On Hamiltonian Labelings of Graphs

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ABSTRACT

For a connected graph G of order n, the detour distance $D(u, v)$ between two vertices u and v in G is the length of a longest $u - v$ path in G. A Hamiltonian labeling of G is a function $c : V(G) \to \mathbb{N}$ such that

$$|c(u) - c(v)| + D(u, v) \geq n$$

for every two distinct vertices u and v of G. The value $\text{hn}(c)$ of a Hamiltonian labeling c of G is the maximum label (functional value) assigned to a vertex of G by c; while the Hamiltonian labeling number $\text{hn}(G)$ of G is the minimum value of a Hamiltonian labeling of G. We present several sharp upper and lower bounds for the Hamiltonian labeling number of a connected graph in terms of its order and other distance parameters.

Key Words: Hamiltonian labeling, detour distance.

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1 Introduction

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest path between these two vertices. The *eccentricity* $e(v)$ of a vertex v in G is the maximum distance from v to a vertex of G. The *radius* $\text{rad}(G)$ of G is the minimum eccentricity among the vertices of G, while the *diameter* $\text{diam}(G)$ of G is the maximum eccentricity among the vertices of G. A vertex v with $e(v) = \text{rad}(G)$ is called a *central vertex* of G. If $d(u, v) = \text{diam}(G)$, then u and v are *antipodal vertices* of G.

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For a connected graph G with diameter d, an *antipodal coloring* of a connected graph G is defined in [2] as an assignment $c : V(G) \to \mathbb{N}$ of colors to the vertices of G such that

$$|c(u) - c(v)| + d(u, v) \geq d$$

for every two distinct vertices u and v of G. In the case of paths of order $n \geq 2$, this gives

$$|c(u) - c(v)| + d(u, v) \geq n - 1.$$

Antipodal colorings of paths gave rise to the more general Hamiltonian colorings of graphs defined in terms of another distance parameter.

The *detour distance* $D(u, v)$ between two vertices u and v in a connected graph G is the length of a longest path between these two vertices. Thus if G is a connected graph of order n, then $d(u, v) \leq D(u, v) \leq n - 1$ for every two vertices u and v in G and $D(u, v) = n - 1$ if and only if G contains a Hamiltonian $u - v$ path. Furthermore $d(u, v) = D(u, v)$ for every two vertices u and v in G if and only if G is a tree. As with standard distance, the detour distance is a metric on the vertex set of a connected graph.

A *Hamiltonian coloring* of a connected graph G of order n is a coloring $c : V(G) \to \mathbb{N}$ of G such that

$$|c(u) - c(v)| + D(u, v) \geq n - 1$$

for every two distinct vertices u and v of G. Consequently, if u and v are distinct vertices such that $|c(u) - c(v)| = k$ for some Hamiltonian coloring c of G, then there is a $u - v$ path in G missing at most k vertices of G. The *value* $\text{hc}(c)$ of a Hamiltonian coloring c of G is the maximum color assigned to a vertex of G. The *Hamiltonian chromatic number* of G is the minimum value of a Hamiltonian coloring of G. Hamiltonian colorings of graphs have been studied in [3, 4, 5, 7, 8] for example.

For a connected graph G with diameter d, a *radio labeling* of G is defined in [1] as an assignment $c : V(G) \to \mathbb{N}$ of labels to the vertices of G such that

$$|c(u) - c(v)| + d(u, v) \geq d + 1$$

for every two distinct vertices u and v of G. Thus for a radio labeling of a graph, colors assigned to adjacent vertices of G must differ by at least d, colors assigned to two vertices at distance 2 must differ by at least $d - 1$, and so on, up to two vertices at distance d (that is, antipodal vertices), whose colors are only required to differ. The *value* $\text{rn}(c)$ of a radio labeling c of G is the maximum color assigned to a vertex of G. The *radio number* of G is the minimum value of a radio labeling of G. In the case of paths of order $n \geq 2$, this gives

$$|c(u) - c(v)| + d(u, v) \geq n.$$

In a similar manner, radio labelings of paths and detour distance in graphs give rise to a related labeling.

A *Hamiltonian labeling* of a connected graph G of order n is defined in [9] as an assignment $c : V(G) \to \mathbb{N}$ of labels to the vertices of G such that

$$|c(u) - c(v)| + D(u,v) \geq n$$

for every two distinct vertices u and v of G. Therefore, in a Hamiltonian labeling of G, every two vertices are assigned distinct labels and two vertices u and v can be assigned consecutive labels in G only if G contains a Hamiltonian $u - v$ path. We can assume that every Hamiltonian labeling of a graph uses the integer 1 as one of its labels. The *value* $\text{hn}(c)$ of a Hamiltonian labeling c of G is the maximum label assigned to a vertex of G by c, that is,

$$\text{hn}(c) = \max\{c(v) : \ v \in V(G)\}.$$

The *Hamiltonian labeling number* $\text{hn}(G)$ of G is defined in [9] as the minimum value of a Hamiltonian labeling of G, that is,

$$\text{hn}(G) = \min\{\text{hn}(c)\},$$

where the minimum is taken over all Hamiltonian labelings c of G. A Hamiltonian labeling c of G with value $\text{hn}(c) = \text{hn}(G)$ is called a *minimum Hamiltonian labeling* of G. Therefore, $\text{hn}(G) \geq n$ for every connected graph G of order n. Among the results obtained in [9] are the following.

Theorem 1.1 [9] Every connected graph of order $n \geq 3$ with Hamiltonian labeling number n is 2-connected.

Theorem 1.2 [9] If G is a Hamiltonian graph of order $n \geq 3$, then $\text{hn}(G) = n$.

It was observed in [9] that the converse of Proposition 1.2 is not true. For example, the Petersen graph P is a non-Hamiltonian graph of order 10 but $\text{hn}(P) = 10$. By Theorem 1.2, if $G = K_n$ or $G = C_n$, where $n \geq 3$, then $\text{hn}(G) = n$. Hamiltonian labeling numbers of complete bipartite graphs were determined.

Theorem 1.3 [9] For integers r and s with $1 \leq r \leq s$,

$$\text{hn}(K_{r,s}) = \begin{cases} r+s & \text{if } r = s \\ (s-1)^2 + s + 1 & \text{if } r = 1 \text{ and } s \geq 2 \\ (s-1)^2 - (r-1)^2 + r + s - 1 & \text{if } 2 \leq r < s. \end{cases}$$

Bounds for the Hamiltonian labeling number of a connected graph were established in terms of its order and Hamiltonian chromatic number.

Theorem 1.4 [9] For every connected graph G of order $n \geq 3$,

$$\text{hc}(G) + 2 \leq \text{hn}(G) \leq \text{hc}(G) + (n-1).$$

Furthermore, for each pair k, n of integers with $2 \leq k \leq n-1$, there exists a Hamiltonian graph G of order n such that $\text{hn}(G) = \text{hc}(G) + k$.

In this work, we establish several sharp upper and lower bound for the Hamiltonian labeling number of a connected graph in terms of its order, diameter, and other distance parameters. We refer to the book [6] for graph theory notation and terminology not described in this paper.

2 Upper Bounds

In this section, we investigate how large the Hamiltonian labeling number of a connected graph of order n can be. In [3, 4] two upper bounds for the Hamiltonian chromatic number of a connected graph were established in terms of its order.

Theorem 2.1 [4] If G is a nontrivial connected graph of order n, then

$$\text{hc}(G) \leq 1 + (n-2)^2.$$

Furthermore $\text{hc}(G) = 1 + (n-2)^2$ if and only if G is a star.

Theorem 2.2 [4] Let G be a connected graph of order $n \geq 4$. If $v_1, v_2, \ldots, v_n$ is any ordering of the vertices of G, then

$$\text{hc}(G) \leq (n-1)^2 + 1 - \sum_{i=1}^{n-1} \min\{D(v_{i+1}, v_i), n/2\}.$$

Theorems 1.4, 2.1, and 2.2 provide the correspondent upper bounds for the Hamiltonian labeling number of a connected graph in terms of its order.

Corollary 2.3 *If G is a nontrivial connected graph of order n, then*

$$\text{hn}(G) \leq n + (n-2)^2.$$

Furthermore, $\text{hn}(G) = n + (n-2)^2$ *if and only if G is a star.*

Corollary 2.4 *Let G be a connected graph of order $n \geq 4$. If $v_1, v_2, \ldots, v_n$ is any ordering of the vertices of G, then*

$$\text{hn}(G) \leq n(n-1) + 1 - \sum_{i=1}^{n-1} \min\{D(v_{i+1}, v_i), n/2\}.$$

The upper bound in Corollary 2.4 is sharp. For example, let $G = K_{1,n-1}$ for some integer $n \geq 4$, and consider the ordering $v_1, v_2, \cdots, v_n$ of the vertices of G, where v_1 is the central vertex of G. Thus $d(v_2, v_1) = 1$ and $d(v_{i+1}, v_i) = 2$ for all $2 \leq i \leq n-1$. Since $n \geq 4$, it follows that $\min\{D(v_{i+1}, v_i), n/2\} = D(v_{i+1}, v_i)$ for $1 \leq i \leq n-1$ and so

$$\sum_{i=1}^{n-1} \min\{D(v_{i+1}, v_i), n/2\} = \sum_{i=1}^{n-1} D(v_{i+1}, v_i) = 1 + 2(n-2) = 2n-3.$$

Thus

$$\begin{aligned} \text{hn}(K_{1,n-1}) &= n(n-1) + 1 - \sum_{i=1}^{n-1} D(v_{i+1}, v_i) \\ &= n(n-1) + 1 - (2n-3) = n + (n-2)^2. \end{aligned}$$

In order to establish an improved upper bound for the Hamiltonian labeling number of a connected graph, we first present some preliminary results. Let G be a connected graph containing an edge e that is not a bridge. Then $G - e$ is connected. For every two distinct vertices u and v in $G - e$, the length of a longest $u - v$ path in $G - e$ does not exceed the length of a longest $u - v$ path in G. Thus every Hamiltonian labeling of $G - e$ is a Hamiltonian labeling of G. This observation yields the following useful lemma.

Lemma 2.5 [9] If e is an edge of a connected graph G that is not a bridge, then

$$\text{hn}(G) \leq \text{hn}(G - e).$$

The following is an immediate consequence of Lemma 2.5.

Corollary 2.6 *If T is a spanning tree of a connected graph G, then*

$$\text{hn}(G) \leq \text{hn}(T).$$

By Corollary 2.6, it will be useful to know how large the Hamiltonian labeling number of a tree of order n can be. It is convenient to introduce some notation. For a Hamiltonian labeling c of a graph G, an ordering $u_1, u_2, \ldots, u_n$ of the vertices of G is called the *c-ordering* of G if

$$1 = c(u_1) < c(u_2) < \cdots < c(u_n) = \text{hn}(c).$$

Next, we present an upper bound for the Hamiltonian labeling number of a nontrivial tree in terms of its order and diameter.

Theorem 2.7 *Let T be a tree of order $n \geq 3$ with* $\operatorname{diam}(T) = d$ *and*

$$k = \min\left(\left\lceil \frac{d-1}{2} \right\rceil, \left\lfloor \frac{n-d+1}{2} \right\rfloor\right).$$

Then

$$\operatorname{hn}(T) \leq \begin{cases} n^2 - 3n + 4 + \frac{3}{2}d - \frac{d^2}{4} + k^2 - kd - k & \text{if } d \text{ is even} \\ n^2 - 3n + 3\frac{1}{4} + d - \frac{d^2}{4} + k^2 - kd & \text{if } d \text{ is odd.} \end{cases}$$

Proof. Let $R: v_0, v_1, \cdots, v_d$ be a path of length d in T and let $h = \lfloor \frac{d+1}{2} \rfloor$. If d is even, then v_h is the only central vertex of T on R and let $v_c = v_h$, while if d is odd, then v_{h-1} and v_h are two central vertices of T on R and let $v_c = v_{h-1}$. Let T_1 and T_2 be two components of $T - v_c v_{c+1}$ such that $v_{c+1} \in V(T_1)$ and $v_c \in V(T_2)$, where then $v_d \in V(T_1)$ and $v_0 \in V(T_2)$. For each i with $1 \leq i \leq h$, let

$$\begin{aligned} V_{1,i} &= \{u \in V(T_1) - V(R) \mid D(v_c, u) = i\} \\ V_{2,i} &= \{u \in V(T_2) - V(R) \mid D(v_c, u) = i\}. \end{aligned}$$

Note that $V_{1,1} = \emptyset$. Suppose that $|V_{1,i}| = n_{1,i} \geq 0$ and $|V_{2,i}| = n_{2,i} \geq 0$. For each i with $1 \leq i \leq h$, let

$$\begin{aligned} V_{1,i} &= \{u_{i,1}, u_{i,2}, \ldots, u_{i,n_{1,i}}\} \\ V_{2,i} &= \{u_{i,n_{1,i}+1}, u_{i,n_{1,i}+2}, \ldots, u_{i,n_{1,i}+n_{2,i}}\}. \end{aligned}$$

That is, we label the vertices in $V_{1,i}$ by $u_{i,s}$, where $1 \leq s \leq n_{1,i}$ and label the vertices in $V_{2,i}$ by $u_{i,t}$, where $n_{1,i} + 1 \leq t \leq n_{1,i} + n_{2,i}$. For example, consider the trees T and T' of order $n = 13$ in Figure 1, where the vertices of R are solid and the edges of the subtree T_1 in each of T and T' are drawn in bold. Since the diameter of T is 6, the only central vertex of T is $v_c = v_3$. In the tree T, $V_{1,1} = \emptyset$, $V_{2,1} = \{u_{1,1}\}$, $V_{1,2} = \{u_{2,1}, u_{2,2}\}$, $V_{2,2} = \{u_{2,3}\}$, $V_{1,3} = \{u_{3,1}\}$ and $V_{2,3} = \{u_{3,2}\}$. Since the diameter of T' is 5, the central vertices of T' are v_2 and v_3. In this case, let $v_c = v_2$. Then $V_{1,1} = \emptyset$, $V_{2,1} = \{u_{1,1}\}$, $V_{1,2} = \{u_{2,1}, u_{2,2}\}$, $V_{2,2} = \{u_{2,3}, u_{2,4}, u_{2,5}\}$, $V_{1,3} = \{u_{3,1}\}$ and $V_{2,3} = \emptyset$.

For each i with $1 \leq i \leq h$, let

$$\ell_i = \min(2i, 2k) \text{ for } 1 \leq i \leq h.$$

We now define a labeling c as follows:

$$c(v_{c+i}) = \begin{cases} 1 & \text{if } i = 1 \\ c(v_{c-(i-1)}) + [n - (2i - 1)] & \text{if } 2 \leq i \leq h \end{cases} \tag{1}$$

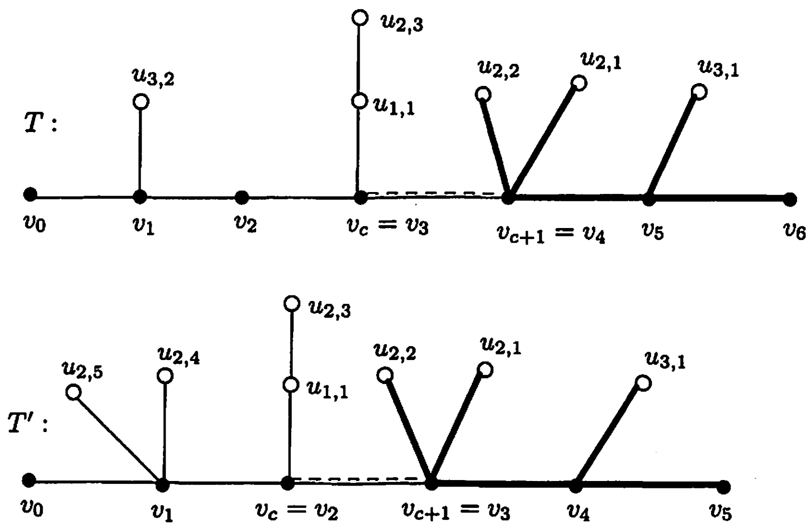

Figure 1: Illustrating the labeling of the vertices of T and T'

$$c(u_{i,s}) = c(v_{c+i}) + s(n-2) \text{ for } 1 \le s \le n_{1,i} \tag{2}$$

$$c(u_{i,t}) = \begin{cases} c(v_{c+i}) + (n - \ell_i) & \text{if } t = 1 \text{ and } V_{1,i} = \emptyset \\ c(u_{i,n_{1,i}}) + (n - \ell_i) & \text{if } t = n_{1,i} + 1 \text{ and } V_{1,i} \neq \emptyset \\ c(u_{i,t-1}) + (n-2) & \text{if } n_{1,i} + 2 \le t \le n_{1,i} + n_{2,i} \end{cases} \tag{3}$$

$$c(v_{c-i}) = \begin{cases} c(v_{c+i}) + (n - \ell_i) & \text{if } V_{2,i} = \emptyset \\ c(u_{i,x}) + (n-2) & \text{if } x = n_{2,i} > 0 \end{cases} \tag{4}$$

$$c(v_c) = \begin{cases} c(v_0) + (n-h) & \text{if } d \text{ is even} \\ c(v_d) + (n-h) & \text{if } d \text{ is odd and } V_{1,h} = \emptyset \\ c(u_{h,x}) + (n-h) & \text{if } d \text{ is odd and } x = n_{1,h} > 0. \end{cases} \tag{5}$$

Therefore,

$$\Omega : x_1, x_2, \ldots, x_n \tag{6}$$

is the c-ordering of the vertices of T. Then when d is even,

$$\Omega : v_{c+1}, V_{2,1}, v_{c-1}, v_{c+2}, V_{1,2}, V_{2,2}, v_{c-2}, v_{c+3}, \ldots, V_{1,h}, V_{2,h}, v_{c-h}, v_c;$$

while when d is odd, then necessarily $V_{2,h} = \emptyset$, and so

$$\Omega : v_{c+1}, V_{2,1}, v_{c-1}, v_{c+2}, V_{1,2}, V_{2,2}, v_{c-2}, v_{c+3}, \ldots, V_{1,h}, v_c.$$

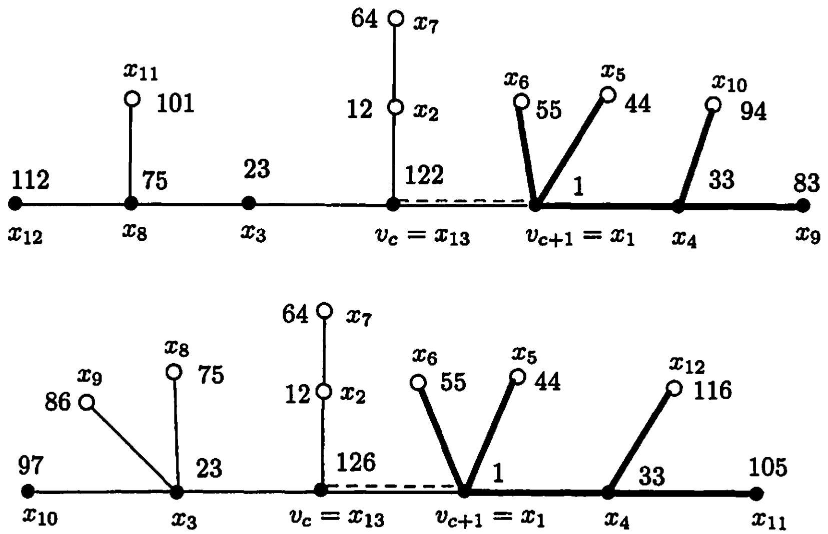

Figure 2: Illustrating the labeling c of the vertices of T

For each of the trees T and T' of Figure 1, such a coloring c is shown in Figure 2 together with the c-ordering $x_1, x_2, \ldots, x_{13}$ of the vertices of the tree.

We now show that c is a Hamiltonian labeling, that is, we show for every pair u, v of distinct vertices of T that

$$|c(u) - c(v)| + D(u, v) \geq n. \tag{7}$$

We begin with $u, v \in V(T) - \{v_c\}$. First assume that u and v are two consecutive vertices in the c-ordering Ω in (6). If $\{u, v\} \subseteq V(R) - \{v_c\}$, then either (i) $u = v_{c-(i-1)}$ and $v = v_{c+i}$, for $2 \leq i \leq h$, or (ii) $u = v_{c-i}$ and $v = v_{c+i}$, where now $1 \leq i \leq h$. If (i) occurs, then $D(v_{c-(i-1)}, v_{c+i}) = 2i-1$ and $|c(v_{c-(i-1)}) - c(v_{c+i})| = n - 2i + 1$ by (1); while if (ii) occurs, then $D(v_{c-i}, v_{c+i}) = 2i$ and $|c(v_{c-i}) - c(v_{c+i})| \geq n - \ell_i \geq n - 2i$ by (4). On the other hand, if $\{u, v\} \not\subseteq V(R)$, then either $D(u, v) \geq 2$ and $|c(u) - c(v)| \geq n - 2$ or $D(u, v) \geq 2i$ and $|c(u) - c(v)| \geq n - \ell_i \geq n - 2i$ by (2)-(4), for $1 \leq i \leq h$. Thus (7) holds in each case.

Next, we assume that u and v are not two consecutive vertices in (6). We show in this case that

$$|c(u) - c(v)| \geq n - 1, \tag{8}$$

which will imply that (7) holds. By (1)-(4) we see that

$$\begin{aligned}|c(u)-c(v)| &\geq [n-(2h-1)]+(n-\ell_h)\\ &\geq 2n-2h+1-2k\end{aligned} \tag{9}$$

(since $\ell_h \leq 2k$). We now consider two cases.

Case 1. $\lceil\frac{d-1}{2}\rceil \leq \lfloor\frac{n-d+1}{2}\rfloor$. Since

$$\frac{d-1}{2} \leq \left\lceil\frac{d-1}{2}\right\rceil \leq \left\lfloor\frac{n-d+1}{2}\right\rfloor \leq \frac{n-d+1}{2},$$

it follows that $n \geq 2d-2 > 2d-1$ and so $n-2d \geq -1$. By (9),

$$\begin{aligned}|c(u)-c(v)| &\geq 2n-2\left(\frac{d+1}{2}\right)+1-2\left(\frac{d}{2}\right)\\ &= 2n-d-1+1-d = 2n-2d \geq n-1.\end{aligned}$$

Case 2. $\lceil\frac{d-1}{2}\rceil \geq \lfloor\frac{n-d+1}{2}\rfloor$. Again by (9),

$$\begin{aligned}|c(u)-c(v)| &\geq 2n-2\left(\frac{d+1}{2}\right)+1-2\left(\frac{n-d+1}{2}\right)\\ &= 2n-d-1+1-n+d-1 = n-1.\end{aligned}$$

Therefore, (7) holds if $u, v \in V(T)-\{v_c\}$ and $u \neq v$.

Finally, we consider those pairs u, v of vertices of T where $u = v_c$ and $v \in V(T)-\{v_c\}$. Note that $v_c = x_n$ in the c-ordering Ω in (6) and $v = x_j$ for some j with $1 \leq j \leq n-1$. If $j \leq n-3$, then $|c(x_n)-c(x_j)| = c(x_n)-c(x_j) \geq c(x_{n-1})-c(x_j) \geq n-1$ by (8) and so (7) holds. If $j = n-1$, then $|c(x_n)-c(x_j)|+D(x_n, x_j) = (n-h)+h = n$. Thus we may assume that $j = n-2$. If d is even, then $D(v_c, v) = h$. It then follows by (4) and (5) that

$$\begin{aligned}|c(v_c)-c(v)|+D(v_c, v) &\geq (n-h)+(n-\ell_h)+h\\ &\geq 2n-\ell_h \geq n.\end{aligned}$$

If d is odd, then $D(v_c, v) \geq h-1$. It then follows by (1) and (5) that

$$\begin{aligned}|c(v_c)-c(v)|+D(v_c, v) &\geq (n-h)+[n-(2h-1)]+(h-1)\\ &\geq 2n-2h \geq 2n-(d+1) \geq n.\end{aligned}$$

Therefore, c is a Hamiltonian labeling, as claimed. It remains to determine the value $\text{hn}(c)$ of c. First, assume that d is even. Then $h = \frac{d}{2}$

and

$$\begin{aligned}
\operatorname{hn}(c) &= c(v_c) = c(v_h) \\
&\leq 1 + \sum_{i=2}^{h}[n-(2i-1)] + \sum_{i=1}^{k}(n-2i) \\
&+ (h-k)(n-2k) + (n-h) + (n-2h-1)(n-2) \\
&= 1 + (nh - h(h+1) + h) - (n-1) + (nk - k(k+1)) \\
&+ (nh - 2hk - nk + 2k^2) + (n-h) \\
&+ (n^2 - 2n - 2nh + 4h - n + 2) \\
&= n^2 - 3n + 4 + 3h - h^2 - 2hk - k + k^2 \\
&= n^2 - 3n + 4 + \frac{3}{2}d - \frac{d^2}{4} - dk - k + k^2.
\end{aligned}$$

Next, assume that d is odd. Then $h = \frac{d+1}{2}$ and

$$\begin{aligned}
\operatorname{hn}(c) &= c(v_c) = c(v_{h-1}) \\
&\leq 1 + \sum_{i=2}^{h}[n-(2i-1)] + \sum_{i=1}^{k}(n-2i) \\
&+ (h-1-k)(n-2k) + (n-h) + (n-2h)(n-2) \\
&= 1 + (nh - h(h+1) + h) - (n-1) + (nk - k(k+1)) \\
&+ (nh - 2hk - n + 2k - nk + 2k^2) + (n-h) \\
&+ (n^2 - 2n - 2nh + 4h) \\
&= n^2 - 3n + 2 + 3h - h^2 - 2hk + k + k^2 \\
&= n^2 - 3n + 2 + \left(\frac{3}{2}d + \frac{3}{2}\right) - \left(\frac{d^2}{4} + \frac{d}{2} + \frac{1}{4}\right) \\
&- k(d+1) + k + k^2 \\
&= n^2 - 3n + 3\frac{1}{4} + d - \frac{d^2}{4} - kd + k^2.
\end{aligned}$$

This completes the proof. ■

If T is a nontrivial tree of order n and diameter 2, then $\operatorname{hn}(T) \leq n + (n-2)^2$ by Theorem 2.7. On the other hand, T is a star and so $\operatorname{hn}(T) = n + (n-2)^2$ by Theorem 1.3. Thus the upper bound in Theorem 2.7 is attainable for $d = 2$ and for all $n \geq 3$. In fact, the upper bound in Theorem 2.7 is also attainable for $d = 3$ and for all $n \geq 4$. A tree of diameter 3 is referred to as a *double star*. Thus a double star T has exactly two non-end-vertices called the *central vertices* of T. The double star whose central vertices have degrees a and b, respectively, is denoted by $S_{a,b}$. We now determine the Hamiltonian labeling of all double stars.

Theorem 2.8 *If $S_{a,b}$ is a double star of order $n = a+b$ where $a \leq b$, then*

$$\text{hn}\,(S_{a,b}) = n + (n-2)^2 - 2(a-1).$$

Proof. It is easy to see that $S_{2,2} = P_4$ and $\text{hn}(S_{2,2}) = 6$, we may assume that $n \geq 5$. Let $u_1, v_1 \in V(S_{a,b})$ be the central vertices of $S_{a,b}$, where $\deg u_1 = a$ and $\deg v_1 = b$ and let $N(u_1) = \{v_1, u_2, u_3, \ldots, u_a\}$ and $N(v_1) = \{u_1, v_2, v_3, \ldots, v_b\}$. We first show that $\text{hn}\,(S_{a,b}) \geq n + (n-2)^2 - 2(a-1)$. Observe that

(a) $D(u,v) = 3$ if and only if $\{u,v\} = \{u_i, v_j\}$ for $2 \leq i \leq a$ and $2 \leq j \leq b$,

(b) $D(v_1, x) = 2$ if and only if $x = u_i$ for $2 \leq i \leq a$.

Let c be a minimum Hamiltonian labeling of $S_{a,b}$ and let $w_1, w_2, \ldots, w_n$ be the c-ordering of the vertices of $S_{a,b}$. Then $v_1 = w_i$ for some i with $1 \leq i \leq n$. Let $S = \{j : D(w_j, w_{j+1}) = 3 \text{ and } 1 \leq j \leq n-1\}$. It then follows by (a) that $|S| \leq 2(a-1)$. We now consider two cases.

Case 1. $\{w_{i-1}, w_{i+1}\} \cap N(u_1) \neq \emptyset$. Then $|S| \leq 2(a-1) - 1$ by (a) and so

$$\begin{aligned}
\text{hn}(c) &\geq 1 + (n-3)(2a-3) + (n-2)(n-(2a-3)-1)\\
&= 1 + (2na - 3n - 6a + 9) + (n^2 - 2na + 2n - 2n + 4a - 4)\\
&= n^2 - 3n - 2a + 6 = n + (n-2)^2 - 2(a-1).
\end{aligned}$$

Case 2. $\{w_{i-1}, w_{i+1}\} \cap N(u_1) = \emptyset$. Then $|S| \leq 2(a-1)$ by (a) and (b). Since $D(w_i, x) = 1$ for each $x \in \{w_{i-1}, w_{i+1}\}$, it follows that $|c(w_i) - c(x)| \geq n-1$. Hence

$$\begin{aligned}
\text{hn}(c) &\geq 1 + (n-3)(2a-2) + (n-1) + (n-2)(n-(2a-2+1+1))\\
&= 1 + (2na - 2n - 6a + 6) + (n-1) + (n^2 - 2na - 2n + 4a)\\
&= n^2 - 3n - 2a + 6 = n + (n-2)^2 - 2(a-1).
\end{aligned}$$

Therefore, $\text{hn}\,(S_{a,b}) = \text{hn}(c) \geq n + (n-2)^2 - 2(a-1)$.

To show that $\text{hn}\,(S_{a,b}) \leq n + (n-2)^2 - 2(a-1)$, we define a labeling c_0 by

$$\begin{aligned}
c_0(v_1) &= 1\\
c_0(u_2) &= 1 + (n-2)\\
c_0(v_i) &= c_0(u_i) + (n-3) \quad \text{for } 2 \leq i \leq a\\
c_0(u_j) &= c_0(v_{j-1}) + (n-3) \quad \text{for } 3 \leq j \leq a\\
c_0(v_k) &= c_0(v_{k-1}) + (n-2) \quad \text{for } a+1 \leq k \leq b\\
c_0(u_1) &= c_0(v_b) + (n-2).
\end{aligned}$$

Then if $\Omega : x_1, x_2, \ldots, x_n$ is the c_0-ordering of $S_{a,b}$, then

$$\Omega : x_1 = v_1, u_2, v_2, u_3, v_3, \ldots, u_a, v_a, v_{a+1}, \ldots v_b, u_1 = x_n,$$

where

$$\begin{aligned} c_0(x_1) &= 1 \\ c_0(x_2) &= 1 + (n-2) \\ c_0(x_i) &= c_0(x_{i-1}) + (n-3) \text{ for } 3 \le i \le 2a-1 \\ c_0(x_j) &= c_0(x_{j-1}) + (n-2) \text{ for } 2a \le i \le n. \end{aligned}$$

Let $x_i, x_j \in V(S_{a,b})$ with $1 \le i < j \le n$. If $j \ge i+2$, then $c(x_j) - c(x_i) \ge 2(n-3) = 2n-6 \ge n-1$; while if $j = i+1$, then $c(x_j) - c(x_i) = n - D(x_i, x_j)$. Thus c_0 is a Hamiltonian labeling of $S_{a,b}$. Furthermore, the value of c_0 is

$$\begin{aligned} \text{hn}\,(c_0) \le c_0(u_1) &= 1 + (n-3)(a-1+a-2) + (n-2)(n-(2a-3)-1) \\ &= 1 + (n-3)(2a-3) + (n-2)(n-2a+2) \\ &= 1 + (2na-3n-6a+9) + (n^2-2na+2n-2n+4a-4) \\ &= n^2 - 3n - 2a + 6 = n + (n-2)^2 - 2(a-1). \end{aligned}$$

Therefore, $\text{hn}\,(S_{a,b}) = n + (n-2)^2 - 2(a-1)$. ■

If T is a double star of order $n \ge 4$, then $\text{hn}(T) \le n + (n-2)^2 - 2$ by Theorem 2.7. On the other hand, if $T = S_{2,n-2}$ for $n \ge 4$, then $\text{hn}(T) = n + (n-2)^2 - 2$ by Theorem 2.8. Thus the upper bound in Theorem 2.7 is attainable for $d = 3$ and for all $n \ge 4$, as claimed.

As a consequence of Corollary 2.6 and Theorem 2.7, we obtain an upper bound for the Hamiltonian labeling number of a connected graph in terms of its order and diameter.

Corollary 2.9 *Let G be a connected graph of order $n \ge 3$ with* $\text{diam}\,(G) = d$ *and*

$$k = \min\left(\left\lceil \frac{d-1}{2} \right\rceil, \left\lfloor \frac{n-d+1}{2} \right\rfloor\right).$$

Then

$$\text{hn}\,(G) \le \begin{cases} n^2 - 3n + 4 + \frac{3}{2}d - \frac{d^2}{4} + k^2 - kd - k & \text{if } d \text{ is even} \\ n^2 - 3n + 3\frac{1}{4} + d - \frac{d^2}{4} + k^2 - kd & \text{if } d \text{ is odd.} \end{cases}$$

3 Lower Bounds

We have already mentioned that a connected graph G of order n has Hamiltonian labeling number n if G is Hamiltonian. Furthermore, $\text{hn}(G) \ge n$

for every nontrivial connected graph G of order n. Next, we provide an improved lower bound for the Hamiltonian labeling number of a nontrivial connected graph in terms of its order. Let G be a connected graph of order $n \geq 5$. For an ordering $s : v_1, v_2, \cdots, v_n$ of the vertices of G, define

$$D(s) = \sum_{i=1}^{n-1} D(v_i, v_{i+1})$$

and

$$D(G) = \max\{D(s) : \ s \text{ is an ordering of } V(G)\}.$$

We now establish a lower bound for the Hamiltonian labeling number of a connected graph G in terms of its order and $D(G)$.

Theorem 3.1 *For a connected graph G of order $n \geq 5$*

$$\text{hn}(G) \geq n(n-1) - D(G) + 1.$$

Proof. Let c be a minimum Hamiltonian labeling of G and let $v_1, v_2, \ldots, v_n$ be the c-ordering of the vertices of G. Certainly,

$$\sum_{i=1}^{n-1} D(v_i, v_{i+1}) \leq D(G).$$

Since c is a Hamiltonian labeling of G, it follows that

$$|c(v_i) - c(v_{i+1})| = c(v_{i+1}) - c(v_i) \geq n - D(v_i, v_{i+1})$$

for $1 \leq i \leq n-1$. Hence

$$\begin{aligned} \text{hn}(c) - 1 &= \sum_{i=1}^{n-1}(c(v_{i+1}) - c(v_i)) = \sum_{i=1}^{n-1} |c(v_i) - c(v_{i+1})| \\ &\geq n(n-1) - \sum_{i=1}^{n-1} D(v_i, v_{i+1}) \geq n(n-1) - D(G). \end{aligned}$$

Therefore, $\text{hn}(G) = \text{hn}(c) \geq n(n-1) - D(G) + 1$. ■

The lower bound in Theorem 3.1 is sharp. For example, if $G = K_{1,n-1}$ for $n \geq 5$, then $D(G) = 2n-3$, if $G = S_{2,n-2}$ for $n \geq 4$, then $D(G) = 2n-1$, and if $G = S_{3,n-3}$ for $n \geq 6$, then $D(G) = 2n+1$ (see [3]). Thus in each case, $\text{hn}(G) = n(n-1) - D(G) + 1$ by Theorems 1.3 and 2.8.

We now present a lower bound for Hamiltonian labeling number of a tree in terms of its order and diameter.

Theorem 3.2 *Let T be a tree of order $n \geq 3$ with* $\mathrm{diam}(T) = d$. *Then*

$$\mathrm{hn}(T) \geq \begin{cases} n^2 - n(d+1) + \frac{d^2+3}{2} & \text{if } d \text{ is odd} \\ n^2 - n(d+1) + \frac{d^2+4}{2} & \text{if } d \text{ is even.} \end{cases}$$

Proof. Let c be a Hamiltonian labeling of G with $\mathrm{hn}(c) = \mathrm{hn}(T)$. Since $\mathrm{diam}(T) = d$, it follows that $d(u,v) = D(u,v) \leq d$ for every two distinct vertices u and v in T and so

$$|c(u) - c(v)| \geq n - d. \tag{10}$$

Let $R : v_0, v_1, \cdots, v_d$ be a path of length d in T and let $r = \mathrm{rad}(T) = \lfloor \frac{d}{2} \rfloor$. If d is odd, then v_r and v_{r+1} are the central vertices of T; while if d is even, then v_r is the only central vertex of T. Define a function w on the set $V(T)$ by

$$w(u) = \begin{cases} \min(d(u,v_0), d(u,v_d)) & \text{if } u \in V(R) \\ 0 & \text{if } u \in V(T) - V(R). \end{cases}$$

Thus $0 \leq w(u) \leq r$ and $w(u) = r$ if and only if u is a central vertex of T. Furthermore,

$$\sum_{u \in V(T)} w(u) = \sum_{u \in V(R)} w(u) = \begin{cases} 2\left(\sum_{i=1}^{r} i\right) = r(r+1) & \text{if } d \text{ is odd} \\ 2\left(\sum_{i=1}^{r-1} i\right) + r = r^2 & \text{if } d \text{ is even.} \end{cases} \tag{11}$$

We claim that

$$|c(u) - c(v)| \geq (n-d) + w(u) + w(v) \tag{12}$$

for every pair u, v of distinct vertices of T. If $u, v \in V(T) - V(R)$, then $w(u) = w(v) = 0$ and (12) holds by (10). Thus we may assume at least one of u and v in $V(R)$, say $u \in V(R)$ and $u = v_s$ for some s with $0 \leq s \leq d$. There are two cases, according to whether $v \in V(R)$ or $v \notin V(R)$.

Case 1. $v \in V(R)$. We may assume, without loss of generality, that $v = v_t$, where $0 \leq s < t \leq d$. There are two subcases.

Subcase 1.1. *The $v_0 - v_s$ subpath of R has length $w(u)$ and the $v_0 - v_t$ subpath of R has length $w(v)$ or the $v_s - v_d$ subpath of R has length $w(u)$ and the $v_t - v_d$ subpath of R has length $w(v)$, say the latter.* Then $w(u) > w(v)$ and $D(u,v) = |w(u) - w(v)| = w(u) - w(v)$. Since $2w(u) \leq 2r \leq d$, it follows that

$$\begin{aligned} |c(u) - c(v)| &\geq n - D(u,v) = n - w(u) + w(v) \\ &= n - d + (d - 2w(u)) + w(u) + w(v) \\ &\geq n - d + w(u) + w(v). \end{aligned}$$

Subcase 1.2. *The* $v_0 - v_s$ *subpath of* R *has length* $w(u)$ *and the* $v_t - v_d$ *subpath of* R *has length* $w(v)$. Then $D(u,v) = d - w(u) - w(v)$ and so $|c(u) - c(v)| \geq n - D(u,v) = n - d + w(u) + w(v)$.

Case 2. $v \notin V(R)$. Thus $w(v) = 0$. We may assume, without loss of generality, that $0 \leq s \leq r = \text{rad}(T)$. Therefore, $D(v_0, u) = w(u)$. There are two subcases.

Subcase 2.1. *The edge sets of the* $u - v$ *path and the* $v_0 - u$ *path are disjoint.* Then

$$D(u,v) + w(u) = D(u,v) + D(v_0,u) \leq d$$

and so $D(v,u) \leq d - w(u) = d - w(u) - w(v)$. Therefore, $|c(u) - c(v)| \geq n - D(u,v) = n - d + w(u) + w(v)$.

Subcase 2.2. *The edge sets of the* $u - v$ *path and the* $v_d - u$ *path are disjoint.* Then

$$D(v_d, v) = D(v_d, u) + D(u,v) = (d - w(u)) + D(u,v) \leq d.$$

As such, $D(u,v) \leq w(u) \leq \frac{d}{2}$. Therefore, $D(u,v) + w(u) \leq d$ and so $D(u,v) \leq d - w(u) = d - w(u) - w(v)$. Then, $|c(u) - c(v)| \geq n - D(u,v) = n - d + w(u) + w(v)$.

Therefore, (12) holds, as claimed. With the aid of (12), we now present a lower bound for the value of c. Let $u_1, u_2, \ldots, u_n$ be the c-ordering of the vertices of T. Then

$$\begin{aligned}
\text{hn}(c) &\geq 1 + \sum_1^{n-1} (c(u_{i+1} - c(u_i)) \\
&\geq 1 + \sum_1^{n-1} (n - d + w(u_i) + w(u_{i+1})) \\
&= 1 + (n-1)(n-d) + \sum_1^{n-1} w(u_i) + \sum_1^{n-1} w(u_{i+1}) \\
&= 1 + (n-1)(n-d) + 2\left[\sum_1^{n} w(u_i)\right] - w(u_1) - w(u_n).
\end{aligned}$$

If d is odd, then $d = 2r + 1$ and $w(u_1) + w(u_n) \leq 2r$. By (11),

$$\begin{aligned}
\text{hn}(c) &\geq 1 + (n-1)(n-d) + 2r(r+1) - 2r \\
&= 1 + (n-1)(n-d) + \frac{(d-1)^2}{2} \\
&= n^2 - n(d+1) + \frac{d^2+3}{2};
\end{aligned}$$

while if d is even, then $d = 2r$ and $w(u_1) + w(u_n) \leq 2r - 1$. By (11),

$$\begin{aligned} \text{hn}(c) &\geq 1 + (n-1)(n-d) + 2r^2 - 2r + 1 \\ &= 2 + (n-1)(n-d) + \frac{d^2}{2} - d \\ &= n^2 - n(d+1) + \frac{d^2+4}{2}. \end{aligned}$$

This completes the proof. ■

If T is a nontrivial tree of order n and diameter 2, then $\text{hn}(T) \geq n + (n-2)^2$ by Theorem 3.2. In this case, $\text{hn}(T) = n + (n-2)^2$ by Theorem 1.3 and so the lower bound in Theorem 3.2 is attainable for $d = 2$ and for all $n \geq 3$. Furthermore, the lower bound in Theorem 3.2 is also attainable for $d = 3$ and for all $n \geq 4$. In this case, $\text{hn}(T) \geq (n-2)^2 + 2$ by Theorem 3.2. On the other hand, if T is a nontrivial tree of order $n \geq 4$ and diameter 3, then T is a double star and so $T = S_{a,b}$ for some integers $a, b \geq 2$ and $a \leq b$. By Theorem 2.8, $\text{hn}(S_{a,b}) = n + (n-2)^2 - 2(a-1)$. Thus $\text{hn}\left(S_{\frac{n}{2},\frac{n}{2}}\right) = (n-2)^2 + 2$ for each even integer $n \geq 4$. Therefore, the lower bound in Theorem 3.2 is attainable for $d = 3$ and for all $n \geq 4$.

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References

[1] G. Chartrand, D. Erwin, F. Harary, and P. Zhang, Radio labelings of graphs. *Bull. Inst. Combin. Appl.* **33** (2001) 77-85.

[2] G. Chartrand, D. Erwin, and P. Zhang, Radio antipodal colorings of graphs. *Math. Bohem.* **127** (2002) 57-69.

[3] G. Chartrand, L. Nebeský, and P. Zhang, Bounds for the Hamiltonian chromatic number of a graph. *Congr. Numer.* **157** (2002) 113-125.

[4] G. Chartrand, L. Nebeský, and P. Zhang, Hamiltonian colorings of graphs. *Discrete Appl. Math.* **146** (2005) 257-272.

[5] G. Chartrand, L. Nebeský, and P. Zhang, On Hamiltonian colorings of graphs. *Discrete Math.* **290** (2005) 133-143.

[6] G. Chartrand and P. Zhang, *Chromatic Graph Theory*. Chapman & Hall/CRC, Boca Raton, FL (2008).

[7] L. Nebeský, Hamiltonian colorings of graphs with long cycles. *Math. Bohem.* **128** (2003) 263-275.

[8] L. Nebeský, The hamiltonian chromatic number of a connected graph without large hamiltonian-connected subgraphs. *Czech. Math. J.* **56** (131) (2006) 317-338.

[9] W. A. Renzema and P. Zhang, Hamiltonian labelings of graphs. *Involve.* To appear.