Remarks on (a, b, k)-Critical Graphs

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Abstract

Let G be a graph of order n, and let a, b, k be integers with $0 \leq a \leq b$, $k \geq 0$. An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for each $x \in V(F)$. Then a graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. In this paper, it is proved that, if a, b, k be nonnegative integers with $2 \leq a < b$ and $b \equiv 0 \pmod{a - 1}$ and $n \geq \frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}$ and binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-b-bk}$, then G is an (a, b, k)-critical graph. Furthermore, it is showed that the result in this paper is best possible in some sense.

1 Introduction

All graphs considered in this paper will be finite and undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x \in V(G)$, the neighborhood $N_G(x)$ of x is the set vertices of G adjacent to x, and the degree $d_G(x)$ of x is $|N_G(x)|$. The minimum vertex degree of $V(G)$ is denoted by $\delta(G)$. For $S \subseteq V(G)$, $N_G(S) = \cup_{x \in S} N_G(x)$ and we denote by $G[S]$ the subgraph of G induced by S, by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A vertex set $S \subseteq V(G)$ is called independent if $G[S]$ has no edges. Let S and T be disjoint subsets of $V(G)$. We denote by $e_G(S, T)$ the number of edges joining S and T. The binding number $bind(G)$ of G is the minimum value of $\frac{|N_G(X)|}{|X|}$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \neq V(G)$. Let a and b be integers with $0 \leq a \leq b$.

An $[a, b]$-factor of graph G is defined as a spanning subgraph F of G such that $a \leq d_F(x) \leq b$ for every vertex x of G, where d_F denotes the degree in F. And if $a = b = k$, then an $[a, b]$-factor is called an k-factor. A graph G is called an (a, b, k)-critical graph if after deleting any k vertices of G the remaining graph of G has an $[a, b]$-factor. If G is an (a, b, k)-critical graph, then we also say that G is (a, b, k)-critical. If $a = b = n$, then an (a, b, k)-critical graph is simply called an (n, k)-critical graph. In particular, a $(1, k)$-critical graph is simply called a k-critical graph. The other terminologies and notations not given in this paper can be found in [1].

Favaron [4] studied the properties of k-critical graphs. Liu and Yu [8] studied the characterization of (n, k)-critical graphs. Enomoto et al [3] gave some sufficient conditions of (n, k)-critical graphs. The characterization of (a, b, k)-critical graph with $a < b$ was given by Liu and Wang [7]. Zhou [10-13] gave some sufficient conditions for graphs to be (a, b, k)-critical. Li [6] gave some sufficient conditions for graphs to be (a, b, k)-critical graphs.

Katerinis and Woodall [5] proved the following results for the existence of k-factors.

Theorem 1. Let $k \geq 2$ be an integer and let G be a graph with $p \geq 4k - 6$ vertices and binding number $b(G)$ such that kp is even and $bind(G) > \frac{(2k-1)(p-1)}{k(p-2)+3}$. Then G has a k-factor.

Theorem 2. If $p > k \geq 2$ and kp is even and $bind(G) > \frac{p-1}{2(\sqrt{kp}-k)}$, then G has a k-factor.

C. Chen [2] showed the following result for the existence of $[a, b]$-factors.

Theorem 3. Let G be a graph of order n, $1 \leq a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-2b+3}$ and $n \geq \frac{(a+b-1)(a+b-2)}{b}$, then G has an $[a, b]$-factor.

S. Zhou [11] gave the following result on (a, b, k)-critical graphs.

Theorem 4. Let G be a graph of order n, and let a, b and k be nonnegative integers such that $1 \leq a < b$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-(a+b)-bk+2}$ and $n \geq \frac{(a+b-1)(a+b-2)}{b} + \frac{bk}{b-1}$, then G is an (a, b, k)-critical graph.

In this paper, we prove the following result, which is an improvement of Theorem 4. We shall use a different technique from [11].

Theorem 5. Let G be a graph of order n, and let a, b and k be nonnegative integers such that $2 \leq a < b$ and $b \equiv 0 \pmod{a-1}$. If the binding number $bind(G) > \frac{(a+b-1)(n-1)}{bn-b-bk}$ and $n \geq \frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}$,

then G is an (a, b, k)-critical graph.

In Theorem 5, if $k = 0$, then we get the following corollary.

Corollary 1. Let G be a graph of order n, and let a, b integers such that $2 \leq a < b$ and $b \equiv 0 \pmod{a - 1}$. If the binding number $bind(G) > \frac{a+b-1}{b}$ and $n \geq \frac{(a+b-1)(2a+b-5)+b}{b}$, then G has an $[a, b]$-factor.

2   The Proof of Theorem 5

The proof of Theorem 5 relies heavily on the following lemmas.

Lemma 2.1.[7] Let a, b, k be nonnegative integers with $1 \leq a < b$, and let G be a graph of order n with $n \geq a + k + 1$. Then G is (a, b, k)-critical if and only if for any $S \subseteq V(G)$ and $|S| \geq k$

$$\sum_{j=0}^{a-1}(a - j)p_j(G - S) \leq b|S| - bk,$$

or

$$a|T| - d_{G-S}(T) \leq b|S| - bk,$$

where $p_j(G - S) = |\{x : d_{G-S}(x) = j\}|$, $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \leq a - 1\}$.

Lemma 2.2.[9] Let G be a graph of order n with $bind(G) > c$. Then $\delta(G) > n - \frac{n-1}{c}$.

Proof of Theorem 5. Suppose that G satisfies the assumption of the theorem, but it is not an (a, b, k)-critical graph. Then by Lemma 2.1, there exists a subset S of $V(G)$ with $|S| \geq k$ such that

$$\delta_G(S, T) = b|S| + d_{G-S}(T) - a|T| \leq bk - 1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, \ d_{G-S}(x) \leq a - 1\}$. We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

If $T = \varnothing$, then by (1), $bk - 1 \geq \delta_G(S, T) = b|S| \geq bk$, a contradiction. Hence, $T \neq \varnothing$. Let

$$h = \min\{d_{G-S}(x) : x \in T\},$$

and

$$m = |\{x : x \in T, d_{G-S}(x) = 0\}|.$$

In view of the definition of T and Lemma 2.2, we obtain

$$0 \leq h \leq a - 1, \quad and$$

$$\delta(G) > n - \frac{n-1}{c} \geq n - \frac{bn - b - bk}{a+b-1} = \frac{(a-1)n + b + bk}{a+b-1}. \qquad (2)$$

According to (2) and $\delta(G) \leq h + |S|$, we get

$$|S| > \frac{(a-1)n + b + bk}{a+b-1} - h. \qquad (3)$$

We shall consider various cases according to the value of h and derive contradictions.

Case 1. $h = 0$.

At first, we prove the following claim.

Claim 1. $\frac{bn - b - bk}{n-1} > 1$.

Proof Since $n \geq \frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}$, then we have

$$
\begin{aligned}
bn - b - bk - (n-1) &= (b-1)n - b - bk + 1 \\
&\geq (b-1)\left(\frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}\right) \\
&\quad -b - bk + 1 \\
&> (b-1)(2a+b-5) + (b-1) + bk - b - bk + 1 \\
&= (b-1)(2a+b-5) \geq 0
\end{aligned}
$$

Thus, we have

$$\frac{bn - b - bk}{n-1} > 1.$$

Let $m = |\{x : x \in T, d_{G-s}(x) = 0\}|$, and let $Y = V(G) \setminus S$. Then $N_G(Y) \neq V(G)$ since $h = 0$. In view of the definition of the binding number $bind(G)$, we get that
$$|N_G(Y)| \geq bind(G)|Y|.$$

Thus, we obtain

$$n - m \geq |N_G(Y)| \geq bind(G)|Y| = bind(G)(n - |S|),$$

that is,

$$|S| \geq n - \frac{n-m}{bind(G)}. \qquad (4)$$

Using $|S| + |T| \leq n$ and (1) and (4) and Claim 1, we have

$$
\begin{aligned}
bk - 1 &\geq \delta_G(S,T) = b|S| + d_{G-s}(T) - a|T| \\
&\geq b|S| - (a-1)|T| - m
\end{aligned}
$$

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$$
\begin{aligned}
&\geq\; b|S| - (a-1)(n-|S|) - m \\
&=\; (a+b-1)|S| - (a-1)n - m \\
&\geq\; (a+b-1)(n - \frac{n-m}{bind(G)}) - (a-1)n - m \\
&=\; bn - (a+b-1)\frac{n-m}{bind(G)} - m \\
&>\; bn - (a+b-1)\frac{(n-m)(bn-b-bk)}{(a+b-1)(n-1)} - m \\
&=\; bn - \frac{(n-m)(bn-b-bk)}{n-1} - m \\
&\geq\; bn - \frac{(n-1)(bn-b-bk)}{n-1} - 1 \\
&=\; bk + b - 1 \\
&>\; bk,
\end{aligned}
$$

a contradiction.

Case 2. $h = 1$.

Subcase 2.1. $|T| \leq \frac{b(n-k-1)}{a+b-1}$.

In view of (1) and (3), we get that

$$
\begin{aligned}
bk - 1 \;&\geq\; \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\
&\geq\; b|S| - (a-1)|T| \\
&>\; b(\frac{(a-1)n+b+bk}{a+b-1} - 1) - (a-1)\frac{b(n-k-1)}{a+b-1} \\
&=\; b\frac{(a-1)n+bk-(a-1)}{a+b-1} - b\frac{(a-1)n-(a-1)k-(a-1)}{a+b-1} \\
&=\; b\frac{(a+b-1)k}{a+b-1} = bk,
\end{aligned}
$$

which is a contradiction.

Subcase 2.2. $|T| \geq \frac{bn-bk+a-1}{a+b-1}$.

Let $u \in T$ such that $d_{G-S}(u) = h = 1$, and put $Y = T - N_{G-S}(u)$. Then $|N_G(Y)| \leq n - 1$ and

$$
|Y| \geq |T| - 1 \geq \frac{bn - bk + a - 1}{a+b-1} - 1 = \frac{bn - bk - b}{a+b-1}.
$$

Thus, we obtain

$$
bind(G) \leq \frac{|N_G(Y)|}{|Y|} \leq \frac{(a+b-1)(n-1)}{bn - bk - b},
$$

which contradicts $bind(G) > \frac{(a+b-1)(n-1)}{bn-bk-b}$.

Subcase 2.3. $\frac{b(n-k-1)}{a+b-1} < |T| < \frac{bn-bk+a-1}{a+b-1}$.

By (3) and $h = 1$, we have that

$$|S| + |T| > \frac{(a-1)n+b+bk}{a+b-1} - 1 + \frac{b(n-k-1)}{a+b-1} = n - 1.$$

According to the integrity of $|S| + |T|$, we obtain

$$|S| + |T| = n. \tag{5}$$

In view of $\frac{b(n-k-1)}{a+b-1} < |T| < \frac{bn-bk+a-1}{a+b-1}$ and (5), we have

$$
\begin{aligned}
\delta_G(S,T) - bk &= b|S| + d_{G-S}(T) - a|T| - bk \\
&= b|S| - (a-1)|T| - bk \\
&\geq b(n - |T|) - (a-1)|T| - bk \\
&= bn - (a+b-1)|T| - bk \\
&> bn - (a+b-1)\frac{bn-bk+a-1}{a+b-1} - bk \\
&= -(a-1),
\end{aligned}
$$

that is,

$$\delta_G(S,T) - bk = b|S| - (a-1)|T| - bk > -(a-1). \tag{6}$$

Since $b \equiv 0 \ (mod\ a-1)$, then by (6) we get that

$$\delta_G(S,T) - bk \geq 0. \tag{7}$$

In view of (7), we obtain
$$\delta_G(S,T) \geq bk,$$

that contradicts (1).

Case 3. $2 \leq h \leq a - 1$.

Subcase 3.1. $|T| \leq h$.

According to (3), we have

$$
\begin{aligned}
\delta_G(S,T) &= b|S| + d_{G-S}(T) - a|T| \\
&\geq b|S| + h|T| - a|T| \\
&= b|S| - (a-h)|T| \\
&> b(\frac{(a-1)n+bk+b}{a+b-1} - h) - (a-h)h,
\end{aligned}
$$

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that is,

$$\delta_G(S,T) > b(\frac{(a-1)n+bk+b}{a+b-1} - h) - (a-h)h. \tag{8}$$

Let $g(h) = b(\frac{(a-1)n+bk+b}{a+b-1} - h) - (a-h)h$. In fact, the function $g(h)$ attains its minimum value at $h = a - 1$ since $2 \le h \le a - 1$. Then, we get

$$g(h) \ge g(a-1).$$

Combining this with (8) and $n \ge \frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}$, we get that

$$
\begin{aligned}
\delta_G(S,T) \quad &> \quad g(h) \ge g(a-1) \\
&= \quad b(\frac{(a-1)n+bk+b}{a+b-1} - (a-1)) - (a-1) \\
&= \quad \frac{(a-1)bn+b^2k+b^2}{a+b-1} - (a-1)(b+1) \\
&\ge \quad \frac{(a-1)((a+b-1)(2a+b-5)+b+bk)+b^2k+b^2}{a+b-1} \\
&\qquad -(a-1)(b+1) \\
&= \quad \frac{(a-1)(a+b-1)(2a+b-5)+(a-1)b+(a-1)bk}{a+b-1} \\
&\qquad +\frac{b^2k+b^2}{a+b-1} - (a-1)(b+1) \\
&= \quad (a-1)(2a+b-5)+bk+b-(a-1)(b+1) \\
&= \quad (a-1)(2a-6)+b+bk \\
&> \quad bk \quad \text{(since } b > a \ge 2 \text{ are integers)},
\end{aligned}
$$

which contradicts (1) again.

Subcase 3.2. $|T| > h$.

Let v be a vertex in T such that $d_{G-S}(v) = h$, and put $Y = T - N_{G-S}(v)$. Then $|Y| \ge |T| - h \ge 1$ and $N_G(Y) \ne V(G)$. Thus, we obtain

$$\frac{n-1}{|T|-h} \ge \frac{|N_G(Y)|}{|Y|} \ge bind(G) > \frac{(a+b-1)(n-1)}{bn-b-bk},$$

that is,

$$|T| < \frac{bn-b-bk}{a+b-1} + h. \tag{9}$$

By (1) and (3) and (9), we have

$$
\begin{aligned}
bk - 1 \quad &\ge \quad \delta_G(S,T) = b|S| + d_{G-S}(T) - a|T| \\
&\ge \quad b|S| + h|T| - a|T|
\end{aligned}
$$

$$\begin{aligned}
&= b|S| - (a-h)|T| \\
&> b\left(\frac{(a-1)n+b+bk}{a+b-1} - h\right) - (a-h)\left(\frac{bn-b-bk}{a+b-1} + h\right).
\end{aligned}$$

Let $f(h) = b\left(\frac{(a-1)n+b+bk}{a+b-1} - h\right) - (a-h)\left(\frac{bn-b-bk}{a+b-1} + h\right)$. Then, we get that

$$bk - 1 \geq \delta_G(S,T) > f(h). \tag{10}$$

In fact, the function $f(h)$ attains its minimum value at $h = 2$ since $2 \leq h \leq a - 1$. Then, we obtain

$$f(h) \geq f(2).$$

Combining this with (10) and $n \geq \frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}$, we have

$$\begin{aligned}
bk - 1 \quad &> \quad f(h) \geq f(2) \\
&= \quad b\left(\frac{(a-1)n+bk+b}{a+b-1} - 2\right) - (a-2)\left(\frac{bn-bk-b}{a+b-1} + 2\right) \\
&= \quad \frac{bn + b^2 + b^2 k + (a-2)b + (a-2)bk}{a+b-1} - 2(a+b-2) \\
&\geq \quad \frac{b\left(\frac{(a+b-1)(2a+b-5)+b}{b} + \frac{bk}{b-1}\right) + (a+b-2)b + (a+b-2)bk}{a+b-1} \\
&\quad -2(a+b-2) \\
&\geq \quad \frac{(a+b-1)(2a+b-5) + bk + (a+b-1)b + (a+b-2)bk}{a+b-1} \\
&\quad -2(a+b-2) \\
&= \quad \frac{(a+b-1)(2a+b-5) + b(a+b-1) + bk(a+b-1)}{a+b-1} \\
&\quad -2(a+b-2) \\
&= \quad bk + 2a + 2b - 5 - 2(a+b-2) \\
&= \quad bk - 1,
\end{aligned}$$

that is a contradiction.

From the argument above, we deduce the contradictions. Hence, G is an (a,b,k)-critical graph.

Remark. Let us show that the condition $bind(G) > \frac{(a+b-1)(n-1)}{bn-b-bk}$ in Theorem 5 can not be replaced by $bind(G) \geq \frac{(a+b-1)(n-1)}{bn-b-bk}$. Let $b > a \geq 2, k \geq 0$ be three integers such that $b + k$ is even, and let $n = \frac{(a+b-1)(2a+b-5)+b+(a+2b-1)k}{b}$ is an integer, and let $l = \frac{2a+b-4+k}{2}$ and $m = n - 2l = n - (2a+b-4+k) = \frac{(a-1)(2a+b-5)+(a+b-1)k}{b}$. Clearly, m is an integer. Let $H = K_m \bigvee lK_2$. Let $X = V(lK_2)$, for any $x \in X$,

then $|N_H(X \setminus x)| = n - 1$. By the definition of $bind(H)$, $bind(H) = \frac{|N_H(X \setminus x)|}{|X \setminus x|} = \frac{n-1}{2l-1} = \frac{n-1}{2a+b-5+k} = \frac{(a+b-1)(n-1)}{bn-b-bk}$. Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$, then $|S| = m \geq k$, $|T| = 2l$. Thus, we obtain

$$
\begin{aligned}
\delta_H(S, T) &= b|S| - a|T| + d_{H-S}(T) \\
&= b|S| - a|T| + |T| = b|S| - (a-1)|T| \\
&= b\frac{(a-1)(2a+b-5) + (a+b-1)k}{b} \\
&\quad -(a-1)(2a+b-4+k) \\
&= bk - (a-1) < bk.
\end{aligned}
$$

By Lemma 2.1, H is not an (a, b, k)-critical graph. In the above sense, the result in Theorem 5 is best possible.

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