Bipartite graphs decomposable into closed trails

Sylwia Cichacz*
AGH University of Science and Technology
Al. Mickiewicza 30, 30-059 Kraków, Poland, cichacz@agh.edu.pl
and
University of Minnesota Duluth
Duluth, MN 55812-3000 U.S.A.

Dalibor Froncek
University of Minnesota Duluth
Duluth, MN 55812-3000 U.S.A., dalibor@d.umn.edu

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Abstract

Let $G = K_{a,b}$, where a, b are even or $G = K_{a,a} - M_{2a}$, where $a \geq 1$ is an odd integer and M_{2a} is a perfect matching in $K_{a,a}$. It has been shown ([3,4]) that G is arbitrarily decomposable into closed trails. Billington asked if the graph $K_{r,s} - F$, where s, r are odd and F is a (smallest possible) spanning subgraph of odd degree, is arbitrarily decomposable into closed trails ([2]).

In this article we answer the question in the affirmative.

1 Introduction

All graphs considered in this paper are simple, finite and undirected. We use standard terminology and notation of graph theory. Consider a graph G, whose number of edges we call the size of G and denote by $\|G\|$. Write $V(G)$ for the vertex set and $E(G)$ for the edge set of graph G.

As in [4] we denote by Lct(G) the set of all integers l such that there is a closed trail of length l in G. A sequence $\tau = (t_1, \ldots, t_p)$ of integers is called *admissible for graph G* if it adds up to $\|G\|$ and $t_i \in$ Lct(G) for all $i \in \{1, \ldots, p\}$. Moreover, if $\tau = (t_1, t_2, \ldots, t_p)$ is an admissible sequence for G and G can be edge-disjointly decomposed into closed trails $T_1, \ldots, T_p$ of lengths $t_1, \ldots, t_p$, respectively, then τ

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is called *realizable in G* and the sequence $(T_1, \dots, T_p)$ is said to be *a G-realization of τ* or *a realization of τ in G*.

If for each admissible sequence τ for a graph G there is a realization of τ in G, then we say that G is *arbitrarily decomposable into closed trails.*

There are some families of graphs that are known to be arbitrarily decomposable into closed trails. The first result on this topic is due to P.N. Balister, who proved that if $G = K_n$ for n odd or $G = K_n - M_n$, where M_n is a perfect matching in K_n, for n even, then G is arbitrarily decomposable into closed trails ([1]). In [4] M. Horňák, M. Woźniak proved an analogous theorem for complete bipartite graphs.

Theorem 1 (M. Horňák, M. Woźniak) *If a, b are positive even integers, then the bipartite graph $K_{a,b}$ is arbitrarily decomposable into closed trails.*

E.J. Billington has put the following open problem in [2]:

Problem 2 (E.J. Billington) *Show that the graf $K_{r,s} - F$, where s, r are odd and F is a (smallest possible) spanning subgraph in which every vertex has odd degree, is arbitrarily decomposable into closed trails.*

This problem is partially solved in [3]. It is shown there that if $a \geq 1$ is an odd integer and M_{2a} is a perfect matching in $K_{a,a}$, then the graph $K_{a,a} - M_{2a}$ is arbitrarily decomposable into closed trails.

Theorem 3 (S. Cichacz, M. Horňák) *If a is an odd integer, $a \geq 1$, then the graph $K'_{a,a} = K_{a,a} - M_{2a}$ is arbitrarily decomposable into closed trails.*

If a and b are odd, then let $K'_{a,b} = K_{a,b} - F$, where F is a (smallest possible) spanning subgraphs with all vertices of odd degrees. Notice that $K'_{a,b}$ is an Eulerian graph. The main goal of our paper is to show that the graph $K'_{a,b}$ for any odd a and b is arbitrarily decomposable into closed trails.

If a, b are odd integers with $b \geqslant a + 4 \geqslant 7$, then a smallest (with respect to size) spanning subgraph of $K_{a,b}$ with all vertices of odd degrees is not unique. Such graphs are of the form $\bigcup_{i=1}^{r} K_{1,b_i}$ where $b_1, \dots, b_r$ are odd positive integers with $\sum_{i=1}^{r} b_i = b$. For $a = 3$ we are dealing with the case $b_i \geqslant \lfloor \frac{b-1}{3} \rfloor$, whereas for $a \geqslant 5$ we assume that $b_i = 1$, for $i = 1, \dots, r-1$ and $b_r = b + 1 - r$.

2 Decomposition of $K'_{a,b}$ into closed trails

Here and subsequently, a closed trail T of length n is regarded as an Eulerian graph (or subgraph) of size n. However, it will be identified with any sequence $(v_0, v_1, \dots, v_{n-1}, v_n)$ of vertices of T such that $v_i v_{i+1}$ are distinct edges of T for $i = 0, 1, \dots, n-1$. Notice that we do not require the v_i to be distinct and certainly $v_0 = v_n$.

Moreover, as in [1] given two edge-disjoint graphs G_1, G_2 which are *not* disjoint on vertices, we shall write $G_1 \cdot G_2$ for their union. Observe that if T_1 and T_2 are closed trails, then $T_1 \cdot T_2$ is a closed trail as well.

As in [3] pick disjoint sets $X^j = \{x_i^j : i \in \{1,\ldots,n\}\}$, $j = 1,2$, and let $X_{p,q}^j = \{x_i^j : i \in \{p, p+1, \ldots, q\}\}$ for $p, q \in \{1,\ldots,n\}$. In this paper the complete bipartite graph $K_{a,b}$ will have the bipartition $\{X_{1,a}^1, X_{1,b}^2\}$ and M_{2a} will be the perfect matching in $K_{a,a}$ consisting of $\{x_i^1, x_i^2\}$ for $1 \leqslant i \leqslant a$.

Denote by $N_G(x)$ the vertex set of all neighbors of the vertex x in a graph G.

The basic idea of our proof is to consider a graph $G = K_{a,b} - F$ as the edge disjoint union $G_1 \cdot G_2$ of two graphs, and given a sequence $\tau = (t_1, \ldots, t_p)$ which is admissible for G, divide it into two sequences $\tau_1 = (t_1, \ldots, t_i)$, $\tau_2 = (t_{i+1}, \ldots, t_p)$ admissible for G_1, G_2, respectively, and then decompose these two graphs separately. It is however obvious that we cannot always simply divide τ into τ_1 and τ_2 as described above. Therefore, we split $t_i = t_i' + t_i''$ at times and search for realizations of $\tau_1' = (t_1, \ldots, t_{i-1}, t_i')$ and $\tau_2' = (t_i'', t_{i+1}, \ldots, t_p)$ in G_1 and G_2, respectively, and finally glue together closed trails of lengths t_i' and t_i'' to form the one of length t_i.

Notice that, if a and b are positive integers, then clearly

$$\begin{array}{lll}\mathrm{Lct}(K_{2,b}) = \{4i : i = 1,2,\ldots,\tfrac{1}{2}b\} & \text{if} & b \text{ is even},\\ \mathrm{Lct}(K_{a,b}) = \{2i : i = 2,3,\ldots,\tfrac{1}{2}(ab-4)\} \cup \{ab\} & \text{if} & a,b \geqslant 4,\ a,b \text{ are even}.\end{array}$$

Let $\tau = (t_1, \ldots, t_p)$ be an admissible sequence for a graph G. We shall write $(t_1^{s_1}, \ldots, t_l^{s_l})$ for the sequence $(\underbrace{t_1,\ldots,t_1}_{s_1},\ldots,\underbrace{t_l,\ldots,t_l}_{s_l})$.

Let $a \geqslant 5$ be odd and the graph $G_{a,5}$ be a subgraph of $K_{a,a+2}$ with the bipartition $\{X_{1,a}^1, X_{a-2,a+2}^2\}$ defined as follows: Let $G^a = K_{a-1,4}$ have the bipartition $\{X_{1,a-1}^1, X_{a-1,a+2}^2\}$, then $E(G_{a,5}) = (E(G^a) \cup \{x_{a-1}^1 x_{a-2}^2, x_a^1 x_{a-2}^2, x_a^1 x_{a-1}^2\}) \setminus \{x_{a-1}^1 x_{a-1}^2\}$. Before we prove the main result we will need the following lemma.

Lemma 4 *If $\tau = (t_1, \ldots, t_r)$ is an admissible sequence for $G_{a,5}$ and $i \in \{a-3, a-2\}$, there is a $G_{a,5}$-realization of τ with a closed trail T_1 of length t_1 passing through x_i^1. Moreover, if $\tau = (t+2, t^{r-1})$, $t \geqslant 10$, $t \equiv 2 \pmod 4$, $a \equiv 1 \pmod t$ and $(i,j) \in \{(1,a-3),(1,a-1),(a-2,a)\}$, there exists a $G_{a,5}$-realization of τ with a closed trail T_1 of length $t+2$ having a subpath of length 4 that joins x_i^1 to x_j^1.*

Proof. Notice that $\|G_{a,5}\| = 4(a-1)+2$, so obviously there exists $t_w \equiv 2 (\mathrm{mod}\, 4)$. Let $t_w' = t_w - 2 \geqslant 4$. The sequence $\tau' = (t_1, \ldots, t_{w-1}, t_w', t_{w+1}, \ldots, t_r)$ is admissible for $G^a = K_{a-1,4}$ and by Theorem 1 there exists a G^a-realization $(T_1, \ldots, T_{w-1}, T_w', T_{w+1}, \ldots, T_r)$ of τ'. One can check that we can permute the set of vertices of the graph G^a in such a way that $\{x_{a-1}^1 x_{a-1}^2\} \subset E(T_w')$. Let us define the T_w so that $V(T_w) = V(T_w') \cup \{x_a^1, x_{a-2}^2\}$ and $E(T_w) = (E(T_w') \cup \{x_{a-1}^1 x_{a-2}^2, x_a^1 x_{a-2}^2, x_a^1 x_{a-1}^2\}) \setminus \{x_{a-1}^1 x_{a-1}^2\}$, then T_w is a closed trail of length t_w and we obtain a $G_{a,5}$-realization of τ.

Notice that $V(T) \cap \{x_1^1, \dots, x_{a-2}^1\} \neq \emptyset$ for any closed trail T in $G_{a,5}$. Therefore, the first part of the lemma follows from the fact that $\{x_1^1, \dots, x_{a-2}^1\}$ is one of the similarity classes of the graph $G_{a,5}$.

Assume now that $\tau = (t+2, t^{r-1})$, $t \geqslant 10$, $t \equiv 2 \pmod 4$ and $a \equiv 1 \pmod t$. Let $\tau' = (t_1', t_2', t_3', t_4', t_5, \dots, t_r)$ and $t_i' = t - 4 \geqslant 6$ for $i = 1, 2, 3, 4$ and $t_i = t$ for $i \geqslant 5$. Notice that for $a = t+1$ we consider $\tau' = (t_1', t_2', t_3', t_4')$. Then by Theorem 1 we can find a $K_{a-5,4}$-realization $(T_1', T_2', T_3', T_4', T_5, \dots, T_r)$, where $K_{a-5,4}$ has the bipartition $\{X_{5,a-1}^1, X_{a-1,a+2}^2\}$. Since $t \equiv 2 \pmod 4$, $|T_i' \cap X_{5,a-1}^1| \geqslant 3$ and $|T_i' \cap X_{a-1,a+2}^2| \geqslant 3$ for all i. As above we can permute the set of vertices of $K_{a-5,4}$ in such a way that $\{x_{a-1}^1 x_{a-1}^2, x_{a-1}^1 x_a^2, x_{a-2}^1 x_a^2\} \subset E(T_1')$ and $x_{a-3}^1 \in V(T_1')$. Let u be a vertex in $N_{T_1'}(x_{a-3}^1)$. Let now $K_{4,4}$ be the bipartite graph with the bipartition $\{X_{1,4}^1, X_{a-1,a+2}^2\}$ and $\tau'' = (t_1'', t_2'', t_3'', t_4'') = (4^4)$. By Theorem 1 there exists a $K_{4,4}$-realization $(T_1'', T_2'', T_3'', T_4'')$ of τ''. We may permute the set of vertices of $K_{4,4}$ such that $\{x_1^1, x_2^1, x_{a-1}^2, u\} \subset V(T_1'')$. Let us define now T_1 so that $E(T_1) = (E(T_1') \cup E(T_1'') \cup \{x_{a-1}^1 x_{a-2}^2, x_a^1 x_{a-2}^2, x_a^1 x_{a-1}^2\}) \setminus \{x_{a-1}^1 x_{a-1}^2\}$. Obviously T_1 is a closed trail of length $t+2$ and we have paths of length 4 between vertices x_{a-3}^1 and x_1^1, x_{a-2}^1 and x_a^1, and x_1^1 and x_{a-1}^1 in the trail T_1.

Notice that for $i = 2, 3, 4$ the trails T_i' and T_i'' have also at least one common vertex in the set $X_{a-1,a+2}^2$. Therefore, we can take $T_i = T_i' \cdot T_i''$ for $i = 2, 3, 4$. ■

Theorem 5 *Let a, b be odd. There exists a (smallest possible) spanning subgraph F of $K_{a,b}$ in which every vertex has odd degree such that $G = K_{a,b} - F$ is arbitrarily decomposable into closed trails.*

Proof. There is no loss of generality in assuming that $a \leqslant b$ for the graph $K_{a,b}$. Assume first that $a = b$. It can be easily seen that for $a = b$ we have $F = M_{2a}$. Graph $G = K_{a,a} - M_{2a}$ is arbitrarily decomposable by Theorem 3.

From now on, let us assume that $a < b$. Notice that $b = a + c$ and c is even. Let $\tau = (t_1, ..., t_p)$ be an admissible sequence for G. We show that there exists a G-realization of τ. We consider the following cases:

Case 1: $a = 3$.
Case 1.1: $b \equiv 3 \pmod 6$. Then $b = 3k$ for some odd integer k. Let

$$E(F) = \{x_1^1 x_1^2, x_2^1 x_2^2, x_3^1 x_3^2, x_1^1 x_4^2, x_2^1 x_5^2, x_3^1 x_6^2, \dots, x_1^1 x_{3k-2}^2, x_2^1 x_{3k-1}^2, x_3^1 x_{3k}^2\}.$$

We can consider now the graph G as an edge disjoint union of graphs $H_1, \dots, H_k$, where $H_r \cong K_{3,3} - M_6 \cong C_6$ and H_r has the bipartition sets $X_{1,3}^1$ and $X_{3(r-1)+1,3(r-1)+3}^2$ for $1 \leqslant r \leqslant k$. If $t_i \equiv 0 \pmod 6$ for every i, then one may check easily that there exists a G-realization of τ.

Suppose now that there exist $t_1, t_2, \dots, t_w$ such that $t_i \not\equiv 0 \pmod 6$ for $1 \leqslant i \leqslant w$. Since $\|G\| = 2b \equiv 0 \pmod 6$, we have $\sum_{i=1}^{w} t_i \equiv 0 \pmod 6$. Let now

$$t_i' = t_i - 4, \quad t_i'' = 4 \quad \text{for} \quad t_i \equiv 4 \pmod 6,$$
$$t_i' = t_i - 8, \quad t_i'' = 8 \quad \text{for} \quad t_i \equiv 2 \pmod 6.$$

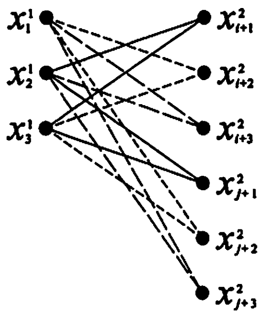

Figure 1: Decomposition of $H_i \cup H_j$ into three cycles C_4.

Let $l_w = \sum_{i=1}^{w} t_i''$. Since $t_i'' \equiv 0 \pmod 4$ for any i, $l_w \equiv 0 \pmod{12}$. As above we can easily obtain a G-realization $(L_w, T_1', \ldots, T_w', T_{w+1}, \ldots, T_p)$ of a sequence $(l_w, t_1', \ldots, t_w', t_{w+1}, \ldots, t_p)$. Notice that a closed trail L_w is isomorphic to $H_1 \cup \ldots \cup H_g$, where each H_i is isomorphic to C_6 and the number g of the cycles is even. There is shown a decomposition of $H_i \cup H_j$ into three cycles C_4 in Figure 1. Since any two closed trails in G have always a common vertex in $X_{1,3}^1$, there exists a decomposition of the closed trail L_w into closed trails $T_1'', \ldots, T_w''$ of lengths $t_1'', \ldots, t_w''$, respectively. Observe that $X_{1,3}^1 \subset V(T_i')$ for any $1 \leqslant i \leqslant w$. In this case T_i' and T_i'' (for $1 \leqslant i \leqslant w$) have two common vertices in $X_{1,3}^1$. It implies that we can denote T_i to be $T_i' \cdot T_i''$ (for $1 \leqslant i \leqslant w$) and we obtain a G-realization of τ.

Case 1.2: $b \equiv 5 \pmod 6$. Then $b = 3k + 2$ for some odd integer k. Let

$$E(F) = \{x_1^1x_1^2, x_2^1x_2^2, x_3^1x_3^2, \ldots, x_1^1x_{3k-2}^2, x_2^1x_{3k-1}^2, x_3^1x_{3k}^2, x_3^1x_{3k+1}^2, x_3^1x_{3k+2}^2\}.$$

Since $\|G\| = 2b \equiv 4 \pmod 6$, we may suppose without loss of generality $t_p \neq 6$. Let $t_p' = t_p - 4$. Notice that we can consider G as a union of graphs $G' = K_{3,b-2} - F'$ and $T_p'' = K_{2,2}$ ($T_p = K_{2,2}$ for $t_p = 4$), with bipartition sets $\{X_{1,3}^1, X_{1,b-2}^2\}$ and $\{X_{1,2}^1, X_{b-1,b}^2\}$, respectively. The spanning subgraph F' of the graph $K_{3,b-2}$ is defined in the same way as in Case 1.1. It implies that a sequence $\tau' = (t_1, \ldots, t_{p-1}, t_p')$ (or $\tau' = (t_1, \ldots, t_{p-1})$ for $t_p = 4$) is G'-realizable. Since T_p' and T_p'' have at least one common vertex in $X_{1,3}^1$, let $T_p = T_p' \cdot T_p''$ and we obtain a G-realization of τ.

Case 1.3: $b \equiv 1 \pmod 6$. Then $b = 3k + 4$ for some odd integer k. Let

$$E(F) = \{x_1^1x_1^2, x_2^1x_2^2, x_3^1x_3^2, \ldots, x_1^1x_{3k-2}^2, x_2^1x_{3k-1}^2, x_3^1x_{3k}^2, x_3^1x_{3k+1}^2, \ldots, x_3^1x_{3k+4}^2\}.$$

In this we can consider G as a union of graphs $G' = K_{3,b-4} - F'$ and $T''_p = K_{2,4}$, with bipartition sets $\{X^1_{1,3}, X^2_{1,b-4}\}$ and $\{X^1_{1,2}, X^2_{b-3,b}\}$, respectively.

As in Case 1.2 we can assume that $t_p \neq 6$. If $t_p \neq 4$ and $t_p \neq 10$, then analogously as in Case 1.2 we obtain a G-realization of τ (taking $t'_p = t_p - 8$). Suppose now that $t_p \in \{4, 10\}$. Since $\|G\| = 2b \equiv 2 \pmod 6$, we may suppose without loss of generality that $t_{p-1} \leq t_p$ and $t_{p-1} \in \{4, 10\}$. Let $t''_{p-1} = t''_p = 4$ and $t'_{p-1} = t_{p-1} - 4$ and $t'_p = t_p - 4$, then a sequence $\tau_2 = (t_1, \ldots, t_{p-2}, t'_{p-1}, t'_p)$ is G'-realizable. By Theorem 1 we obtain a $K_{2,4}$-realization of $\tau_3 = (t''_{p-1}, t''_p)$. Taking $T_{p-1} = T'_{p-1} \cdot T''_{p-1}$ ($T_{p-1} = T''_{p-1}$ for $t_{p-1} = 4$) and $T_p = T'_p \cdot T''_p$ ($T_p = T''_p$ for $t_p = 4$) we obtain a G-realization of τ.

We assume from now on that the sequence τ is nondecreasing, i.e., $t_1 \leqslant \cdots \leqslant t_p$.

Case 2: $a \geqslant 5$ and $c \geqslant 4$.
Let F be a subgraph of $K_{a,b}$ with $V(F) = X^1_{1,a} \cup X^2_{1,b}$ and with $E(F) = \{x^1_1 x^2_1, x^1_2 x^2_2, \ldots, x^1_a x^2_a, x^1_a x^2_{a+1}, \ldots, x^1_a x^2_{b-1}, x^1_a x^2_b\}$. Notice that F is an edge disjoint union of graphs M_{2a} and $K_{1,c}$ and obviously F is a smallest possible spanning subgraphs with all vertices of odd degrees in $K_{a,b}$. Observe that the graph $G = K_{a,b} - F$ can be viewed as an edge disjoint union of graphs $G_1 = K_{a-1,c}$ and $G_2 = K_{a,a} - M_{2a}$ with bipartitions $\{X^1_{1,a-1}, X^2_{a+1,b}\}$ and $\{X^1_{1,a}, X^2_{1,a}\}$, respectively. Moreover, the graphs G_1 and G_2 are arbitrarily decomposable into closed trails by Theorems 1 and 3.

Let $s_i = t_1 + t_2 + \ldots + t_i$, $i = 1, 2, \ldots, p$.

Case 2.1: For some i, $s_i = \|G_1\| = (a-1)c$. Then we can find a realization of $\tau_1 = (t_1, \ldots, t_i)$ in G_1 by Theorem 1 and then decompose G_2 into closed trails of lengths $t_{i+1}, \ldots, t_p$ by Theorem 3.

Case 2.2: For some i, $s_{i-1} \leqslant (a-1)c - 4$ and $s_i \geqslant (a-1)c + 4$ (take $s_0 = 0$ if $i = 1$). Let $\tau_1 = (t_1, \ldots, t_{i-1}, t'_i)$ and $\tau_2 = (t''_i, t_{i+1}, \ldots, t_p)$, where $t'_i = (a-1)c - s_{i-1} \geqslant 4$ and $t''_i = t_i - t'_i \geqslant 4$. By Theorem 1, we may decompose $G_1 \cong K_{a-1,c}$ into closed trails $T_1, \ldots, T_{i-1}, T'_i$ of lengths $t_1, \ldots, t_{i-1}, t'_i$, respectively. Then we may also find a G_2-realization $(T''_i, T_{i+1}, \ldots, T_p)$ of τ_2 by Theorem 3. Since $t''_i \geqslant 4$, we have $V(T''_i) \cap X^1_{1,a-1} \neq \emptyset$. It follows that we may carry out these decompositions of G_1 and G_2 in such a way that T'_i and T''_i have a common vertex. If we denote T_i to be $T'_i \cdot T''_i$ then $(T_1, T_2, \ldots, T_p)$ is a G-realization of τ.

Case 2.3: For some i, $s_i = (a-1)c + 2$. Since $(a-1)c \equiv 0 \pmod 4$ and τ is nondecreasing, $t_i \geqslant 6$. Denote $t'_i = t_i - 2$, $t'_{i+1} = t_{i+1} + 2$. Then, since $t'_i \geqslant 4$, $t'_{i+1} \geqslant 6$, we may find a G_1-realization of $\tau_1 = (t_1, \ldots, t_{i-1}, t'_i)$ from Theorem 1 and a G_2-realization of $\tau_2 = (t'_{i+1}, t_{i+2}, \ldots, t_p)$ from Theorem 3. Recall that $G_1 \cong K_{a-1,c}$ and $G_2 \cong K_{a,a} - M_{2a}$ Let $(T_1, T_2, \ldots, T_{i-1}, T'_i)$ and

$(T'_{i+1}, T_{i+2}, \dots, T_p)$ be G_1 and G_2-realizations of τ_1 and τ_2, respectively, such that $\|T'_i\| = t'_i$ and $\|T'_{i+1}\| = t'_{i+1}$.

Observe that $G_2 \cong K_{a,a} - M_{2a}$ is symmetric with respect to parts of its bipartition. It is easy to verify that we may write T'_{i+1} as $(v_1, \dots, v_{t'_{i+1}+1})$ with $v_1 \in X^1_{1,a-1}$ and $v_5 \in X^1_{1,a-1}$ (it is enough to permute the vertices of $X^1_{1,a}$ and $X^2_{1,a}$) and $v_1 \neq v_5$ (even if $t'_{i+1} \equiv 0 \pmod 4$). We may also write T'_i as $(w_1, \dots, w_{t'_i+1})$ with $w_1 \in X^1_{1,a-1}$ and $w_3 \in X^1_{1,a-1}$. Notice that without loosing generality we may choose the realization of τ_2 in such a way that $w_1 = v_1$ and $w_3 = v_5$. In such a case, if we denote $T_i = (v_1, v_2, v_3, v_4, v_5, w_4, \dots, w_{t'_i+1})$ and $T_{i+1} = (v_1, w_2, v_5, \dots, v_{t'_{i+1}+1})$, then T_i and T_{i+1} are closed trails of lengths t_i and t_{i+1}, respectively (see Figure 2), and the sequence $(T_1, \dots, T_p)$ is a G-realization of τ.

Case 2.4: For some i, $s_i = (a-1)c - 2$. It follows $t_i \geqslant 6$. Let us introduce

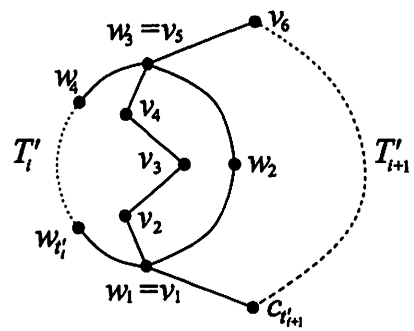

Figure 2: Intersecting closed trails.

$t'_i = t_i + 2 \geqslant 8$, $t'_p = t_p - 2 \geqslant 4$. Let $(T_1, \dots, T_{i-1}, T'_i)$ and $(T_{i+1}, \dots, T_{p-1}, T'_p)$ be G_1 and G_2-realizations of $\tau_1 = (t_1, \dots, t_{i-1}, t'_i)$ and $\tau_2 = (t_{i+1}, \dots, t_{p-1}, t'_p)$, respectively, such that $\|T'_i\| = t'_i$ and $\|T'_p\| = t'_p$.

Assume first that we can write T'_i as $(w_1, \dots, w_{t'_i+1})$ with $w_1, w_5 \in X^1_{1,a-1}$ and $w_1 \neq w_5$. We write T'_p as $(v_1, \dots, v_{t'_p+1})$ with $v_1, v_3 \in X^1_{1,a-1}$. As above we can choose the realization of τ_2 in such a way that $w_1 = v_1$ and $w_5 = v_3$. In such a case, if we denote $T_i = (v_1, v_2, v_3, w_6, \dots, w_{t'_i+1})$ and $T_p = (v_1, w_2, w_3, w_4, v_3, \dots, v_{t'_p+1})$, then T_i and T_p are closed trails of lengths t_i and t_p.

Suppose now that we cannot write T'_i in such a way. It follows that $t'_i \equiv 0 \pmod 4$ and $w_1 = w_5$, $w_3 = w_7$. If now $t_p \geqslant 8$ ($t'_p \geqslant 6$), then as in Case 2.3 we may write T'_p as $(v_1, \dots, v_{t'_p+1})$ with $v_1, v_5 \in X^1_{1,a-1}$ and $v_1 \neq v_5$. Moreover, since we can permute the vertices of $K_{a-1,c}$, we may assume $w_1 = v_1$ and $w_7 = v_5$. In such a case, if we denote $T_i = (v_1, v_2, v_3, v_4, v_5, w_8, \dots, w_{t'_i+1})$ and $T_p = (v_1, w_2, w_3, w_4, w_5, w_6, v_5, \dots, v_{t'_p+1})$, then T_i and T_p are closed trails of lengths t_i and t_p. If $t_p = 6$ (it follows that $t_{i+1} = 6$) and $t_1 = 4$, then $s_i - t_1 = (a-1)c - 6$, $s_{i+1} - t_1 = (a-1)c$ and we continue the proof the same way as in Case 2.1 (we build up a G-realization of τ from a G_1-realization of $(t_2, \dots, t_{i+1})$ and a G_2-realization of $(t_1, t_{i+2}, \dots, t_p)$). The last missing case is $t_1 = \cdots = t_p = 6$, but then $(a-1)c \equiv 2 \pmod 6$, $(a-1)(a+c) \equiv 0 \pmod 6$ and $a(a-1) \equiv 4 \pmod 6$, a contradiction.

Case 3: $a \geqslant 5$ and $c = 2$. Notice that there exists a unique F in $K_{a,a+2}$ and $E(F) = \{x_1^1x_1^2, x_2^1x_2^2, \dots, x_a^1x_a^2, x_a^1x_{a+1}^2, x_a^1x_{a+2}^2\}$.

Let $K'_{a,a+2} = K_{a,a+2} - ((a-1)K_2 \cup K_{1,3})$.

Assume first that $t_j \equiv 0 \pmod 4$ for all j, it follows that there exists i such that $s_{i-1} \leqslant 2(a-1) - 4$ ($s_0 = 0$) and $s_i = 2(a-1)$ or $s_i \geqslant 2(a-1)+4$. Let $\tau_1 = (t_1, \dots, t_{i-1}, t'_i)$ and $\tau_2 = (t''_i, t_{i+1}, \dots, t_p)$, where $t'_i = 2(a-1) - s_{i-1} \geqslant 4$ and $t''_i = t_i - t'_i$. Notice that $t'_i, t''_i \equiv 0 \pmod 4$. Let us consider now G as a union of $G_1 = K_{a-1,2}$ and $G_2 = K_{a,a} - M_{2a}$ with bipartitions $\{X^1_{1,a-1}, X^2_{a+1,a+2}\}$ and $\{X^1_{1,a}, X^2_{1,a}\}$, respectively. Now using the same arguments as in Case 2.1 ($t''_i = 0$) or 2.2 ($t''_i \geqslant 4$) we obtain a G-realization of τ.

Suppose now that there exists j with $t_j \equiv 2 \pmod 4$. Notice that we can consider the graph G also as a union of graphs $G_1 = G_{a,5}$ and $G_2 = K'_{a-2,a}$ with the bipartitions $\{X^1_{1,a}, X^2_{a-2,a+2}\}$ and $\{X^1_{1,a}, X^2_{1,a-2}\}$, respectively. We will argue now by the induction hypothesis. By Case 1 the graph $K'_{3,5}$ is arbitrarily decomposable into closed trails. Let $a \geqslant 5$ and consider the following subcases:

Case 3.1: For some i, $s_i = \|G_1\| = 4(a-1)+2$. Then we can find a realization of $\tau_1 = (t_1, \dots, t_i)$ in G_1 by Lemma 4 and then decompose G_2 into closed trails of lengths $t_{i+1}, \dots, t_p$ by the induction hypothesis.

Case 3.2: For some i, $s_{i-1} \leqslant 4(a-1) - 2$ and $s_i \geqslant 4(a-1)+6$ ($s_0 = 0$). Let $\tau_1 = (t_1, \dots, t_{i-1}, t'_i)$ and $\tau_2 = (t''_i, t_{i+1}, \dots, t_p)$, where $t'_i = 2(a-1)+2-s_{i-1} \geqslant 4$ and $t''_i = t_i - t'_i \geqslant 4$. Then we may find a G_2-realization $(T''_i, T_{i+1}, \dots, T_p)$ of τ_2 by induction. Notice that if there exists a vertex $v \in V(T''_i)$ such that $v \in X^1_{a-2,a}$ then we can permute the set of vertices of G_2 such that $v = x^1_{a-2}$, whereas if $v \in X^1_{1,a-3}$, then we permute the set of vertices of G_2 such that $v = x^1_{a-3}$.

Now by Lemma 4 we decompose G_1 into closed trails $T_1, \dots, T_{i-1}, T'_i$ of lengths $t_1, \dots, t_{i-1}, t'_i$ so that we determine if either $x^1_{a-2} \in V(T'_i)$ or $x^1_{a-3} \in V(T'_i)$. Letting $T_i = T'_i \cdot T''_i$ we obtain a G-realization of τ.

Case 3.3: For some i, $s_i = 4(a-1)+4$. Assume first that $t_i = 4$, then obviously $t_{i-1} = 4$ and there exists r with $t_r \equiv 2 \pmod 4$. It follows that $s_{i-2} = 4(a-1)-4$ and $s_{i-2} + t_r = 4(a-1)+2$ or $s_{i-2} + t_r \geqslant 4(a-1)+6$ and we continue the proof the same way as in Case 3.1 or Case 3.2, respectively.

Suppose $t_i \geqslant 6$. If $t_p > t_i$ then $s_{i-1} + t_p \geqslant 4(a-1)+6$ and we can use similar arguments as in Case 3.2. From now on we consider the case where

$t_i = \cdots = t_p = t \equiv 2 \pmod 4$ (recall that not all t_i's are divisible by 4). If there is $l \in \{1, \ldots, i-1\}$ with $6 \leqslant t_l \leqslant t-2$, then $s_i - t_l \leqslant 4(a-1) - 2$ and $s_{i+1} - t_l \geqslant 4(a-1) + 6$. Henceforth we assume that $t_k \in \{4, t\}$ for $k = 1, \ldots, i$. Suppose first that $t_2 = 4$. If $t = 6$, then $\sum_{k=3}^{i+1} t_k = 4(a-1) + 2$. If $t \geqslant 10$, then $\sum_{k=3}^{i} t_k = 4(a-1) - 4 \leqslant 4(a-1) - 2$ and $\sum_{k=3}^{i+1} t_k \geqslant 4(a-1) + 6$. In the remaining case we have $t_2 = t$, $t_1 \in \{4, t\}$ and there is $d \in \{0, 4\}$ such that $4a \equiv d \pmod t$ and $(a-1)(a+2) = a^2 + a - 2 \equiv d \pmod t$. Consequently, $4a = d + \alpha t$ and $a^2 + a - 2 = d + \beta t$ for some $\alpha, \beta \in \mathbb{Z}$, $(d + \alpha t)^2 + 4(d + \alpha t) - 32 = 16(d + \beta t)$, $t(\alpha^2 t + 2\alpha d + 4\alpha - 16\beta) = 32 + 12d - d^2 \in \{32, 64\}$ and $t|32$ in contradiction with $t \equiv 2 \pmod 4$.

Case 3.4: For some i, $s_i = 4(a-1)$. Let $t = t_1$. If $t_p \geqslant t + 6$, then $s_i - t_1 = 4(a-1) - t \leqslant 4(a-1) - 2$ and $s_i - t_1 + t_p \geqslant 4(a-1) + 6$. Suppose therefore $t_p \leqslant t + 4$. If there is $l \in \{1, \ldots, i\}$ with $t_l = t_p - 2$, then $s_i - t_l + t_p = 4(a-1) + 2$. If there is $l \in \{i+1, \ldots, p\}$ with $t_l = t + 2$, then $s_i - t_1 + t_l = 4(a-1) + 2$. Henceforth we assume that $t_k \in \{t, t+4\}$ for any k and, consequently, $t \equiv 2 \pmod 4$. If $t_2 = t$ and $t_{p-1} = t + 4$, then $s_i - t_1 - t_2 + t_{p-1} = s_i - t + 4 \leqslant 4(a-1) - 2$ and $s_i - t_1 - t_2 + t_{p-1} + t_p = 4(a-1) + 8 \geqslant 4(a-1) + 6$. Thus, we are left with the cases $\tau = (t, (t+4)^{p-1})$, $\tau = (t^{p-1}, t+4)$ and $\tau = (t^p)$. If $\tau = (t^{p-1}, t+4)$, then $4(a-1) \equiv 0 \pmod t$, $(a-1)(a+2) \equiv 4 \pmod t$, $t = 2(2q+1)$ with $q \in \mathbb{Z}$, $(2q+1)|(a-1)|(a-1)(a+2)$, $(2q+1)|t|[(a-1)(a+2) - 4]$ and $(2q+1)|4$, a contradiction. If $\tau = (t, (t+4)^{p-1})$, then $4(a-1) \equiv t \pmod{t+4}$, $(a-1)(a+2) = a^2 + a - 2 \equiv t \pmod{t+4}$, there are $\alpha, \beta \in \mathbb{Z}$ with $4a = (\alpha+1)(t+4)$, $a^2 + a + 2 = (\beta+1)(t+4)$, $[(\alpha+1)(t+4)]^2 + 4(\alpha+1)(t+4) + 32 = 16(\beta+1)(t+4)$, $(t+4)[16(\beta+1) - (\alpha+1)^2(t+4) - 4(\alpha+1)] = 32$ and $(t+4)|32$ in contradiction with $t \equiv 2 \pmod 4$.

For $\tau_3 = (t^p)$ we obtain $a \equiv 1 \pmod t$.

Assume first that $t \geqslant 10$. Let us introduce $t'_i = t + 2$, $t'_{i+1} = t - 2$. Let $(T_1, \ldots, T_{i-1}, T'_i)$ and $(T'_{i+1}, T_{i+2}, \ldots, T_p)$ be G_1 and G_2-realizations of $\tau_1 = (t_1, \ldots, t_{i-1}, t'_i)$ and $\tau_2 = (t'_{i+1}, t_{i+1} \ldots, t_p)$.

We can write T'_{i+1} as $(v_1, \ldots, v_{t'_{i+1}+1})$ with $v_1, v_3 \in X^1_{1,a}$. Notice that we can permute the set of vertices of G_2 such that if $v_1, v_3 \in X^1_{a-2,a}$ then $v_1 = x^1_{a-2}, v_3 = x^1_a$, if $v_1, v_3 \in X^1_{1,a-3}$ then $v_1 = x^1_{a-3}, v_3 = x^1_1$, if $v_1 \in X^1_{1,a-3}$ and $v_3 \in X^1_{a-2,a}$ then $v_1 = x^1_1, v_3 = x^1_{a-1}$. By Lemma 4 we can write T'_i as $(w_1, \ldots, w_{t'_i+1})$ with $w_1, w_5 \in X^1_{1,a}$, $w_1 \neq w_5$ and $w_1 = v_1$, $w_5 = v_3$. In such a case, if we denote $T_i = (v_1, v_2, v_3, w_6, \ldots, w_{t'_i+1})$ and $T_{i+1} = (v_1, w_2, w_3, w_4, v_3, \ldots, v_{t'_{i+1}+1})$, then T_i and T_{i+1} are closed trails of lengths t.

If $t = 6$, then the sequence $\tau = (6^9)$ is realizable in $K'_{7,9}$, see Figure 3.

For $a \geqslant 13$ we can consider the graph $G = K'_{a,a+2}$ as an edge disjoint union of graphs $G_3 = K'_{7,9}$, $G_4 = K'_{a-6,a-6}$, $G_5 = K_{a-7,8}$ and $G_6 = K_{6,a-7}$ with the bipartitions $\{X^1_{a-6,a}, X^2_{a-6,a+2}\}$, $\{X^1_{1,a-6}, X^2_{1,a-6}\}$, $\{X^1_{1,a-7}, X^2_{a-5,a+2}\}$ and $\{X^1_{a-5,a}, X^2_{1,a-7}\}$ respectively. Graphs G_4, G_5, G_6 are decomposable into

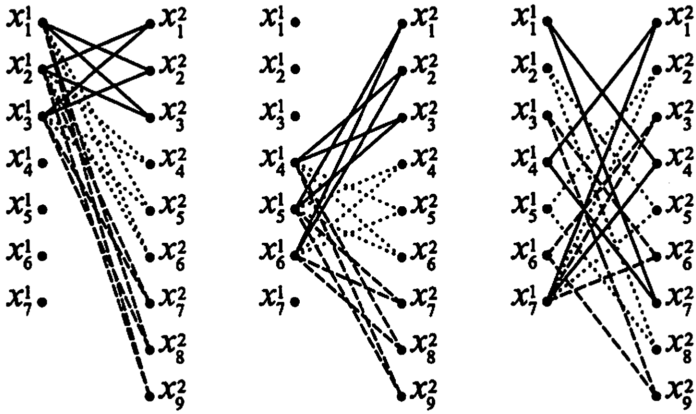

Figure 3: The $K'_{7,9}$-realization of $\tau = (6^9)$.

closed trails of lengths 6 (by Theorems 1 or 3) so the sequence $\tau = (6^p)$ is G-realizable. ■

Notice that the Theorem 5 is not generally true for all smallest spanning subgraphs with all degrees odd. For instance, consider $G = K_{3,b} - F$ with $b \geq 9$ odd, where $E(F) = \{x_1^1x_1^2, x_2^1x_2^2\} \cup \{x_3^1x_i^2 : i = 3, 4, \dots, n\}$. Then $\|G\| = 2n$ and the sequence $\tau = (6, 6, 2n - 12)$ is admissible for G, but is not realizable in G.

Problem 6 *Characterize all spanning subgraphs F with all degrees odd for which $K_{a,b} - F$ is arbitrarily decomposable into closed trails.*

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