every two distinct colors i and j used in the coloring, there exist adjacent vertices of G colored i and j. A complete coloring in which k colors are used is a *complete k-coloring*. If a graph G has a complete k-coloring, then the size of G is at least $\binom{k}{2}$. While the minimum positive integer k for which a graph G has a complete k-coloring is the chromatic number $\chi(G)$, the maximum positive integer k for which a graph G has a complete k-coloring is the *achromatic number* $\psi(G)$. Achromatic number was introduced by Harary, Hedetniemi, and Prins [4]. Among the results obtained on the achromatic number of a graph is the following by Geller and Kronk [3] concerning the deletion of a vertex from a graph.

Theorem 1.1 *For each vertex v in a nontrivial graph G,*

$$\psi(G) - 1 \leq \psi(G - v) \leq \psi(G).$$

As a consequence of Theorem 1.1, it follows that if H is an induced subgraph of a graph G, then $\psi(H) \leq \psi(G)$. In particular, if an end-vertex is deleted from the path P_n of order $n \geq 2$, then $\psi(P_n) - 1 \leq \psi(P_{n-1}) \leq \psi(P_n)$. Hell and Miller [5], in fact, established the following.

Theorem 1.2 *For $n \geq 2$, $\psi(P_n) = \max\left\{k : \left(\lfloor \frac{k}{2} \rfloor + 1\right)(k-2) + 2 \leq n\right\}$.*

A *Grundy coloring* of a graph G is a proper vertex coloring of G using the integers $1, 2, \ldots, k$ as colors for some positive integer k and having the property that for every two colors i and j with $i < j$, every vertex colored j has a neighbor colored i. The maximum positive integer k for which a graph G has a Grundy k-coloring is the *Grundy number* of G and is denoted by $\Gamma(G)$. The Grundy number of a graph appears to have been introduced by Christen and Selkow in [2]. Since every Grundy coloring of a nonempty graph G is both a proper vertex coloring and a complete coloring, it follows that

$$2 \leq \chi(G) \leq \Gamma(G) \leq \psi(G).$$

In [6] Zaker also gave upper bounds for the Grundy number of a graph G in terms of the degrees, girth, and clique partition number of G. Since a vertex v assigned the color $\Gamma(G)$ in a Grundy coloring of G has a neighbor colored i for each positive integer i less than $\Gamma(G)$, it follows that $\Delta(G) \geq \deg v \geq \Gamma(G) - 1$ and so

$$\Gamma(G) \leq \Delta(G) + 1.$$

For the graph G of size $m = 10$ in Figure 1, a 3-coloring, a Grundy 4-coloring, and a complete 5-coloring of G are shown. Thus $\chi(G) \leq 3$, $\Gamma(G) \geq 4$, and $\psi(G) \geq 5$. Since G contains an odd cycle, $\chi(G) = 3$; since $\Delta(G) = 3$, $\Gamma(G) = 4$; and since the size $m < \binom{6}{2}$, $\psi(G) = 5$.

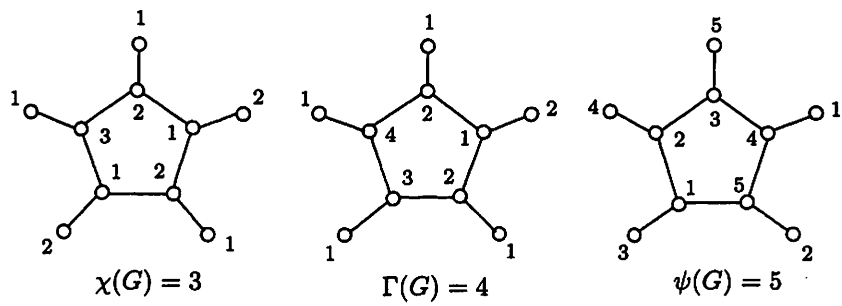

Figure 1: Complete and Grundy colorings

Bhave [1] showed that for every two integers a and b with $2 \leq a \leq b$, there exists a graph G with $\chi(G) = a$ and $\psi(G) = b$. Although the graphs G constructed in [1] are, for the most part, disconnected, there is also a *connected* graph G with $\chi(G) = a$ and $\psi(G) = b$. Christen and Selkow [2] showed that a graph G has a Grundy k-coloring if and only if $\chi(G) \leq k \leq \Gamma(G)$. In this paper, we determine all triples (a, b, c) of integers with $2 \leq a \leq b \leq c$ for which there exists a connected graph G such that $\chi(G) = a$, $\Gamma(G) = b$, and $\psi(G) = c$. A triple for which such a graph G exists will be called a *realizable triple*. After proving a necessary condition and describing the building blocks of the construction, the characterization is stated and proved in Theorem 2.9. Moreover, the construction is shown to provide a graph of minimum order for all triples (a, b, c) with $a \leq b = c$ in Theorem 2.10. In fact, Theorem 2.10 is tight in a much wider sense, as will be discussed in a forthcoming paper.

2 Realizable Triples

We begin by determining those realizable triples (a, b, c) of integers with $a = b = c \geq 2$. Since $\chi(K_a) = \Gamma(K_a) = \psi(K_a) = a$ for each integer $a \geq 2$, we have the following observation.

Observation 2.1 *For each integer $a \geq 2$, the triple (a, a, a) is realizable.*

We now consider those triples (a, b, c) of integers with $a = b = 2$. First, we present three useful lemmas. Since the proofs of these lemmas are straightforward, the proofs are omitted.

Lemma 2.2 *If G contains P_4 as an induced subgraph, then $\Gamma(G) \geq 3$.*

Lemma 2.3 *If G is a complete bipartite graph, then $\chi(G) = \Gamma(G) = \psi(G) = 2$.*

Lemma 2.4 *Let G be a connected graph. Then G is a complete bipartite graph if and only if G contains neither K_3 nor P_4 as an induced subgraph.*

Proposition 2.5 *A triple $(2, 2, c)$ is realizable if and only if $c = 2$.*

Proof. By Observation 2.1, the triple $(2, 2, 2)$ is realizable. For the converse, if G is a connected graph with $\chi(G) = \Gamma(G) = 2$, then G is a bipartite graph that contains no P_4 as an induced subgraph by Lemma 2.2. It then follows by Lemma 2.4 that G is a complete bipartite graph. Therefore, $\psi(G) = 2$ by Lemma 2.3. ∎

We continue to consider realizable triples (a, b, c) of integers for which $a = b$. In view of Proposition 2.5, we may assume that $a = b \geq 3$.

Proposition 2.6 *Every triple (a, a, c) with $3 \leq a \leq c$ is realizable.*

Proof. If $a = c$, then $\chi(K_a) = \Gamma(K_a) = \psi(K_a) = a$ by Observation 2.1. Thus, we may assume that $a < c$. By Theorem 1.2, for a given integer c, there is a positive integer ℓ such that $\psi(P_\ell) = c$. Let F be the graph obtained from K_a by identifying an end-vertex of the path P_ℓ with a vertex of K_a. Thus $\psi(F) \geq c$ as F contains P_ℓ as an induced subgraph. It then follows by Theorem 1.1 that there exists an integer $k \leq \ell$ such that identifying an end-vertex of P_k with a vertex of K_a results in a graph G with $\psi(G) = c$ (see Figure 2). Since K_a is the only block of G that is not acyclic, $\chi(G) = \chi(K_a) = a$.

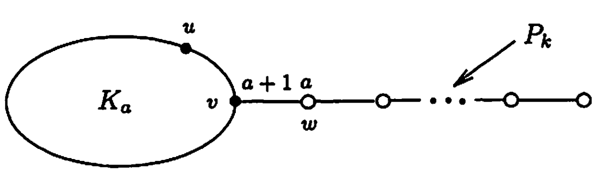

Figure 2: Graphs G with $\chi(G) = \Gamma(G) = a$ and $\psi(G) = c$

It remains to show that $\Gamma(G) = a$. Since the maximum degree of G is a and $\chi(G) = a$, it follows that $\Gamma(G) = a$ or $\Gamma(G) = a + 1$. Assume, to the contrary, that $\Gamma(G) = a + 1$. Then there exists a Grundy $(a + 1)$-coloring of G. Since v is the only vertex of degree a in G, it follows that v is the only vertex that is assigned the color $a+1$. Only the neighbor w of v with degree 2 can be colored a, for if some neighbor u of v having degree $a-1$ is colored a, then for at least one color i in the set $\{1, 2, \ldots, a-1\}$, there is no neighbor of u that is colored i. Thus as claimed, w is colored a. Since $a \geq 3$, there is no neighbor of w that is colored 1 or 2. This contradicts the assumption that the coloring is a Grundy $(a+1)$-coloring of G. Thus, as claimed, $\Gamma(G) = a$. ∎

We now show that all triples (a, b, c) with $2 = a < b \leq c$ are realizable.

Proposition 2.7 *Every triple* $(2, b, c)$ *with* $3 \leq b \leq c$ *is realizable.*

Proof. Let H be the graph obtained from $K_{b-1,b-1}$ whose partite sets are

$$U_1 = \{u_1, u_2, \ldots, u_{b-1}\} \text{ and } U_2 = \{v_1, v_2, \ldots, v_{b-1}\}$$

by removing the $b-2$ edges u_iv_i for $1 \leq i \leq b-2$. Then clearly $\chi(H) = 2$. We show that $\Gamma(H) = \psi(H) = b$, starting with the Grundy number.

Let $f_1 : V(H) \to \{1, 2, \ldots, b\}$ be a b-coloring of vertices of G defined by

$$f(x) = \begin{cases} i & \text{if } x \in \{u_i, v_i\} \text{ and } 1 \leq i \leq b-2 \\ b-1 & \text{if } x = u_{b-1} \\ b & \text{if } x = v_{b-1}. \end{cases}$$

Then f is a Grundy b-coloring and so $\Gamma(H) \geq b$. On the other hand, $\Gamma(H) \leq \Delta(H) + 1 = b$ and so $\Gamma(H) = b$ as claimed.

To show that $\psi(H) = b$, it is sufficient to show that $\psi(H) \leq b$ since $\psi(H) \geq \Gamma(H) = b$. Assume, to the contrary, that $\psi(H) = b' \geq b+1$ and consider a complete b'-coloring with the color classes $S_1, S_2, \ldots, S_{b'}$. If $|S_i| = 1$ for some i $(1 \leq i \leq b')$, then let $S_i = \{v\}$ and observe that v must be adjacent to at least one vertex in each color class S_j $(1 \leq j \leq b'$ and $j \neq i)$, implying that $\deg v \geq b' - 1 > b - 1 = \Delta(H)$, which is impossible. Hence $|S_i| \geq 2$ for every i. However, this implies that the order of H is $\sum_{i=1}^{b'} |S_i| \geq 2b' > 2b - 2$, which is also impossible. Therefore, $\chi(H) = 2$ and $\Gamma(H) = \psi(H) = b$.

Proceeding as in the proof of Proposition 2.6, we can establish the existence of an integer k such that identifying an end-vertex of P_k with the vertex u_1 of H results in a graph G with $\psi(G) = c$. Since G is bipartite, $\chi(G) = 2$. To see that $\Gamma(G) = b$, note first that $\Gamma(G) \leq b$ since $\Delta(G) = \Delta(H) = b-1$. Furthermore, since H is an induced subgraph of G, it follows that $b = \Gamma(H) \leq \Gamma(G)$. Therefore, $\Gamma(G) = b$ and so G has the desired complete coloring numbers. ∎

We next show for every graph G that each of the chromatic number, the Grundy number, and the achromatic number of $G + K_1$ exceeds the corresponding numbers of G by exactly 1.

Proposition 2.8 *For every graph G,*

$$\chi(G+K_1) = \chi(G)+1, \ \ \Gamma(G+K_1) = \Gamma(G)+1, \\ \text{and } \ \psi(G+K_1) = \psi(G)+1.$$

Proof. Let G be a graph with $\chi(G) = a$, $\Gamma(G) = b$, and $\psi(G) = c$ and construct $G + K_1$ from G by adding a new vertex v and joining v to each vertex of G. Then clearly $\chi(G + K_1) = a + 1$.

If $f : V(G) \to \{1, 2, \ldots, c\}$ is a complete c-coloring of G, then observe that the coloring $f_1 : V(G + K_1) \to \{1, 2, \ldots, c + 1\}$ given by

$$f_1(x) = \begin{cases} f(x) & \text{if } x \in V(G) \\ c+1 & \text{if } x = v \end{cases}$$

is a complete $(c+1)$-coloring of $G+K_1$ and so $\psi(G+K_1) \geq c+1$. On the other hand, $\psi(G+K_1) \leq \psi(G)+1$ by Theorem 1.1 and so $\psi(G+K_1) = c+1$.

Similarly, we can define a Grundy $(b + 1)$-coloring of $G + K_1$ from a Grundy b-coloring of G and so $\Gamma(G + K_1) \geq b + 1$. Now assume, to the contrary, that there exists a Grundy b'-coloring $g : V(G + K_1) \to \{1, 2, \ldots, b'\}$ with the color classes $S_1, S_2, \ldots, S_{b'}$, where $b' \geq b+2$. Assume that $S_i = \{x \in V(G + K_1) : g(x) = i\}$ for each i $(1 \leq i \leq b')$. Since v is adjacent to every vertex of G and g is a proper coloring, the color class to which v belongs must be a singleton set. Suppose that $g(v) = j$ and consider the coloring $g_1 : V(G) \to \{1, 2, \ldots, b' - 1\}$ defined by

$$g_1(x) = \begin{cases} g(x) & \text{if } x \in S_i \text{ and } 1 \leq i \leq j-1 \\ g(x)-1 & \text{if } x \in S_i \text{ and } j+1 \leq i \leq b'. \end{cases}$$

Then observe that g_1 must be a Grundy $(b' - 1)$-coloring of G and so $\Gamma(G) \geq b' - 1 > b$, which is a contradiction. Therefore, $\Gamma(G+K_1) = b+1$. ∎

We are now prepared to present a characterization of all realizable triples (a, b, c) of integers with $2 \leq a \leq b \leq c$.

Theorem 2.9 *Let a, b, and c be integers with $2 \leq a \leq b \leq c$. Then there exists a connected graph G with $\chi(G) = a$, $\Gamma(G) = b$, and $\psi(G) = c$ if and only if $a = b = c = 2$ or $b \geq 3$.*

Proof. First, let G be a connected graph such that $\chi(G) = a$, $\Gamma(G) = b$, and $\psi(G) = c$. Thus $b = 2$ or $b \geq 3$. If $b = 2$, then $a = b = c = 2$ by Proposition 2.5.

For the converse, let a, b, and c be integers with $2 \leq a \leq b \leq c$. If $a = b = c \geq 2$, then $G = K_a$ has the desired complete coloring numbers by Observation 2.1. If $a = b \geq 3$, then the result follows by Proposition 2.6.

Thus, we may assume that $2 \leq a < b \leq c$. By Proposition 2.7, there exists a connected graph G such that $\chi(G) = 2$, $\Gamma(G) = b - a + 2$, and

$\psi(G) = c - a + 2$. If $a = 2$, this gives the desired result; while if $a \geq 3$, then by Proposition 2.8, it follows that $G + K_{a-2}$ is a connected graph possessing the complete coloring numbers

$$\chi(G + K_{a-2}) = \chi(G) + (a - 2) = a$$
$$\Gamma(G + K_{a-2}) = \Gamma(G) + (a - 2) = b$$
$$\psi(G + K_{a-2}) = \psi(G) + (a - 2) = c,$$

completing the proof. ∎

The graph G with $\chi(G) = a$ and $\Gamma(G) = \psi(G) = b > a$ constructed in the proof of Theorem 2.9 has order $2(b - a + 1) + (a - 2) = 2b - a$. Hence among all graphs with chromatic number a and achromatic number b, there is one of order $2b - a$. In fact, this is the minimum order of such a graph. Let $\omega(G)$ denote the *clique number* of G, which is the maximum order among the complete subgraphs of G.

Theorem 2.10 *The order of every connected graph G is at least $2\psi(G) - \omega(G)$.*

Proof. Assume, to the contrary, that there is a connected graph G of order $n < 2b - a$ with $\omega(G) = a$ and $\psi(G) = b$. Let $S_1, S_2, \ldots, S_b$ be the color classes of a complete b-coloring of G and let $n = 2b - a - \ell$ for some positive integer ℓ. Since $n = 2b - a - \ell = 2[b - (a + \ell)] + (a + \ell)$, at least $a + \ell$ of the b color classes consist of a single vertex, say $S_i = \{x_i\}$ for $i \in \{1, 2, \ldots, a + \ell\}$. Since this coloring is complete, the $a + \ell$ vertices $x_1, x_2, \ldots, x_{a+\ell}$ must form a complete graph $K_{a+\ell}$ in G and so $\omega(G) \geq a + \ell > a$, which is a contradiction. ∎

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