Triangle-Free Graphs

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Abstract

We address the problem: for which values of d and n does there exist a triangle-free regular graph of degree d on n vertices? A complete solution is given.

1 Introduction

We assume the standard ideas of graph theory; in particular, a graph is called *regular* if every vertex has the same degree, and *triangle-free* if it contains no 3-cycle as a subgraph — that is, there are no three mutually adjacent vertices. We shall refer to a regular graph of degree d, with n vertices, as a (d, n)-graph.

We wish to address the problem: for which values of d and n does there exist a triangle-free (d, n)-graph?

2 Even order

There is a very easy upper bound for d.

Theorem 1 *If G is a (d, n)-graph, $n < 2d$, then G contains a triangle.*

Proof. Write $N(x)$ for the set of vertices adjacent to x (the *open neighborhood* of x). Suppose G is triangle-free. If x and y are adjacent vertices, then they can have no common neighbor, so $N(x)\backslash\{y\}$ and $N(y)\backslash\{x\}$ are disjoint sets of size $d-1$. So G has at least $2(d-1)+2 = 2d$ vertices. □

So a triangle-free (d, n)-graph satisfies $d \leq n/2$.

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On the other hand, suppose n is even — say $n = 2m$ — and $d \leq m = n/2$. A regular bipartite graph of degree d is easily constructed: the following example will be useful later. The vertices are $\{x_1, x_2, \ldots, x_n\}$. If $1 \leq i \leq m$ the neighbors of x_i are $x_{m+i+1}, x_{m+i+2}, \ldots, x_{m+i+d}$, with subscripts reduced modulo m to the range $(m+1 \ldots 2m)$; this rule describes all the edges. This regular bipartite graph, which we shall denote $K_{m,m}^{[d]}$, is obviously triangle-free. So:

Theorem 2 *There is a triangle-free (d, n)-graph whenever $d \leq n/2$, for even n.* □

3 Odd order

For odd n the lower bound can be refined, using the following result of Andrásfai, Erdös and Sòs ([1], Theorem 1.1):

Theorem 3 *If a graph G on n vertices is K_r-free with minimal degree greater than*

$$\frac{3r-7}{3r-4}n$$

then G is $(r-1)$-chromatic. □

When $r = 3$, this says that a triangle-free n-vertex graph with minimum degree greater than $\frac{2}{5}n$ can be 2-colored. A 2-coloring induces a bipartition of the vertices; in a regular bipartite graph the two parts must be of equal size. (If the parts contain n_1 and n_2 vertices then the number of edges equals $n_1 d$, and also equals $n_2 d$.) So

Theorem 4 *When n is odd, there is no triangle-free (d, n)-graph with $d > 2n/5$.* □

Putting it the other way, any triangle-free (d, n)-graph satisfies $n \geq 5d/2$.

We now prove that this bound is tight. The proof splits into three cases, which we present as separate Lemmas.

Lemma 4.1 *When d is even, there is a triangle-free (d, n)-graph when $n = 5d/2 + 2t, 0 \leq t \leq d/2$.*

Proof. For convenience, write $d = 2k$. We construct a graph with $5k + 2t$ vertices, where t is a non-negative integer and $t \leq k$. The vertices comprise five sets X_1, X_2, X_3, X_4, X_5 of size k and two sets Y_1, Y_2 of size t. There are edges joining each member of Y_1 to every member of $X_1 \cup X_3$, joining each member of Y_2 to every member of $X_2 \cup X_4$, and joining each member of X_5 to every member of $X_1 \cup X_4$. To these are added the edges of a $K_{k,k}$ with bipartition $\{X_2, X_3\}$, a $K_{k,k}^{[k-t]}$ with bipartition $\{X_1, X_2\}$, and a $K_{k,k}^{[k-t]}$ with bipartition $\{X_3, X_4\}$. □

When d is a multiple of 4, $5d/2$ is even, and n is even, so the above construction is not interesting; however, only some small modifications are required:

Lemma 4.2 *When d is even, there is a triangle-free (d,n)-graph when $n = 5d/2 + 2t + 1, 0 \leq t \leq d/2$.*

Proof. Again we write $d = 2k$. Our graph has $5k + 2t + 1$ vertices comprising four sets X_1, X_2, X_3, X_4 of size k, a set X_5 of size $k-1$ and two sets Y_1, Y_2 of size $t + 1$. There are edges joining each member of Y_1 to every member of $X_1 \cup X_3$, joining each member of Y_2 to every member of $X_2 \cup X_4$, and joining each member of X_5 to every member of $X_1 \cup X_4$. To these are added the edges of a $K_{k,k}^{[k-1]}$ with bipartition $\{X_2, X_3\}$ and $K_{k,k}^{[k-t]}$'s with bipartitions $\{X_1, X_2\}$ and $\{X_3, X_4\}$. □

Lemma 4.3 *When d is even, there is a triangle-free (d,n)-graph whenever $d \leq (n+1)/3$.*

Proof. Suppose H is an n-cycle $(v_1, v_2, \ldots, v_n), n$ odd. Write H^i for the graph formed by joining v_j to v_{j+i} for every j, with subscripts reduced modulo n as necessary. (In other words, H^i is the union of the cycles

$$(v_1, v_{1+i}, v_{1+2i}, \ldots), (v_2, v_{2+i}, v_{2+2i}, \ldots), \ldots$$

— a single cycle when n and i are coprime. Then

$$H \cup H^3 \cup H^5 \cup \ldots \cup H^{d-1}$$

is triangle-free for even $d \leq (n+1)/3$. □

This covers all the cases outside the range of the two earlier lemmas. So:

Theorem 5 *When n is odd, there is a triangle-free (d,n)-graph for all even $d \leq \frac{1}{3}n + 1$.* □

4 Variability of triangle-free graphs

The graphs constructed in Lemmas 4.1 and 4.2 are never isomorphic to those of Lemma 4.3 (except for the trivial case $d = 2, n = 5$, where the graph is a 5-cycle). In the graphs of Lemma 4.3, no two vertices ever have the same open neighborhood. In the $(4, 11)$-graph of Lemma 4.2, the members of X_1 have the same neighbors (as do the members of X_4); in all other cases, the members of X_5 have this property.

Meringer[2] has tabulated small regular graphs with given minimum girth; of course, "girth at least 4" means "triangle-free". Results in this section that cite the exact number of isomorphism classes are taken from that web page.

For given n and d, how many triangle-free (d, n)-graphs exist? We only know the answer in three cases. Up to isomorphism there are precisely two triangle-free $(4, 11)$-graphs, one triangle-free $(6, 15)$-graph and six triangle-free $(6, 17)$-graphs.

Figure 1 shows the two graphs cited in [2]; the one in the left is the one constructed in Lemma 4.2 while the one on the right comes from Lemma 4.3.

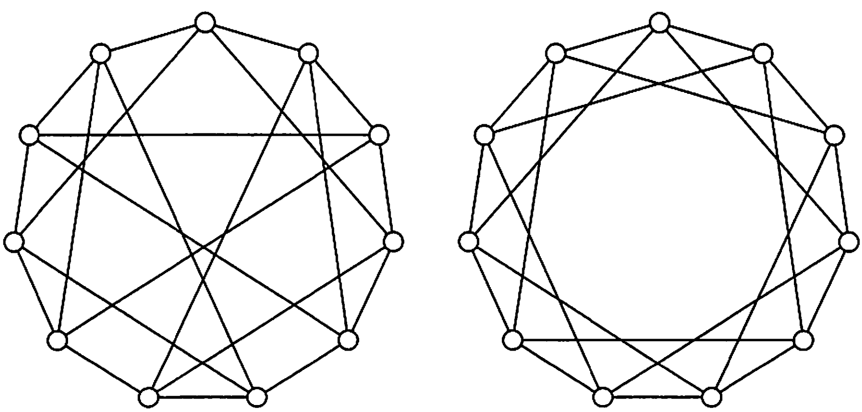

Figure 1: The two triangle-free $(11, 4)$-graphs

The numbers of classes increase rapidly; there are 31 classes of triangle-free $(4, 13)$ graphs and 1606 classes of triangle-free $(4, 15)$-graphs. The number of triangle-free $(6, 19)$-graphs is not known.

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References

[1] B. Andrásfai, P. Erdös and V. T. Sòs, On the connection between chromatic number, maximal clique and minimal degree of a graph. *Discrete Math.* 8, (1974), 205–218.

[2] M. Meringer, http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html