Sarvate–Beam Quad Systems for $v = 6$

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Abstract

A *Sarvate–Beam Quad System* $SB(v, 4)$ is a set V of v elements and a collection of 4-subsets of V such that each distinct pair of elements in V occurs i times for every i in the list 1, 2, ..., $\binom{v}{2}$. In this paper, we completely enumerate all Sarvate–Beam Quad Systems for $v = 6$.

1 Sarvate–Beam Designs

The present problem under consideration has its roots in papers published recently by D. Sarvate and W. Beam, R. Stanton and others. In these papers, Sarvate and Beam introduced a new type of combinatorial object called an *adesign*.

Definition 1.
An adesign $AD(v, k)$ *is a set* V *of* v *elements and a collection of* k*-subsets of* V *(called* blocks*) such that each distinct pair of elements in* V *occurs in a different number of blocks. A* strict *adesign* $SAD(v, k)$ *is an adesign such that exactly one pair of elements occurs* i *times for every* i *in the list* 1, 2, ..., $\binom{v}{2}$.

Definition 1 was given by Sarvate and Beam [2], although the term "frequency" was used by Dukes [1] to refer to the number of blocks containing each distinct pair of points from V. We note that this *distinct frequency* condition distinguishes an adesign from a balanced incomplete block design (BIBD). The following definition was also given by Sarvate and Beam [2]:

Definition 2.
An aPBD(v, K) *is a set V of v elements and a collection of subsets of V such that every pair of distinct elements of V occurs a distinct number of times, and the size of any block is in K.*

We think it is natural to introduce the following definition, although it does not appear in the literature:

Definition 3.
A strict *aPBD $SaPBD(v, K)$ is a set V of v elements and a collection of subsets of V such that every pair of distinct elements of V occurs exactly once from the list* 1, 2, ..., $\binom{v}{2}$, *and the size of any block is in K.*

The definition of "strict adesign" was renamed by Stanton [4] as a *Sarvate-Beam design* (or *SB design*). Stanton [4] also introduced the term "SB Triple System", referring to a $SAD(v, 3)$. He generalized his terminology in [6] to a "SB Quad System", which is a $SAD(v, 4)$. Dukes [1] improved the notation, and labels it $SB(v, k)$. Thus, we have $SB(v, 3)$ for SB Triple Systems and $SB(v, 4)$ for SB Quad Systems. The notation can further be used to denote a $SaPBD(v, K)$ by $SB(v, K)$.

2 Preliminaries

Sarvate and Beam [2] proved the following:

Theorem 4.
For a $SB(v, 3)$, it must be true that $v \equiv 0, 1 \pmod 3$.

The following mimics the proof of the former:

Theorem 5.
For a $SB(v, \{3, 4\})$, it must be true that $v \equiv 0, 1 \pmod 3$.

Proof. The number of pairs in a block of cardinality 3 is $\binom{3}{2} = 3$; the number of pairs in a block of cardinality 4 is $\binom{4}{2} = 6$. Let a be the number of blocks of cardinality 3, and let b be the number of blocks of cardinality 4. Hence, there are a total of $3a + 6b = 3(a + 2b)$ pairs.

For a $SB(v, K)$, the required number of pairs to be covered is $1 + 2 + \ldots + \binom{v}{2} = v(v-1)(v^2 - v + 2)/8$. Therefore, for a $SB(v, \{3, 4\})$ to exist, we must have $3(a + 2b) = v(v-1)(v^2 - v + 2)/8$. Hence, 3 must divide $v(v-1)(v^2 - v + 2)$ since $\gcd(3, 8) = 1$.

We see that either $3|v(v-1)$ or $3|v^2 - v + 2$. One can show that 3 never divides $v^2 - v + 2$ by checking all three cases of $v \equiv 0, 1, 2 \pmod 3$. Thus, $3|v(v-1)$. That is, $3|v$ or $3|v-1$. Hence, $v \equiv 0, 1 \pmod 3$ for a $SB(v, \{3, 4\})$ to exist.

■

Using Stanton's [4] notation, some examples of SB designs are provided:

Example 6.
$1234 + 2(135) + 3(145) + 234 + 4(235) + 6(245) + 5(345)$ *is a (non-strict) aPBD*$(5, \{3, 4\})$.

Example 7.
$1234 + 134 + 145 + 2(146) + 2(156) + 3(235) + 2(236) + 3(245) + 3(246) + 3(256) + 4(345) + 4(346) + 5(356) + 5(456)$ *is a* $SB(6, \{3, 4\})$.

Stanton [4] goes on to make a new definition:

Definition 8.
An SB design is called restricted *if only blocks beginning with 1 or 2 (up to isomorphism) are allowable.*

Stanton [4] gave examples of restricted $SB(6, 3)$ and $SB(7, 3)$. Also, Stanton [5] gave an example of a restricted $SB(8, \{2, 3\})$.

3 SB Quad Systems

Stanton [6] analyzes SB Quad Systems. He gives the following example:

Example 9.
$1234 + 2(1345) + 2(1356) + 5(2345) + 6(2346) + 3(2356) + 2456$ *is a restricted* $SB(6, 4)$.

The following example is given by Sarvate and Beam [3]:

Example 10.
$1236 + 1356 + 3(1456) + 2(2346) + 3(2356) + 5(2456) + 8(3456)$ *is a (non-strict)* $AD(6, 4)$.

Stanton [7] outlined a general procedure for finding bounds on possible values of v for restricted SB Triple Systems. He generalized the procedure to SB Quad Systems in [6], and proved the following:

Theorem 11.
The only restricted $SB(v, 4)$ *that possibly exist are for* $v \in \{6, 9\}$.

4 New Results

In his analysis of restricted SB Quad Systems, Stanton [6] assigns the frequency 1 to the block 1234, a_1 to the block 1345, a_2 to the block 1346, a_3 to the block 1356, a_4 to the block 1456, b_1 to the block 2345, b_2 to the block 2346, b_3 to the block 2356, b_4 to the block 2456 and 0 to all other blocks.

Stanton then labels the frequency of the pair $\{a, b\}$ by $f(ab)$, noting that $f(12) = 1$, $f(13) = 1 + a_1 + a_2 + a_3$, $f(14) = 1 + a_1 + a_2 + a_4$, $f(15) = a_1 + a_3 + a_4$, $f(16) = a_2 + a_3 + a_4$, $f(23) = 1 + b_1 + b_2 + b_3$, $f(24) = 1 + b_1 + b_2 + b_4$, $f(25) = b_1 + b_3 + b_4$, $f(26) = b_2 + b_3 + b_4$, $f(34) = 1 + a_1 + a_2 + b_1 + b_2$, $f(35) = a_1 + a_3 + b_1 + b_3$, $f(36) = a_2 + a_3 + b_2 + b_3$, $f(45) = a_1 + a_4 + b_1 + b_4$, $f(46) = a_2 + a_4 + b_2 + b_4$ and $f(56) = a_3 + a_4 + b_3 + b_4$. We consider the existence of a restricted $SB(6,4)$ to be equivalent to finding an ordered 9-tuple $(1, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4)$ satisfying the above conditions and such that $1 \le f(ab) \le 15 = \binom{6}{2}$ with each $f(ab)$ distinct.

Stanton also introduces parameters $A = \sum a_i$ and $B = \sum b_i$ and shows that $A + B = 19$. He further shows that $A \geq 4$. Assuming that $A \leq B$, this leaves 6 cases of possibilities for restricted $SB(6,4)$: $(A, B) \in \{(4,15), (5,14), (6,13), (7,12), (8,11), (9,10)\}$.

Each of these cases requires checking large numbers of possible 9-tuples that can produce restricted $SB(6,4)$. It is a simple matter to write a computer code to check all of the conditions for each possible 9-tuple. When we did this, we got the following results:

(A, B)	(4, 15)	(5, 14)	(6, 13)	(7, 12)	(8, 11)	(9, 10)
solutions	16	0	0	0	0	0
nonisomorphic	4	0	0	0	0	0

These solutions correspond to the 9-tuples $(1,0,1,0,3,4,0,8,3)$, $(1,0,1,0,3,7,0,6,2)$, $(1,0,2,0,2,6,5,1,3)$ and $(1,1,2,0,1,0,2,8,5)$. Thus we have shown the following:

Theorem 12.
There are 4 non-isomorphic restricted $SB(6,4)$.

To extend his analysis to unrestricted SB Quad Systems, Stanton additionally assigns the frequency d_1 to the only other possible block 3456 (keeping the other assignments in the restricted case the same). Again denoting the frequency of the pair $\{a, b\}$ by $f(ab)$, he notes that $f(34) = 1 + a_1 + a_2 + b_1 + b_2 + d_1$, $f(35) = a_1 + a_3 + b_1 + b_3 + d_1$, $f(36) = a_2 + a_3 + b_2 + b_3 + d_1$, $f(45) = a_1 + a_4 + b_1 + b_4 + d_1$, $f(46) = a_2 + a_4 + b_2 + b_4 + d_1$ and $f(56) = a_3 + a_4 + b_3 + b_4 + d_1$ (with the other frequencies in the restricted case the same). We consider the existence of an unrestricted $SB(6,4)$ to be equivalent to finding an ordered 10-tuple $(1, a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, d_1)$ satisfying the above conditions and such that $1 \le f(ab) \le 15$ with each $f(ab)$ distinct.

Stanton also introduced the parameter $D = \sum d_i = d_1$ and noted that $A + B + D = 19$. Retaining the restriction $A \ge 4$ and the assumption that $A \le B$, this again leaves 6 cases of possibilities for unrestricted $SB(6,4)$: $(A, B + D) \in \{(4,15),(5,14),(6,13),(7,12),(8,11),(9,10)\}$.

Each of these cases requires checking large numbers of possible 10-tuples that can produce unrestricted $SB(6,4)$. We generalized the computer code used in the unrestricted case to check all of the conditions for each possible 10-tuple. When we did this, we got the following results:

$(A, B+D)$	(4, 15)	(5, 14)	(6, 13)	(7, 12)	(8, 11)	(9, 10)
solutions	24	4	0	0	0	0
nonisomorphic	6	1	0	0	0	0

These solutions correspond to the 10-tuples $(1,0,1,0,3,4,2,1,6,2)$, $(1,0,2,0,2,1,6,4,1,3)$, $(1,0,2,0,2,5,1,8,0,1)$, $(1,0,2,0,2,7,4,1,2,1)$, $(1,1,2,0,1,0,6,4,2,3)$, $(1,1,2,0,1,6,1,6,0,2)$ and $(1,0,1,0,4,8,0,3,0,3)$. Thus we have shown the following:

Theorem 13.
There are 7 non–isomorphic unrestricted $SB(6,4)$.

We note that there are no restricted $SB(6,4)$ when $A = 5$, yet there exists an unrestricted $SB(6,4)$ when $A = 5$. This was an unexpected yet pleasant discovery!

Theorems 12 and 13 can be combined into one summary result (the main result of the paper):

Theorem 14.
There are 11 non–isomorphic $SB(6,4)$.

The computer code used to produce this result can be found at the website `http://www.suu.edu/faculty/hein/professional.html` (We also furnish the results of the code on the website.) It is written in C++, and produces 5 files — 4 are auxiliary files used by the program, and a file called `u64solns.dat` with all 44 solutions. (These solutions were checked by hand to produce the non–isomorphic solutions). The total runtime for the code on our local server is 0.366 second.

5 Open Problems

Stanton [4, 5, 6, 7] states several open questions:

- Find the number of non–isomorphic $SB(6,3)$.
- Find the number of non–isomorphic restricted $SB(6,3)$.
- Find the number of non–isomorphic $SB(7,3)$.
- Find the number of non–isomorphic restricted $SB(7,3)$.
- Find the number of non–isomorphic restricted $SaPBD(8,\{2,3\})$.
- Find the number of non–isomorphic $SB(9,4)$.
- Find the number of non–isomorphic restricted $SB(9,4)$.
- Find the number of non–isomorphic $SB(v,4)$ for $v > 9$.

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