Neighbor-Distinguishing Vertex Colorings of Graphs

Gary Chartrand
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008

Futaba Okamoto
Mathematics Department
University of Wisconsin - La Crosse
La Crosse, WI 54601

Ping Zhang
Department of Mathematics
Western Michigan University
Kalamazoo, MI 49008

Abstract

Recently four new vertex colorings of graphs (in which adjacent vertices may be colored the same) were introduced for the purpose of distinguishing every pair of adjacent vertices. For each graph and for each of these four colorings, the minimum number of required colors never exceeds the chromatic number of the graph. In this paper, we summarize some of the results obtained on these colorings and introduce some relationships among them.

Key Words: neighbor-distinguishing coloring, set coloring, metric coloring, multiset coloring, sigma coloring.

AMS Subject Classification: 05C15, 05C20.

1 Introduction

The subject of graph colorings goes back to 1852 when the young British mathematician Francis Guthrie observed that the counties in a map of England could be colored with four colors so that every two adjacent counties are colored differently. This led to the Four Color Problem of determining whether the regions of *every* plane map could be colored with four or fewer colors in such a way that every two adjacent regions are colored differently. Of course, the Four Color Problem has an affirmative solution, as was announced in 1976 by Kenneth Appel and Wolfgang Haken. As a consequence of the resulting Four Color Theorem, it is possible to distinguish every two adjacent regions of every plane map M by coloring the regions of M with at most four colors.

For example, consider the map M of Figure 1(a). By the Four Color Theorem, there is a proper coloring of the regions of M with four colors, say 1, 2, 3, 4, that is, adjacent regions are colored differently. Such a coloring is shown in Figure 1(b). Therefore, every proper coloring of a map distinguishes every pair of adjacent regions. A different coloring of M is given in Figure 1(c), using the colors 1, 2, 3. Here too every two adjacent regions of M are distinguished from each other. In this case, the sets of colors of the neighboring regions of every two adjacent regions of M are different. A third coloring of M is given in Figure 1(d), using the colors 1 and 2. In this case as well, every two adjacent regions of M are distinguished from each other, where here the sums of the colors of the neighboring regions of every two adjacent regions of M are different.

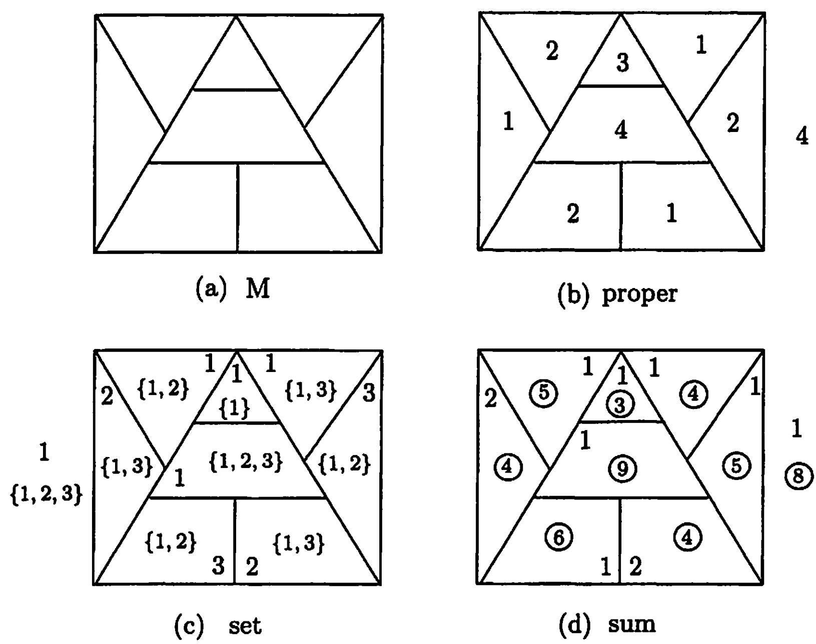


Figure 1: Three colorings of the regions of a map M

Figure 1 therefore shows that it is possible to color the regions of a map M with fewer colors than that required of a proper coloring and still distinguish every two adjacent regions. As a second example, consider the map M shown in Figure 2(a). Here, as well, the regions of M can be properly colored with four colors, as shown in Figure 2(b), but with no fewer colors.

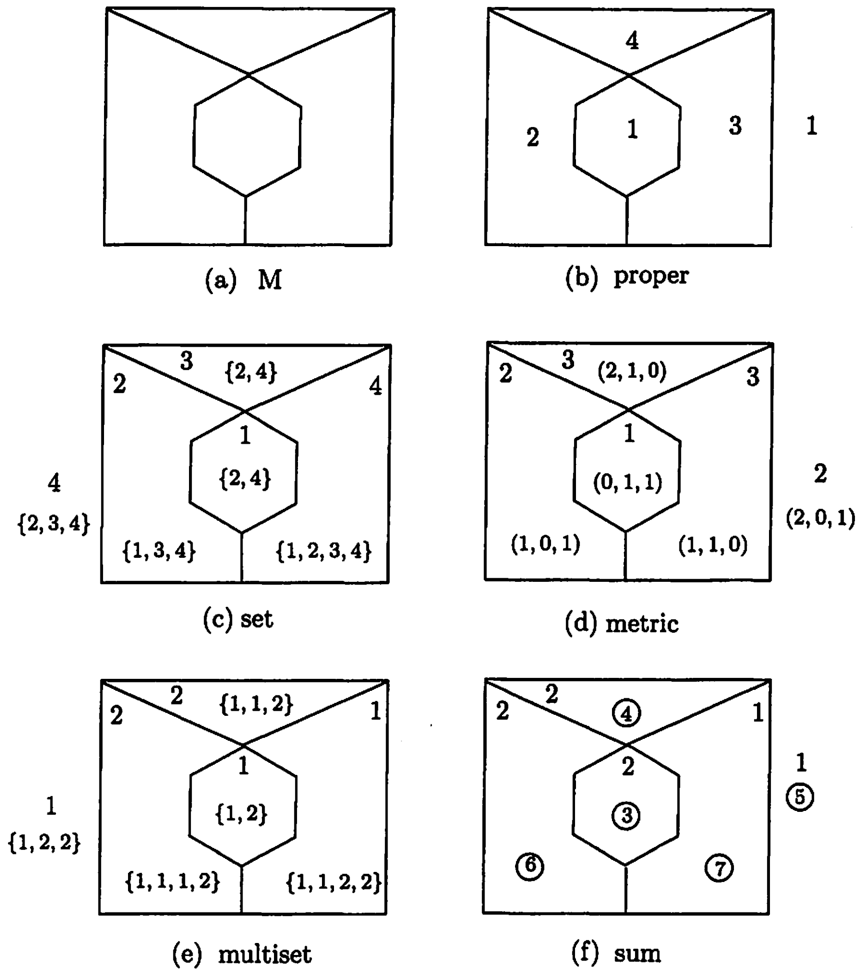


Figure 2: Four colorings of the regions of a map M

While there exists a 4-coloring of the regions of M shown in Figure 2(c) so that the sets of colors of the neighboring regions of every two adjacent regions of M are different, there is no such 3-coloring and consequently there is no improvement in the number of colors needed for this map. On the other hand, there is a 3-coloring of the regions of M (using the colors 1, 2, 3), as shown in Figure 2(d), such that for every two adjacent regions of M, their distances to a nearest region of some color are not the same. Each region of M is labeled with a triple (a_1, a_2, a_3), where a_i $(1 \le i \le 3)$ is the distance from that region to a nearest region colored i. Thus adjacent regions can be distinguished by this coloring. There is no such 2-coloring that accomplishes this however. On the other hand, there does exist a 2-coloring of the regions of M with the colors 1 and 2, as shown in Figure 2(e), so that the multisets of the colors of the neighboring regions of every two adjacent regions of M are different. If the colors of the neighboring regions were summed, then we do not distinguish every two adjacent regions of M by this coloring. However, if we were to replace the color 1 by 2 in the centermost region, then the sums of the colors of the neighboring regions are different for every two adjacent regions of M and once again adjacent regions of M are distinguished by this coloring.

The four types of colorings of the regions of a map that we have just described can be used to distinguish every pair of adjacent regions. These colorings give rise to four vertex colorings of graphs that can be used to distinguish every pair of adjacent vertices, that is, all four vertex colorings are *neighbor-distinguishing*. These four vertex colorings are described in the four succeeding sections where previous results on them are summarized and new results comparing these parameters are presented.

2 Set Colorings

Before discussing the first of the four neighbor-distinguishing vertex colorings described in the introduction, it is convenient to introduce some additional notation. Let $\mathbb{N}$ denote the set of natural numbers (positive integers) and for a positive integer k, let

$$\mathbb{N}_k = \{1, 2, \ldots, k\}.$$

For a vertex coloring $c : V(G) \to \mathbb{N}$ of a graph G and a set $S \subseteq V(G)$, define $c(S)$ as the set of colors assigned to the vertices of S by c, that is,

$$c(S) = \{c(v) : v \in S\}.$$

We refer to the book [6] for graph theory notation and terminology not described in this paper.

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be assigned the same color. For a vertex v in a graph G, let $N(v)$ be the neighborhood of v (the set of all vertices adjacent to v in G). The *neighborhood color set* $\mathrm{NC}(v) = c(N(v))$ is the set of colors of the neighbors of v. The coloring c is called a *set coloring* if $\mathrm{NC}(u) \neq \mathrm{NC}(v)$ for every pair u, v of adjacent vertices of G. The minimum number of colors required of such a coloring is called the *set chromatic number* of G and is denoted by $\chi_s(G)$. These concepts were introduced and studied in [2] and studied further in [7, 8]. For example, consider the graph G of Figure 3, where a set 3-coloring of G is also shown together with the neighborhood color set of each vertex of G. While the chromatic number of G is 4, its set chromatic number is 3.

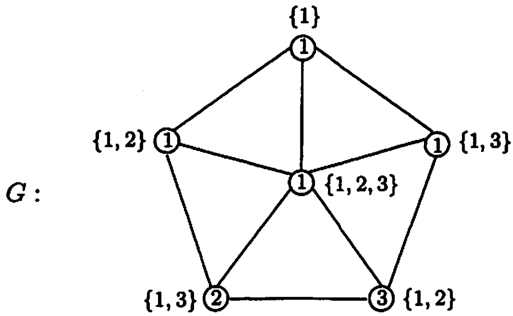


Figure 3: A 4-chromatic graph G with set chromatic number 3

Let c be a proper k-coloring of a k-chromatic graph G. Suppose that u and v are adjacent vertices of G. Since $c(u) \in \text{NC}(v)$ and $c(u) \notin \text{NC}(u)$, it follows that $\text{NC}(u) \neq \text{NC}(v)$. Hence every proper k-coloring of G is also a set k-coloring of G. Therefore, for every connected graph G of order n,

$$1 \leq \chi_s(G) \leq \chi(G) \leq n. \qquad (1)$$

Among the results obtained in [2] is the following.

Theorem 2.1 [2] *A graph G has set chromatic number 2 if and only if G is bipartite.*

An immediate consequence of Theorem 2.1 is stated next.

Theorem 2.2 [2] *If G is a 3-chromatic graph, then $\chi_s(G) = 3$.*

Of course, every complete k-partite graph has chromatic number k. This is also true for the set chromatic number.

Theorem 2.3 [2] *For every complete k-partite graph G, $\chi_s(G) = k$.*

By Theorem 2.3, the complete k-partite graph $K_{1,1,\ldots,1,n-(k-1)}$ has set chromatic number k. Thus every pair k, n of integers with $2 \leq k \leq n$ can be realized as the set chromatic number and the order, respectively, of some connected graph. Furthermore, $\chi_s(G) = 1$ if and only if $G = K_1$. Therefore, we have the following.

Theorem 2.4 *For positive integers k and n, there is a connected graph G of order n with $\chi_s(G) = k$ if and only if $k = n = 1$ or $2 \le k \le n$.*

For a k-chromatic graph G of order n, there are certain values of k which also imply that $\chi_s(G) = k$ as well.

Theorem 2.5 [2] *If G is a connected graph of order n such that $\chi(G) \in \{1, 2, 3, n-1, n\}$, then $\chi_s(G) = \chi(G)$.*

The *clique number* $\omega(G)$ of a graph G is the order of a largest clique (complete subgraph) in G. While the clique number of G is a lower bound for the chromatic number of G, this is not the case for the set chromatic number.

Theorem 2.6 [2] *For every graph G,*

$$\chi_s(G) \ge 1 + \lceil \log_2 \omega(G) \rceil .$$

The lower bound given in Theorem 2.6 for the set chromatic number of a graph was shown to be sharp in [2]. Figure 4 shows a graph G with $\omega(G) = 4$ and $\chi_s(G) = 3$, and so $\chi_s(G) = 3 = 1 + \lceil \log_2 4 \rceil$.

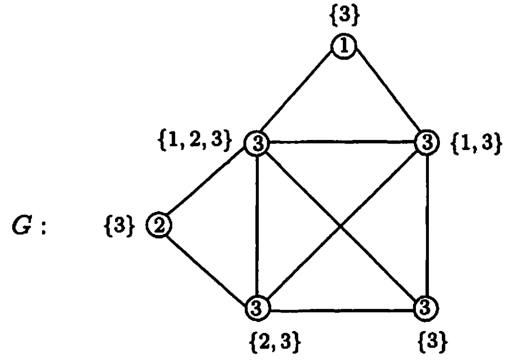


Figure 4: A graph G with $\chi_s(G) = 1 + \lceil \log_2 \omega(G) \rceil$.

It is well known that if v is a vertex of a nontrivial graph G, then either $\chi(G - v) = \chi(G)$ or $\chi(G - v) = \chi(G) - 1$, which is also the case when an edge is deleted from a nonempty graph G. For the set chromatic number, a much different situation can occur.

Theorem 2.7 [2] *If v is a vertex of a nontrivial graph G, then*

$$\chi_s(G) - 1 \le \chi_s(G - v) \le \chi_s(G) + \deg v.$$

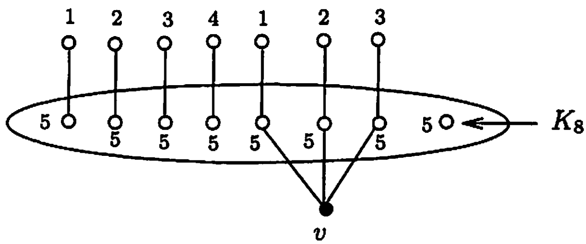


Figure 5: A graph G with a vertex v such that $\chi_s(G-v) = \chi_s(G) + \deg v$

The upper and lower bounds for the set chromatic number of a graph in Theorem 2.7 are both sharp. Figure 5 shows a graph G with $\chi_s(G) = 5$ and having a vertex v of degree 3 such that $\chi_s(G-v) = 8 = \chi_s(G) + \deg v$.

Theorem 2.8 [2] *If e is an edge of a nonempty graph G, then*

$$|\chi_s(G) - \chi_s(G-e)| \leq 2.$$

In the case of Theorem 2.8, however, it is not known if there is a graph G and an edge e of G such that $|\chi_s(G) - \chi_s(G-e)| = 2$. However if $e = uv$ is not a bridge in G such that the distance between u and v in $G-e$ is at least 4, then

$$|\chi_s(G) - \chi_s(G-e)| \leq 1.$$

Figure 6 shows a 5-chromatic graph G with $\chi_s(G) = 4$ and three edges e_{-1}, e_0, and e_1 such that $\chi_s(G-e_i) = \chi_s(G) + i$ for $i \in \{-1, 0, 1\}$.

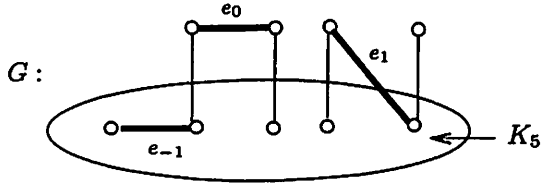


Figure 6: A graph G with $\chi_s(G-e_i) = \chi_s(G) + i$ for $i \in \{-1, 0, 1\}$

While it is not known whether there is a graph G with $\chi_s(G) = a$ and $\chi(G) = b$ for *all* pairs a, b of integers with $2 \leq a \leq b$, it is known if $a \geq 1 + \log_2 b$. Should there exist a graph G with $\chi_s(G) = a$ and $\chi(G) = b$ where $a \geq 3$ and $a < 1 + \log_2 b$, then it follows by Theorem 2.6 that $\omega(G) < b$.

Theorem 2.9 [7] *For each pair a, b of integers with $2 \leq a \leq b \leq 2^{a-1}$, there exists a connected graph G with $\chi_s(G) = a$ and $\chi(G) = b$.*

3 Metric Colorings

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path. For a set $S \subseteq V(G)$ and a vertex v of G, the *distance* $d(v, S)$ *between* v *and* S is defined as

$$d(v, S) = \min\{d(v, x) : x \in S\}.$$

Then $0 \leq d(v, S) \leq \text{diam}(G)$, where $d(v, S) = 0$ if and only if $v \in S$. Suppose that $c : V(G) \to \mathbb{N}_k$ is a k-coloring of G for some positive integer k where adjacent vertices may be colored the same and let $V_1, V_2, \ldots, V_k$ be the resulting color classes. A k-vector called the *metric color code* can be associated with each vertex v of G, which is denoted by $\text{code}_\mu(v)$ and defined by

$$\text{code}_\mu(v) = (a_1, a_2, \cdots, a_k) = a_1 a_2 \cdots a_k,$$

where for each integer i with $1 \leq i \leq k$, $a_i = d(v, V_i)$. If $\text{code}_\mu(u) \neq \text{code}_\mu(v)$ for every two adjacent vertices u and v of G, then c is called a *metric coloring* of G. The minimum k for which G has a metric k-coloring is called the *metric chromatic number* of G and is denoted by $\mu(G)$. These concepts were introduced and studied in [4]. For example, the graph G of Figure 7 has chromatic number 4 and metric chromatic number 3. A metric 3-coloring of G is shown in Figure 7 together with the metric color code of each vertex of G.

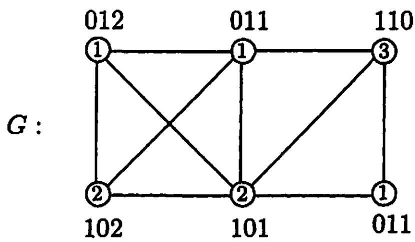


Figure 7: A 4-chromatic graph G with $\mu(G) = 3$

Let c be a proper k-coloring of a nontrivial connected graph G with resulting color classes $V_1, V_2, \ldots, V_k$ and let u and v be two adjacent vertices of G. Then $u \in V_i$ and $v \in V_j$ for some $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$. Suppose that $\text{code}_\mu(u) = (a_1, a_2, \cdots, a_k)$ and $\text{code}_\mu(v) = (b_1, b_2, \cdots, b_k)$. Then $a_i = b_j = 0$ and $a_j = b_i = 1$. Thus $\text{code}_\mu(u) \neq \text{code}_\mu(v)$ and so c is also a metric coloring of G. Consequently,

$$2 \leq \mu(G) \leq \chi(G) \leq n \tag{2}$$

for every nontrivial connected graph G of order n. We now describe a number of the results presented in [4]. As is the case with proper colorings, only bipartite graphs have metric 2-colorings.

Theorem 3.1 [4] *A nontrivial connected graph G has metric chromatic number 2 if and only if G is bipartite.*

The following is an immediate consequence of Theorem 3.1.

Corollary 3.2 [4] *Let G be a connected graph. If $\chi(G) = 3$, then $\mu(G) = 3$.*

The complete multipartite graphs also have equal chromatic number and metric chromatic number of a graph.

Theorem 3.3 [4] *For every complete k-partite graph G where $k \geq 2$, $\mu(G) = k$.*

The clique number is also not a lower bound for the metric chromatic number of a graph.

Theorem 3.4 [4] *For every nontrivial connected graph G,*

$$\mu(G) \geq 1 + \lceil \log_2 \omega(G) \rceil.$$

The lower bound for the metric chromatic number of a graph in Theorem 3.4 is sharp. Consider the graph G of order 11 shown in Figure 8 consisting of a complete subgraph H of order 8, where

$$V(H) = \{v_{ijk} : i, j, k \in \{0, 1\}\},$$

and three additional vertices x, y, and z, where x is adjacent to v_{ijk} if and only if $i = 1$, y is adjacent to v_{ijk} if and only if $j = 1$, and z is adjacent to v_{ijk} if and only if $k = 1$. (Many edges of G belonging to the subgraph H have been omitted in Figure 8.) By Theorem 3.4, $\mu(G) \geq 1 + \lceil \log_2 \omega(G) \rceil = 4$. Since the 4-coloring defined by $c(x) = 1$, $c(y) = 2$, $c(z) = 3$, and $c(v_{ijk}) = 4$ for all $i, j, k \in \{0, 1\}$ is a metric coloring, it follows that $\mu(G) = 4$.

While the removal of a vertex from a given graph can never result in a graph with a larger chromatic number than that of the given graph, this is not the case for the metric chromatic number.

Theorem 3.5 [4] *If v is a vertex that is not a cut-vertex of a connected graph G, then*

$$\mu(G - v) \leq \mu(G) + \deg v.$$

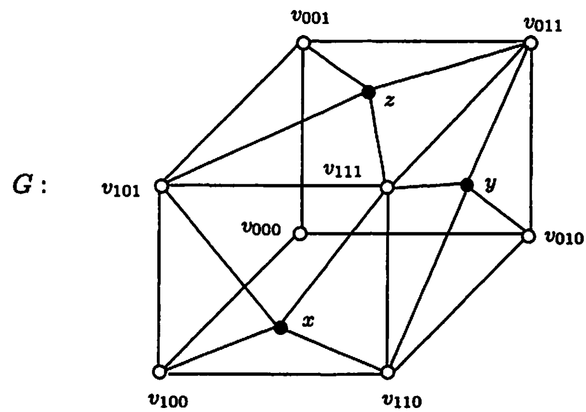


Figure 8: An 8-chromatic graph G with $\mu(G) = 1 + \lceil \log_2 \omega(G) \rceil = 4$

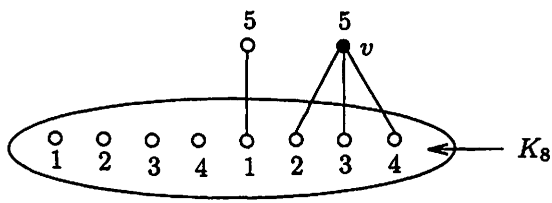


Figure 9: A graph G and a vertex v of G with $\mu(G - v) = \mu(G) + \deg v$

The upper bound for $\mu(G - v)$ in Theorem 3.5 is sharp. For example, Figure 9 shows a graph G and a vertex v with $\deg v = 3$ such that $\mu(G) = 5$ and $\mu(G - v) = 8 = \mu(G) + \deg v$. A metric 5-coloring of G is shown in Figure 9 as well.

Similar to Theorem 2.9 for set colorings, each pair a, b of integers with $2 \leq a \leq b$ can be realized as the metric chromatic number and chromatic number, respectively, of a connected graph under some restrictions for b in terms of a.

Theorem 3.6 [4] *For each pair a, b of integers with $2 \leq a \leq b \leq 2^{a-1}$, there exists a connected graph G with $\mu(G) = a$ and $\chi(G) = b$.*

Here too, it is not known if there is a graph G with $\mu(G) = a$ and $\chi(G) = b$ where $a \geq 3$ and $b > 2^{a-1}$. If such a graph G exists, then it follows by Theorem 3.4 that $\omega(G) < b$. In particular, it is not known if there is a 5-chromatic graph whose metric chromatic number is 3.

If c is a set k-coloring of a connected graph G, then $\mathrm{NC}(x) \neq \mathrm{NC}(y)$ for every two adjacent vertices x and y of G. Thus there is some color i that belongs to exactly one of NC(x) and NC(y). This implies that $\text{code}_\mu(x)$ and $\text{code}_\mu(y)$ differ in the i-th coordinate and so $\text{code}_\mu(x) \neq \text{code}_\mu(y)$. Thus c is also a metric coloring and so

$$\mu(G) \leq \chi_s(G). \tag{3}$$

Therefore, if G is a bipartite graph or a complete graph, then $\mu(G) = \chi_s(G)$. We next describe an infinite class of graphs G for which $\mu(G) < \chi_s(G)$.

For two graphs F and H, the composition $G = F[H]$ of F and H is obtained from F and H by replacing each vertex v of F by a copy H_v of H such that if x and y are two adjacent vertices in F, then each vertex in H_x is adjacent to every vertex in H_y in G. We now compute the metric chromatic numbers of all such graphs, where F is a cycle and H is a complete graph.

Theorem 3.7 *For $G = C_p[K_q]$, where $p \geq 3$ and $q \geq 1$,*

$$\mu(G) = \begin{cases} 2 & \text{if } p \text{ is even and } q = 1 \\ 3 & \text{if } p \text{ is odd and } q = 1 \\ 3q & \text{if } p = 3 \\ q+2 & \text{otherwise.} \end{cases}$$

Proof. If $q = 1$, then $G = C_p$; while if $p = 3$, then $G = K_{3q}$. Hence we may assume that $p \geq 4$ and $q \geq 2$. Let $C_p : u_1, u_2, \ldots, u_p, u_{p+1} = u_1$ be a p-cycle and let $H_1, H_2, \ldots, H_p$ be vertex-disjoint graphs, where $H_i = K_q$ and $V(H_i) = V_i = \{v_{i,1}, v_{i,2}, \ldots, v_{i,q}\}$ for $1 \leq i \leq p$. The graph $G = C_p[K_q]$ is constructed from $H_1, H_2, \ldots, H_p$ by joining v_{i_1,j_1} and v_{i_2,j_2} if and only if $u_{i_1}u_{i_2} \in E(C_p)$.

We show that $\mu(G) \leq q+2$ by defining a metric $(q+2)$-coloring on G. First consider the q-coloring $c : V(G) \to \mathbb{N}_q$ given by $c(v_{i,j}) = j$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Let $c_1 : V(G) \to \mathbb{N}_{q+2}$ be a coloring such that for $v \neq v_{p,q}$,

$$c_1(v) = \begin{cases} q+1 & \text{if } v \in \{v_{p-2,q}, v_{p-1,q}\} \\ q+2 & \text{if } v = v_{p-1,q-1} \\ c(v) & \text{otherwise} \end{cases}$$

and

$$c_1(v_{p,q}) = \begin{cases} q+1 & \text{if } p = 4 \\ q+2 & \text{if } p \geq 5. \end{cases}$$

Figure 10 shows the coloring c_1 for $C_4[K_4]$ and $C_5[K_4]$. Observe that c_1 is a metric coloring of G and so $\mu(G) \leq q+2$.

We next show that $\mu(G) \geq q+2$. If $c : V(G) \to \mathbb{N}_k$ is a metric coloring, then no two vertices in V_i can be colored the same for $1 \leq i \leq p$. Hence $k \geq q$. Since $\omega(G) = 2q$ and $q \geq 2$, it follows that

$$\chi(G) \geq 2q > q+1.$$

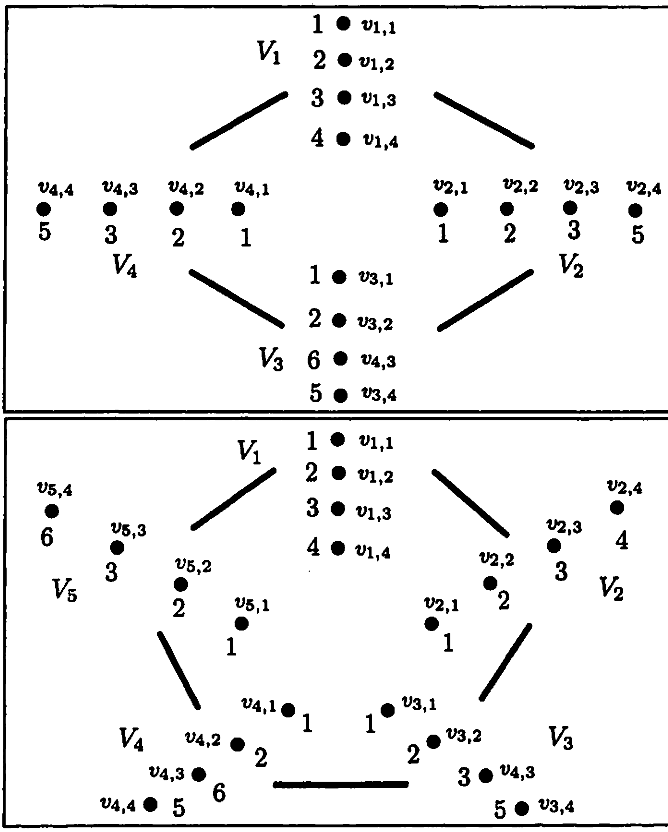


Figure 10: Metric colorings of $C_4[K_4]$ and $C_5[K_4]$

Hence if $k \leq q+1$, then there exists a pair of adjacent vertices that are assigned the same color, say $c(v_{1,1}) = c(v_{2,1}) = 1$. Furthermore, we may assume, without loss of generality, that $c(V_1) = \mathbb{N}_q$. If $k = q$, then $c(V_2) = \mathbb{N}_q$ and

$$\text{code}_\mu(v_{1,1}) = \text{code}_\mu(v_{2,1}) = (0, 1, 1, \cdots, 1),$$

which is a contradiction. On the other hand, if $k = q+1$, then $V_i \neq \mathbb{N}_q$ for some $i \geq 2$ and so we may assume that $V_2 = \mathbb{N}_{q+1} - \{q\}$. However then,

$$\text{code}_\mu(v_{1,1}) = \text{code}_\mu(v_{2,1}) = (0, 1, 1, \cdots, 1),$$

which is again a contradiction. Therefore, $k \geq q+2$ for every metric k-coloring of G. ■

We now determine the set chromatic numbers of the graphs $G = C_p[K_q]$ for even integers p.

Theorem 3.8 *For $G = C_p[K_q]$ where $p \geq 4$ is even and $q \geq 1$,*

$$\chi_s(G) = \chi(G) = 2q.$$

Proof. First assume that G has been constructed as described in the proof of Theorem 3.7. Since $\omega(G) = 2q$, it follows that $\chi(G) \geq 2q$. Since any coloring c of G for which

$$c(V_i) = \begin{cases} \mathbb{N}_q & \text{if } i \text{ is odd} \\ \mathbb{N}_{2q} - \mathbb{N}_q & \text{if } i \text{ is even} \end{cases}$$

is a proper $2q$-coloring, $\chi(G) = 2q$.

Now assume, to the contrary, that $\chi_s(G) = k < 2q$ and let $c : V(G) \to \mathbb{N}_k$ be a set k-coloring. No two vertices belonging to V_i $(1 \leq i \leq p)$ can be assigned the same color. Without loss of generality, let $c(V_1) = \mathbb{N}_q$. Since c is not a proper coloring, we may assume that $c(v_{1,1}) = c(v_{2,1}) = 1$. Note that $|c(V_1) \cap c(V_2)| \leq 1$, for otherwise, assume that $c(v_{1,2}) = c(v_{2,2})$. Then $\mathrm{NC}(v_{1,1}) = \mathrm{NC}(v_{1,2})$ (and $\mathrm{NC}(v_{2,1}) = \mathrm{NC}(v_{2,2})$), a contradiction. Since $|c(V_1) \cap c(V_2)| \leq 1$ and $k < 2q$, it follows that $k = 2q - 1$ and $c(V_1 \cup V_2) = \mathbb{N}_{2q-1}$. However, this implies that

$$\mathrm{NC}(v_{1,1}) = \mathrm{NC}(v_{2,1}) = \mathbb{N}_{2q-1},$$

which is a contradiction. Therefore, $2q \leq \chi_s(G) \leq \chi(G) = 2q$ and we obtain the desired result. ∎

The following result is a consequence of Theorems 3.7 and 3.8.

Corollary 3.9 *For each nonnegative integer ℓ, there exists a connected graph G such that $\chi_s(G) - \mu(G) = \ell$.*

We now describe an infinite class of graphs G for which $\mu(G) < \chi_s(G) < \chi(G)$.

Theorem 3.10 *For each integer $k \geq 3$, there exists a connected graph G such that*

$$\mu(G) = k, \ \ \chi_s(G) = 2^{k-2} + k - 1, \ \ \text{and} \ \ \chi(G) = 2^{k-1}.$$

Proof. Let $H = K_{2^{k-1}}$ with $V(H) = U \cup W$, where $U = \{u_1, u_2, \ldots, u_{2^{k-2}}\}$ and $W = \{w_1, w_2, \ldots, w_{2^{k-2}}\}$. Let $S_1, S_2, \ldots, S_{2^{k-2}}$ be the 2^{k-2} subsets of $\mathbb{N}_{k-2}$, where $S_1 = \emptyset$. A graph G is constructed from H by adding $k-2$ new vertices $v_1, v_2, \ldots, v_{k-2}$ to H and joining v_i to u_j and w_j if and only if $i \in S_j$ for $1 \leq i \leq k-2$ and $2 \leq j \leq 2^{k-2}$ (see Figure 11 for $k = 4$). Hence G is a graph of order $2^{k-1} + k - 2$ with $\omega(G) = \chi(G) = 2^{k-1}$.

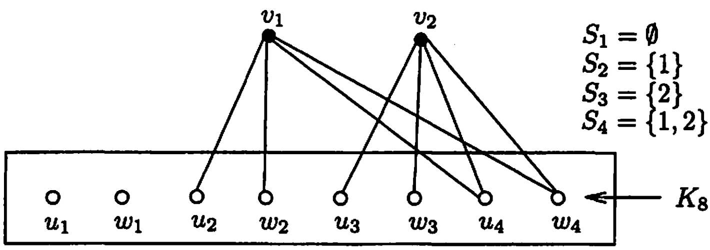


Figure 11: The graph G for $k = 4$

We first show that $\mu(G) = k$. That $\mu(G) \geq k$ follows by Theorem 3.4. Since the k-coloring $c_\mu : V(G) \to \mathbb{N}_k$ given by

$$c_\mu(x) = \begin{cases} i & \text{if } x = v_i \ (1 \leq i \leq k-2) \\ k-1 & \text{if } x \in U \\ k & \text{if } x \in W \end{cases}$$

is a metric coloring (see Figure 12 for $k = 4$), it follows that $\mu(G) = k$.

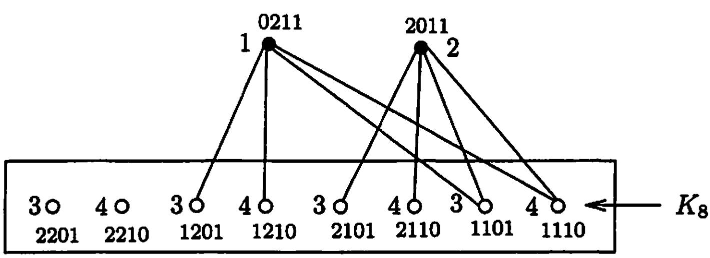


Figure 12: A metric 4-coloring of the graph G for $k = 4$

It remains to show that $\chi_s(G) = 2^{k-2} + k - 1$. Since the $(2^{k-2} + k - 1)$-coloring $c_s : V(G) \to \mathbb{N}_{2^{k-2}+k-1}$ defined by

$$c_s(x) = \begin{cases} i & \text{if } x = v_i \ (1 \leq i \leq k-2) \\ k-1 & \text{if } x \in U \\ k-1+i & \text{if } x = w_i \ (1 \leq i \leq 2^{k-2}) \end{cases}$$

is a set coloring, it follows that $\chi_s(G) \leq 2^{k-2} + k - 1$ (see Figure 13 for $k = 4$).

Assume, to the contrary, that there exists a set ℓ-coloring of G using the colors in $\mathbb{N}_\ell$, where $\ell \leq 2^{k-2} + k - 2$. Permuting the colors in $\mathbb{N}_\ell$ if necessary, we can obtain a set ℓ-coloring $c : V(G) \to \mathbb{N}_\ell$ such that $c(V(H)) = \mathbb{N}_{\ell'}$ for some $\ell' \leq \ell$. Since $2^{k-2} + k - 2 < 2^{k-1}$ for $k \geq 3$, it follows that c is not a

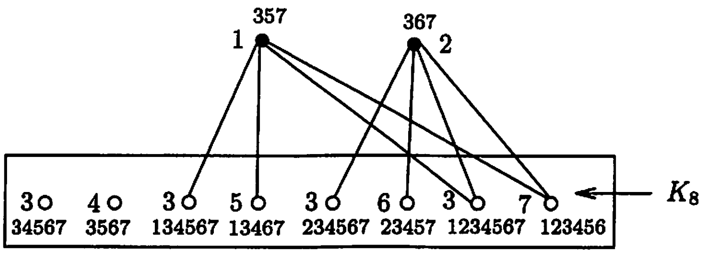


Figure 13: A set 7-coloring of the graph G for $k = 4$

proper coloring. Because the remaining $\ell - \ell'$ colors are used for the $k - 2$ vertices $v_1, v_2, \ldots, v_{k-2}$, it follows that $0 \le \ell - \ell' \le k - 2$. Let X be the subset of $V(H)$ such that for every $x \in X$ there exists a vertex $y \in X - \{x\}$ for which $c(y) = c(x)$. Then

$$|X| \ge 2^{k-1} + 1 - \ell' \ge 2^{k-1} + 1 - (2^{k-2} + k - 2) > 2^{k-3}.$$

Since $\mathbb{N}_{\ell'} \subseteq \mathrm{NC}(x)$ for every $x \in X$ and there are $2^{\ell-\ell'}$ subsets of $\mathbb{N}_\ell$ that contain $\mathbb{N}_{\ell'}$ as a subset, it follows that

$$2^{k-3} < |X| \le 2^{\ell-\ell'} \le 2^{k-2}$$

and so $\ell' = \ell - k + 2 \le (2^{k-2} + k - 2) - k + 2 = 2^{k-2}$. This, however, implies that

$$2^{k-2} + 1 \le 2^{k-1} + 1 - \ell' \le |X| \le 2^{k-2},$$

which is a contradiction. Hence $\chi_s(G) \ge 2^{k-2} + k - 1$ and so $\chi_s(G) = 2^{k-2} + k - 1$. ■

4 Multiset Colorings

For a connected graph G, let $c : V(G) \to \mathbb{N}_k$, where k is a positive integer, be a coloring of the vertices of G where adjacent vertices may be colored the same. The coloring c is called a *multiset coloring* if the multisets of colors of the neighbors of every two adjacent vertices of G are different, that is, for every two adjacent vertices u and v, there exists a color i such that the number of neighbors of u colored i and the number of neighbors of v colored i are not the same. For a vertex v of G, the multiset $M(v)$ of colors of the neighbors of v can be represented by a k-vector. The *multiset color code* of v is the k-vector

$$\mathrm{code}_m(v) = (a_1, a_2, \cdots, a_k) = a_1 a_2 \cdots a_k,$$

where a_i is the number of occurrences of i in $M(v)$, that is, the number of vertices adjacent to v that are colored i for $1 \leq i \leq k$. Therefore,

$$\sum_{i=1}^{k} a_i = \deg v.$$

Thus a vertex coloring of G is a multiset coloring if every two adjacent vertices have distinct multiset color codes. Hence every multiset coloring of a graph G is neighbor-distinguishing. The *multiset chromatic number* $\chi_m(G)$ of G is the minimum positive integer k for which G has a multiset k-coloring. These concepts were introduced and studied in [3]. For the 4-chromatic graph G of Figure 7, we saw that its metric chromatic number of G is 3. In fact, the multiset chromatic number of this graph is 2. Figure 14 shows a multiset 2-coloring of G together with the multiset color code of each vertex of G.

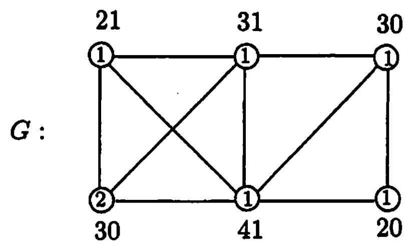


Figure 14: A multiset 2-coloring of a 4-chromatic graph G

Suppose that c is a proper vertex k-coloring of a graph G. If u is a vertex of G and $c(u) = i$ for some integer i $(1 \leq i \leq k)$, then the i-th coordinate of the color code of u is 0. On the other hand, if v is a neighbor of u, then the i-th coordinate of the color code of v is at least 1, implying that $\text{code}_m(u) \neq \text{code}_m(v)$. Hence every proper coloring of G is a multiset coloring. Therefore, for every graph G of order n,

$$1 \leq \chi_m(G) \leq \chi(G) \leq n. \tag{4}$$

If u and v are vertices (adjacent or not) of a graph G such that $\deg u \neq \deg v$, then necessarily $\text{code}_m(u) \neq \text{code}_m(v)$. On the other hand, if G contains two adjacent vertices u and v with $\deg u = \deg v$, then in order for c to be a multiset coloring, c must assign at least two distinct colors to the neighbors of u and v. Thus we have the following observation from [3].

Observation 4.1 [3] *The multiset chromatic number of a graph G is* 1 *if and only if every two adjacent vertices of G have distinct degrees.*

The multiset chromatic number of every complete multipartite graph was also determined in [3]. If every partite set of a complete k-partite graph G has n vertices, then we write $G = K_{k(n)}$, where then $K_{n(1)} = K_n$ and $K_{1(n)} = \overline{K}_n$. For t distinct positive integers $n_1, n_2, \ldots, n_t$, the complete multipartite graph containing k_i partite sets of cardinality n_i for $1 \leq i \leq t$ is denoted by $K_{k_1(n_1),k_2(n_2),\ldots,k_t(n_t)}$. For positive integers ℓ and n,

$$f(\ell, n) = \binom{n + \ell - 1}{\ell - 1}$$

is then the number of n-element multisubsets of an ℓ-element set.

Theorem 4.2 [3] *For positive integers k and n, the multiset chromatic number of the regular complete k-partite graph $K_{k(n)}$ is the unique positive integer ℓ for which*

$$f(\ell - 1, n) < k \leq f(\ell, n).$$

Corollary 4.3 *If $G = K_{k_1(n_1),k_2(n_2),\ldots,k_t(n_t)}$, where $n_1, n_2, \ldots, n_t$ are t distinct positive integers, then*

$$\chi_m(G) = \max\{\chi_m(K_{k_i(n_i)}) : 1 \leq i \leq t\}.$$

We have already observed that $\chi_m(G) \leq \chi(G)$ for every graph G. In fact, every pair a, b of positive integers with $a \leq b$ can be realized as the multiset chromatic number and chromatic number, respectively, of some graph.

Theorem 4.4 [3] *For each pair a, b of positive integers with $a \leq b$, there exists a connected graph G such that $\chi_m(G) = a$ and $\chi(G) = b$.*

Since every vertex coloring of a graph G results in every two adjacent vertices with different degrees having distinct multiset color codes, it follows that determining the multiset chromatic number of a graph is of greatest interest and most challenging when the graph in question has many pairs of adjacent vertices having the same degree. For this reason, the greatest emphasis has been placed on studying the multiset chromatic numbers of regular graphs. A familiar class of regular graphs are powers of cycles. In particular, for a connected graph G of order n and a positive integer k, the *k-th power G^k* of G is that graph whose vertex set is that of G and where two vertices u and v are adjacent in G^k if $1 \leq d_G(u, v) \leq k$. Thus $G^1 = G$ and $G^k = K_n$ if $k \geq \text{diam}(G)$. The following result is a consequence of Theorem 4.2.

Theorem 4.5 [3] *For each integer $k \geq 2$,*

$$\chi_m(C_{2k}^{k-1}) = \left\lceil \frac{-1 + \sqrt{8k + 1}}{2} \right\rceil.$$

Theorem 4.6 [3] *Let $p \geq 2$ be an integer. If $(3p) \mid n$ and $n \geq 6p$, then*

$$\chi_m(C_n^k) \leq 3$$

for $2p-1 \leq k \leq \lfloor \frac{5p-1}{2} \rfloor$.

Conjecture 4.7 [3] *For every integer $k \geq 3$, there exists an integer $f(k)$ such that $\chi_m(C_n^k) = 3$ for all $n \geq f(k)$.*

While $f(k) = 2k+2$ for $k = 3, 4$, we believe that $f(k) > 2k+2$ for sufficiently large k.

For nearly every pair k, n of positive integers with $k \leq n$, there is a connected graph G of order n having multiset chromatic number k.

Theorem 4.8 [3] *Let k and n be integers with $1 \leq k \leq n$. Then there exists a connected graph G of order n with $\chi_m(G) = k$ if and only if $k \neq n-1$.*

Since a set coloring of a connected graph G is a multiset coloring of G, it follows that

$$\chi_m(G) \leq \chi_s(G).$$

Next we show that every pair a, b of positive integers with $a \leq b$ can be realized as the multiset chromatic number and set chromatic number, respectively, of some connected graph.

Theorem 4.9 *For each pair a, b of positive integers with $a \leq b$, there exists a connected graph G such that $\chi_m(G) = a$ and $\chi_s(G) = b$.*

Proof. If $a = b$, then the complete graph K_a has the desired property. Thus, we may assume that $a < b$. We consider two cases, according to whether $a = 1$ or $a \geq 2$.

Case 1. $a = 1$. Then $b \geq 2$. We show that there is a graph H such that $\chi_m(H) = 1$ and $\chi_s(H) = b \geq 2$. We begin by constructing a graph F. Let $S_1, S_2, \ldots, S_{2^{b-1}}$ be the 2^{b-1} subsets of $\mathbb{N}_{b-1}$, where $|S_1| \leq |S_2| \leq \cdots \leq |S_{2^{b-1}}|$. Hence $S_1 = \emptyset$ and $S_{2^{b-1}} = \mathbb{N}_{b-1}$. Then the graph F is obtained from $K_{2^{b-1}}$ with $V(K_{2^{b-1}}) = U = \{u_1, u_2, \ldots, u_{2^{b-1}}\}$ by adding pairwise disjoint sets $W_2, W_3, \ldots, W_{2^{b-1}}$ to $K_{2^{b-1}}$, where $W_i = \{w_{i,1}, w_{i,2}, \ldots, w_{i,|S_i|}\}$, and joining each vertex in W_i to u_i for each i with $2 \leq i \leq 2^{b-1}$. Since $|S_i| \leq i-1$ for $1 \leq i \leq 2^{b-1}$, we can add more pendant edges at each vertex u_i, if necessary, to obtain the graph H such that $\deg_H u_i = 2^{b-1} - 2 + i$ for $1 \leq i \leq 2^{b-1}$. Figure 15 shows the graph H for $b = 4$. Let $X = V(H) - V(F)$. For $b = 4$, the set X consists of the solid vertices shown in Figure 15, while the vertex set $V(F)$ of F consists of all hollow vertices.

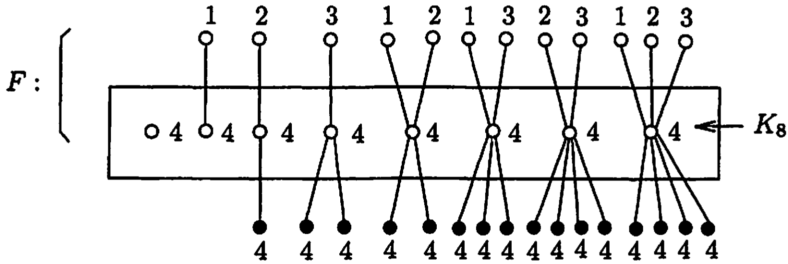


Figure 15: The graph H in Case 1 for $a = 1$ and $b = 4$

Since every two adjacent vertices in H have different degrees, $\chi_m(H) = 1$ by Observation 4.1. It remains only to show that $\chi_s(H) = b$. Because $\omega(H) = 2^{b-1}$, it follows by Theorem 2.6 that $\chi_s(H) \geq b$. On the other hand, consider the coloring $c_1 : V(H) \to \mathbb{N}_b$ of H that assigns (i) the color b to each vertex in $U \cup X$ and (ii) the colors in S_i to the $|S_i|$ end-vertices in W_i for $2 \leq i \leq 2^{b-1}$. Figure 15 shows such a coloring for $b = 4$. Then $\mathrm{NC}(u_i) = S_i \cup \{b\}$ for $1 \leq i \leq 2^{b-1}$. Since $|\mathrm{NC}(u_i)| \geq 2$ for $2 \leq i \leq 2^{b-1}$ and $|\mathrm{NC}(x)| = 1$ for each end-vertex in H, it follows that c_1 is a set b-coloring. Therefore, $\chi_s(H) = b$.

Case 2. $a \geq 2$. Then $b \geq 3$. We now construct a graph G from the graph H in Case 1 and the complete graph K_a with $V(K_a) = Y = \{y_1, y_2, \ldots, y_a\}$ by joining each vertex y_i to the vertex $w_{2,1}$ in H for $1 \leq i \leq a$ (see Figure 16 for $a = 3$ and $b = 4$). Observe that two vertices are adjacent and have the same degree if and only if both vertices belong to Y. Therefore, no multiset coloring can assign the same color to two distinct vertices in Y and so $\chi_m(G) \geq |Y| = a$. Since a coloring that assigns (i) the color i to the vertex y_i in Y for $1 \leq i \leq a$ and (ii) the color 1 to the remaining vertices is a multiset a-coloring, it then follows that $\chi_m(G) = a$. To verify that $\chi_s(G) = b$, observe first that $\chi_s(G) \geq b$ again by Theorem 2.6. On the other hand, the coloring $c_2 : V(G) \to \mathbb{N}_b$ such that c_2 restricted to $V(H)$ is the coloring c_1 mentioned above and $c_2(y_i) = i$ for $1 \leq i \leq a$ is a set b-coloring of G. Figure 16 shows such a coloring for $a = 3$ and $b = 4$. Thus $\chi_s(G) = b$, as desired. ■

For every connected graph G, we know that

$$\chi_m(G) \leq \chi_s(G) \leq \chi(G) \quad \text{and} \quad \mu(G) \leq \chi_s(G) \leq \chi(G).$$

However, there is still the question of the relationship between $\mu(G)$ and $\chi_m(G)$.

The multiset chromatic number of the Petersen graph P is 2. A multiset 2-coloring of P is shown in Figure 17(a). Since $\chi(P) = 3$, the metric chromatic number of P is either 2 or 3 and, in fact, $\mu(P) = 3$. A

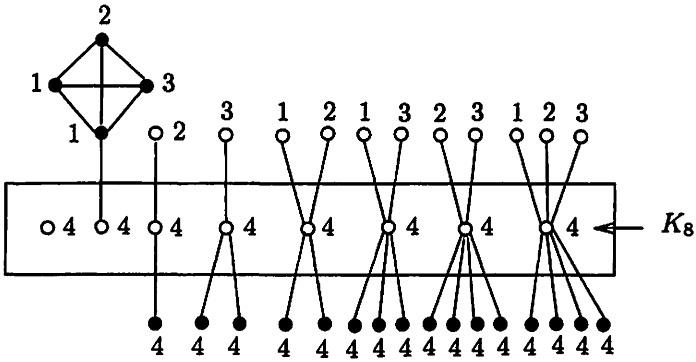


Figure 16: The graph G in Case 2 of the proof of Theorem 4.9 for $a = 3$ and $b = 4$

metric 3-coloring of P is shown in Figure 17(b). To see that $\mu(P) = 3$, assume, to the contrary, that there is a metric 2-coloring c of P using the colors 1 and 2. Then every vertex of P colored 1 has metric color code $(0, 1)$ or $(0, 2)$. Let $C : v_1, v_2, \ldots, v_5, v_1$ be a 5-cycle of P. At least three vertices of C are colored the same, say 1. If all five vertices of C are colored 1, then two consecutive vertices of C must have the same metric color code, a contradiction. If exactly four vertices of C are colored 1, say $c(v_3) = 2$, then $\text{code}_\mu(v_2) = \text{code}_\mu(v_4) = (0, 1)$, implying that $\text{code}_\mu(v_1) = \text{code}_\mu(v_5) = (0, 2)$, a contradiction. Hence exactly three vertices of C are colored 1. If $c(v_1) = c(v_5) = 2$, then $\text{code}_\mu(v_1) = \text{code}_\mu(v_5) = (1, 0)$, a contradiction. If $c(v_2) = c(v_4) = 2$, then $\text{code}_\mu(v_1) = \text{code}_\mu(v_5) = (0, 1)$, another contradiction. Thus $\mu(P) = 3$.

The Petersen graph is not the only graph whose metric chromatic number exceeds its multiset chromatic number. For example, for each integer $n \geq 3$,

$$\mu(K_{1,n-1}) = 2 = \chi_m(K_{1,n-1}) + 1,$$

while

$$\mu(G) = n - 1 = \chi_m(G) + 1$$

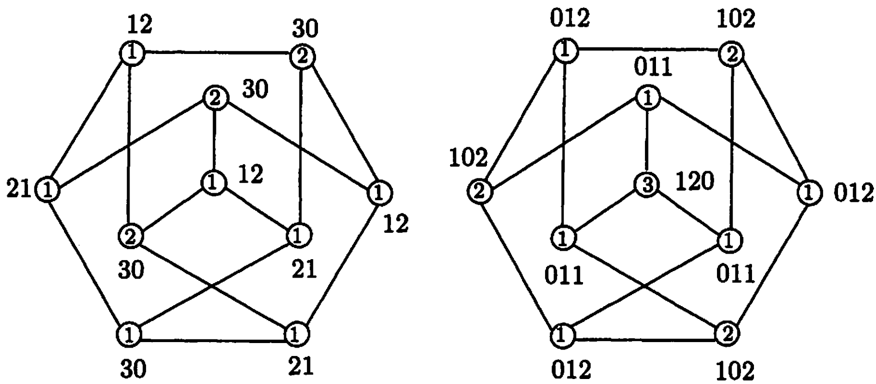
for $G \in \{K_n - e, (K_{n-2} \cup K_1) + K_1\}$.

In fact, $\mu(G) - \chi_m(G)$ can be arbitrary large. For a graph G, its *corona* $\text{cor}(G)$ is that graph obtained from G by adding a pendant edge at each vertex of G. It was shown in [2] and [9] that

$$\chi_m(\text{cor}(K_n)) = \left\lceil \frac{1 + \sqrt{4n - 3}}{2} \right\rceil$$

and

$$\chi_s(\text{cor}(K_n)) = \chi(\text{cor}(K_n)) = n.$$



(a) (b)

Figure 17: A multiset 2-coloring and a metric 3-coloring of the Petersen graph P

We now determine the metric chromatic number of $\text{cor}(K_n)$. Observe that $\mu(\text{cor}(K_1)) = \mu(K_2) = 2$.

Proposition 4.10 *For $n \geq 2$,*

$$\mu(\text{cor}(K_n)) = \lceil 2\sqrt{n}\, \rceil - 1.$$

Proof. Let $G = \text{cor}(K_n)$, $\mu(G) = k$, and consider a metric k-coloring $c : V(G) \to \mathbb{N}_k$ such that $c(V(K_n)) = \mathbb{N}_\ell$. Since $\chi_s(G) = \chi(G) = n$, it follows that $1 \leq \ell \leq k \leq n$.

There are at least $\lceil n/\ell \rceil$ vertices in K_n that are assigned the same color, implying that at least $\lceil n/\ell \rceil - 1$ colors not in $\mathbb{N}_\ell$ are needed for the end-vertices of G so that each vertex belonging to K_n has a distinct code. Hence

$$k \geq \min\{\ell + \lceil n/\ell \rceil - 1 : \ 1 \leq \ell \leq n\}.$$

Let f be a function from $[1, n]$ (as a subset of $\mathbb{R}$) to $\mathbb{R}$ defined by

$$f(x) = x + \tfrac{n}{x} - 1$$

and observe that f is continuous on $[1, n]$ and attains its global minimum at $\sqrt{n}$. Hence

$$k \geq \lceil f(\sqrt{n}\,) \rceil = \lceil 2\sqrt{n}\, \rceil - 1.$$

We now consider two cases.

Case 1. $\sqrt{n} \in \mathbb{Z}$. Then $k \geq 2\sqrt{n} - 1$. Construct G from K_n with

$$V(K_n) = \{u_{i,j} : \ 1 \leq i, j \leq \sqrt{n}\}$$

by adding n end-vertices. Label the end-vertex joined to the vertex $u_{i,j}$ by $w_{i,j}$. Then the coloring $c_1 : V(G) \to \mathbb{N}_{2\sqrt{n}-1}$ given by

$$c_1(x) = \begin{cases} j & \text{if } x = u_{i,j} \\ \sqrt{n}+i & \text{if } x = w_{i,j} \text{ and } i \neq \sqrt{n} \\ 1 & \text{if } x = w_{\sqrt{n},j} \end{cases}$$

is a metric $(2\sqrt{n}-1)$-coloring of G (see Figure 18 for $n = 9$). Hence $k = 2\sqrt{n} - 1$.

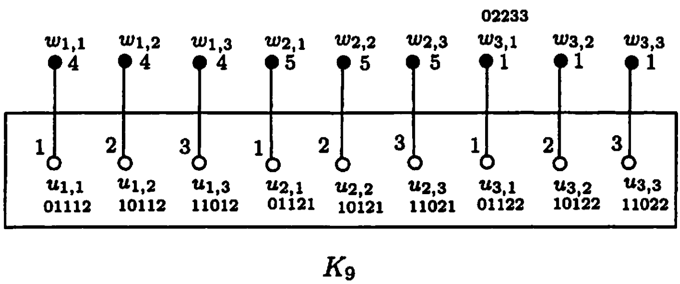


Figure 18: A set 5-coloring of the graph $\mathrm{cor}(K_9)$

Case 2. $\sqrt{n} \notin \mathbb{Z}$. Let $p = \lfloor \sqrt{n} \rfloor$. Then $p^2 + 1 \leq n \leq (p+1)^2 - 1$. Also,

$$k \geq \lceil 2\sqrt{n} \rceil - 1 \geq 2\sqrt{n} - 1 > 2p - 1,$$

that is,

$$k \geq \lceil 2\sqrt{n} \rceil - 1 \geq 2p.$$

We consider two subcases.

Subcase 2.1. $p^2 + 1 \leq n \leq p(p+1)$. Write $n = p^2 + r$, where $1 \leq r \leq p$. Construct G from K_n with

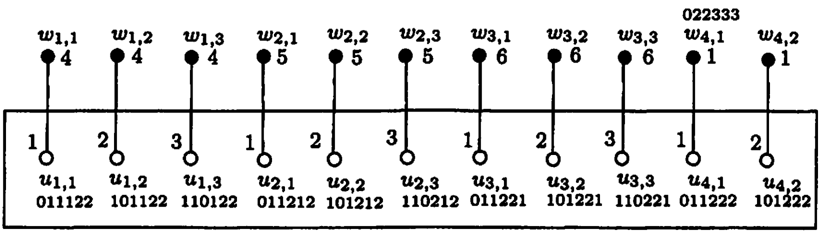
$$V(K_n) = \{u_{i,j} : 1 \leq i,j \leq p\} \cup \{u_{p+1,j} : 1 \leq j \leq r\}$$

by adding n end-vertices. Label the end-vertex joined to the vertex $u_{i,j}$ by $w_{i,j}$. Then the coloring $c_2 : V(G) \to \mathbb{N}_{2p}$ given by

$$c_2(x) = \begin{cases} j & \text{if } x = u_{i,j} \\ p+i & \text{if } x = w_{i,j} \text{ and } i \neq p+1 \\ 1 & \text{if } x = w_{p+1,j} \end{cases}$$

is a metric $(2p)$-coloring of G (see Figure 19 for $n = 11$). Hence

$$2p \leq \lceil 2\sqrt{n} \rceil - 1 \leq k \leq 2p$$



K_{11}

Figure 19: A set 6-coloring of the graph $\mathrm{cor}(K_{11})$

and so $k = \lceil 2\sqrt{n} \rceil - 1$.

Subcase 2.2. $p(p+1)+1 \le n \le (p+1)^2 - 1$. Write $n = p(p+1) + r$, where $1 \le r \le p$. Construct G from K_n with

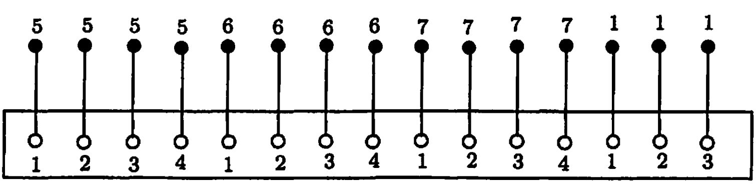
$$V(K_n) = \{u_{i,j}: \ 1 \le i \le p, 1 \le j \le p+1\} \cup \{u_{p+1,j}: \ 1 \le j \le r\}$$

by adding n end-vertices. Label the end-vertex joined to the vertex $u_{i,j}$ by $w_{i,j}$. Then the coloring $c_3 : V(G) \to \mathbb{N}_{2p+1}$ given by

$$c_3(x) = \begin{cases} j & \text{if } x = u_{i,j} \\ p+1+i & \text{if } x = w_{i,j} \text{ and } i \ne p+1 \\ 1 & \text{if } x = w_{p+1,j} \end{cases}$$

is a metric $(2p+1)$-coloring of G (see Figure 20 for $n = 15$). Hence

$$2p \le \lceil 2\sqrt{n} \rceil - 1 \le k \le 2p+1.$$



K_{15}

Figure 20: A set 7-coloring of the graph $\mathrm{cor}(K_{15})$

We now show that $\lceil 2\sqrt{n} \rceil - 1 > 2p$. Let $\sqrt{n} - p = \alpha$. Then $0 < \alpha < 1$ and

$$p(p+1) + 1 \le n = (p+\alpha)^2.$$

Hence

$$0 < 1 - \alpha^2 \leq (2\alpha - 1)p$$

and since $p \geq 1$, it follows that $2\alpha > 1$. Then

$$\lceil 2\sqrt{n} \rceil - 1 \geq 2\sqrt{n} - 1 = 2(p + \alpha) - 1 > 2p.$$

Therefore,

$$2p + 1 \leq \lceil 2\sqrt{n} \rceil - 1 \leq k \leq 2p + 1$$

and so $k = \lceil 2\sqrt{n} \rceil - 1$. ∎

On the other hand, not every metric coloring is a multiset coloring. For example, for the path $P_5 : v_1, v_2, v_3, v_4, v_5$ of order 5, the 2-coloring c with $c(v_1) = c(v_2) = c(v_3) = c(v_4) = 1$ and $c(v_5) = 2$ is a metric coloring which is not a multiset coloring. Of course, this does not imply that $\mu(P_5) < \chi_m(P_5)$ and, in fact, $\chi_m(P_5) = \mu(P_5) = 2$ since P_5 is bipartite.

While we have seen graphs G for which $\mu(G) = \chi_m(G)$ and graphs G for which $\mu(G) > \chi_m(G)$ (indeed for which $\mu(G)$ is considerably larger than $\chi_m(G)$), we do not know if these are the only possibilities.

Problem 4.11 *Does there exist a graph G for which $\mu(G) < \chi_m(G)$?*

5 Sigma Colorings

For a nontrivial connected graph G, let $c : V(G) \to \mathbb{N}$ be a vertex coloring of G where adjacent vertices may be colored the same. For a set S of integers, let $\sigma(S)$ denote the sum of all elements in S. The *color sum* $\sigma(v)$ of v is the sum of the colors of the vertices in $N(v)$, that is, $\sigma(v) = \sigma(M(v))$, where $M(v)$ is the multiset of colors of the neighbors of v (as defined in Section 4). If $\sigma(x) \neq \sigma(y)$ for every two adjacent vertices x and y of G, then c is called a *sigma coloring* of G. The minimum number of colors required in a sigma coloring of a graph G is called the *sigma chromatic number* of G and is denoted by $\sigma(G)$. These concepts were introduced and studied in [5].

A graph G with chromatic number 3 is shown in Figure 21(a) along with a proper coloring of G using the colors 1, 2, 3. Since $\sigma(u) = \sigma(v) = \sigma(y) = 5$, this coloring is not a sigma coloring. However, if we were to interchange the colors 2 and 3 (see Figure 21(b)), a sigma coloring results.

While, as we have seen, not every proper coloring of a graph is a sigma coloring, it is the case that *some* proper coloring of a graph G using $\chi(G)$ colors *is* a sigma coloring. From this, it follows that $\sigma(G) \leq \chi(G)$ for every nontrivial connected graph G. In fact, the sigma chromatic number equals the multiset chromatic number for every nontrivial connected graph. To see this, the following lemma is useful.

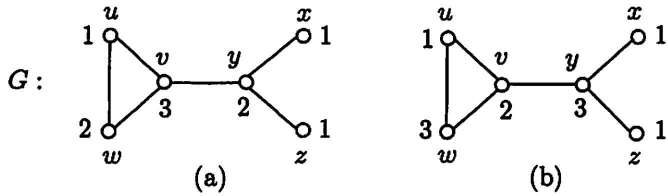


Figure 21: A non-sigma coloring and a sigma coloring of a graph

Lemma 5.1 [5] *For integers $k \geq 1$ and $N \geq 1$, let $\mathfrak{A}_k = \{a_1, a_2, \ldots, a_k\}$ be a set of k positive integers such that $a_{i+1} \geq Na_i + 1$ for $1 \leq i \leq k-1$. Then for every two distinct multisets X and Y of cardinality at most N whose elements belong to $\mathfrak{A}_k$, $\sigma(X) \neq \sigma(Y)$.*

Theorem 5.2 *For every nontrivial connected graph G, $\chi_m(G) = \sigma(G)$.*

Proof. Since every sigma coloring of G is a multiset coloring of G, it follows that $\chi_m(G) \leq \sigma(G)$. It only remains therefore to show that $\chi_m(G) \geq \sigma(G)$. Suppose that $\chi_m(G) = k$ and $\Delta(G) = \Delta$. Let c be a multiset k-coloring of G using the colors $1, 2, \ldots, k$. Now let $\mathfrak{A}_k = \{a_1, a_2, \ldots, a_k\}$ be a set of k integers, where the elements a_i $(1 \leq i \leq k)$ are defined recursively by (i) $a_1 \geq 1$ and (ii) once a_{i-1} is defined for an integer i with $2 \leq i \leq k$, a_i is an integer such that $a_i \geq \Delta a_{i-1} + 1$. Thus $a_1 < a_2 < \cdots < a_k$. Define a k-coloring c' of G by

$$c'(v) = a_{c(v)} \text{ for } v \in V(G) \text{ and } 1 \leq i \leq k.$$

We show that c' is a sigma coloring of G. Let x and y be two adjacent vertices of G. Then $M(x) \neq M(y)$. Let S_x be the multisubsets of $\mathfrak{A}_k$ obtained from $M(x)$ by replacing each element $i \in M(x)$ by a_i. Similarly, S_y is the multisubset of $\mathfrak{A}_k$ obtained from $M(y)$ by replacing each element $i \in M(x)$ by a_i. Thus S_x and S_y are two distinct multisubsets of $\mathfrak{A}_k$. Since $|S_x| \leq \Delta$ and $|S_y| \leq \Delta$, it follows by Lemma 5.1 that $\sigma(S_x) \neq \sigma(S_y)$. ■

While $\sigma(G) = \chi_m(G)$ for every graph G, there are major differences between these two colorings. In any multiset coloring of a graph G, it is not important which colors are used; that is, if c is a multiset k-coloring of a graph G, then any k positive integers can be used for the colors. As we saw in Figure 21, this is not the case for a sigma k-coloring of G, however. For example, if $G = K_{10(3)}$, then $\sigma(G) = \chi_m(G) = 3$ by Theorem 4.2. Of course, there is a multiset 3-coloring of G using the colors 1, 2, 3. There is, however, no such sigma 3-coloring. In fact, there is no sigma 3-coloring that uses any three of the four colors of the set $\{1, 2, 3, 4\}$. On the other hand, there is a sigma 3-coloring of G using the colors 1, 2, 5.

By Theorem 4.2, $\sigma(K_{5(2)}) = 3$ and $\sigma(K_{3(3)}) = 2$ and so by Corollary 4.3, $\sigma(K_{5(2),3(3)}) = 3$. In Figure 22, a sigma 3-coloring of the complete 8-partite graph $K_{5(2),3(3)}$ is given using the colors 1, 2, 3, where the color sums of the vertices in each partite set are given as well. The colors assigned to the vertices in this coloring are not interchangeable, however, as every non-identity permutation ϕ of the colors 1, 2, 3 produces a 3-coloring that is not a sigma coloring, as is also shown in Figure 22.

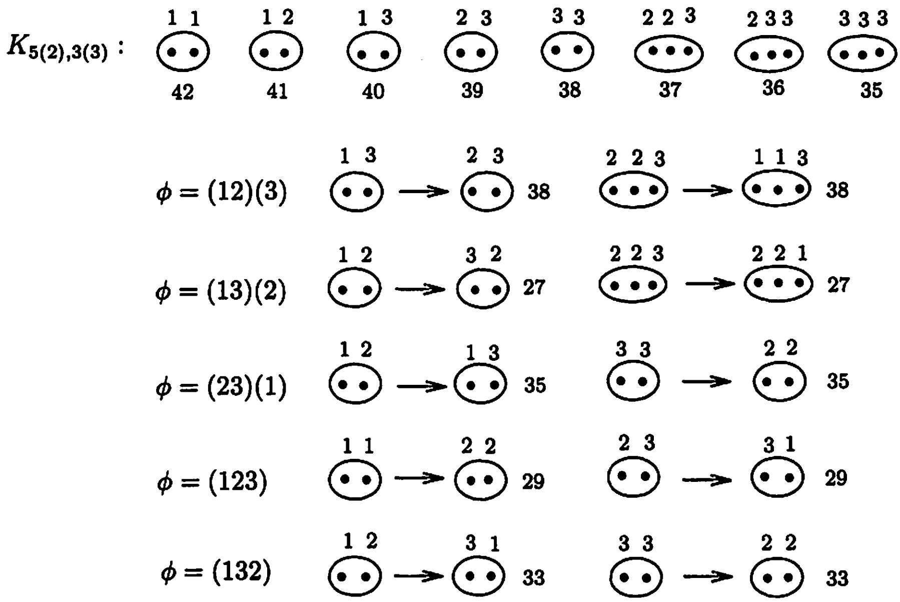


Figure 22: 3-Colorings of $K_{5(2),3(3)}$

For a nontrivial connected graph G with $\sigma(G) = k$ and a sigma k-coloring c of G, let min(c) be the smallest color assigned by c to a vertex of G and max(c) the largest such color. It was shown in [5] that if $\sigma(G) = k$, then there is always some sigma k-coloring c of G such that min$(c) = 1$. A question of interest concerns the minimum value of max(c) over all sigma k-colorings c of G for which $\sigma(G) = k$. This minimum value is called the *sigma value* $\nu(G)$ of G. That is, for a connected graph G with $\sigma(G) = k$,

$$\nu(G) = \min\{\max(c)\},$$

where the minimum is taken over all sigma k-colorings c of G. Thus, $\nu(G) \geq \sigma(G)$ for every nontrivial connected graph G. For example, if $G = K_{10(3)}$, then $\sigma(G) = 3$ and $\nu(G) = 5$.

A nontrivial connected graph G is called *sigma continuous* if $\nu(G) = \sigma(G)$, that is, if $\sigma(G) = k$, then there is a sigma k-coloring of G using the colors in $\mathbb{N}_k$. Thus $G = K_{10(3)}$ is not sigma continuous. It was shown in [5] that there are several well-known classes of sigma continuous graphs, including cycles.

If G is a bipartite graph, then $\sigma(G) \leq 2$. Whether every such graph is sigma continuous is not known, however.

Problem 5.3 [5] *Is every bipartite graph sigma continuous?*

It has been shown that there is an important class of sigma continuous bipartite graphs, however.

Theorem 5.4 [5] *Every tree is sigma continuous.*

By Theorem 5.4, there are infinitely many connected sigma continuous graphs with sigma chromatic number 2. In fact, even more can be said.

Theorem 5.5 [5] *For each integer $k \geq 2$, there is a connected sigma continuous graph with sigma chromatic number k.*

As a consequence of Theorem 4.3, for integers n_1 and n_2 where $1 \leq n_1 < n_2$ and $k_i = n_i + 1$ $(i = 1, 2)$, if $G = K_{k_1(n_1),k_2(n_2)}$, then $\sigma(G) = 2$. The admissible sigma 2-colorings of G were established in [5].

Theorem 5.6 [5] *For integers n_1 and n_2 with $1 \leq n_1 < n_2$ and $k_i = n_i + 1$ $(i = 1, 2)$, let $G = K_{k_1(n_1),k_2(n_2)}$. For positive integers a and b, there exists a sigma 2-coloring of G using the colors a and $a + b$ if and only if $(a + b)n_1 < an_2$ or $a(n_2 - n_1) \not\equiv 0 \pmod{b}$.*

By Theorem 5.6, there exist connected graphs with sigma chromatic number 2 that is not sigma continuous. In fact, more can be said.

Theorem 5.7 [5] *For each integer $k \geq 2$, there is a connected graph with sigma chromatic number k that is not sigma continuous.*

Another parameter of interest was introduced in [5]. For a sigma coloring c of G, the *sigma range* $\rho(G)$ of G is defined by

$$\rho(G) = \min\{\max(c)\}$$

over all sigma colorings c of G. Hence the sigma range of G is the smallest positive integer k for which there exists a sigma coloring of G using colors from the set $\mathbb{N}_k$, while the sigma value of G is the smallest positive integer k for which there exists a sigma coloring of G using $\sigma(G)$ colors from the set $\mathbb{N}_k$. Therefore, for every graph G,

$$\sigma(G) \leq \rho(G) \leq \nu(G). \tag{5}$$

As an example, consider $G = K_{10(3)}$. We saw that $\sigma(G) = 3$ and $\nu(G) = 5$. We now show that $\rho(G) = 4$. Since there is no sigma coloring using the colors 1, 2, 3, it follows that $\rho(G) \geq 4$. To show that $\rho(G) \leq 4$, let $V_1, V_2, \ldots, V_{10}$ be the partite sets of G and let $A_1, A_2, \ldots, A_{10}$ be the following 3-element multisubsets of $\{1, 2, 3, 4\}$:

$$\{1,1,1\}, \{1,1,2\}, \{1,1,3\}, \{1,1,4\}, \{1,2,4\},$$
$$\{1,3,4\}, \{1,4,4\}, \{2,4,4\}, \{3,4,4\}, \{4,4,4\}.$$

Since $\sigma(A_i) \neq \sigma(A_j)$ for $1 \leq i < j \leq 10$, the 4-coloring of G that assigns the three colors in A_i to the three vertices in V_i for each i with $1 \leq i \leq 10$ is a sigma coloring of G using the colors $1, 2, 3, 4$. Thus $\rho(G) = 4$. Therefore, if $G = K_{10(3)}$, then $\sigma(G) = 3$, $\rho(G) = 4$, and $\nu(G) = 5$.

It was shown in [5] that $\nu(G) - \rho(G)$ can be arbitrarily large for some connected graphs G. Such is also the case for $\rho(G) - \sigma(G)$. The following problem appears in [5].

Problem 5.8 [5] *Which ordered triples of positive integers can be realized as $(\sigma(G), \rho(G), \nu(G))$ for some graph G?*

6 Epilogue

We have seen that for each of the four neighbor-distinguishing vertex colorings

set colorings, metric colorings, multiset colorings, sigma colorings,

the number of colors required to color the vertices of a graph need never exceed the chromatic number of the graph. Thus we have the following.

Four Four Color Theorems Let G be a nontrivial connected graph.

(1) If G is planar, then $\chi(G) \leq 4$.

(2) If G is planar, then $\chi_s(G) \leq 4$.

(3) If G is planar, then $\mu(G) \leq 4$.

(4) If G is planar, then $\chi_m(G) \leq 4$.

From what we saw in Section 5, statement (4) can be replaced by

(4′) If G is planar, then $\sigma(G) \leq 4$.

Of course, statements (2), (3), and (4) (and (4′)) are all corollaries of statement (1) (*the* Four Color Theorem). We therefore close with the following.

Problem *Does there exist a proof of any of the statements* (2), (3), (4), *or* (4′), *that does not use the original Four Color Theorem and that is not computer-aided?*

7 Acknowledgments

We are grateful to the referee whose valuable suggestions resulted in an improved paper.

References

[1] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiz, and F. Saba, Irregular networks. *Congr. Numer.* **64** (1988) 197-210.

[2] G. Chartrand, F. Okamoto, C. W. Rasmussen, and P. Zhang, The set chromatic number of a graph. *Discuss. Math. Graph Theory.* **29** (2009) 545-561.

[3] G. Chartrand, F. Okamoto, E. Salehi, and P. Zhang, The multiset chromatic number of a graph. *Math. Bohem.* **134** (2009) 191-209.

[4] G. Chartrand, F. Okamoto, and P. Zhang, The metric chromatic number of a graph. *Australas. J. Combin.* 44 (2009) 273-286.

[5] G. Chartrand, F. Okamoto, and P. Zhang, The sigma chromatic number of a graph. *Graphs Combin.* To appear.

[6] G. Chartrand and P. Zhang, *Chromatic Graph Theory.* Chapman & Hall/CRC Press, Boca Raton (2009).

[7] R, Gera, F. Okamoto, C. W. Rasmussen, and P. Zhang, Set colorings in perfect graphs. *Math. Bohem.* To appear.

[8] F. Okamoto, C. W. Rasmussen, and P. Zhang, Set vertex colorings and joins of graphs. *Czechoslovak Math. J.* **59** (2009) 929-941.

[9] F. Okamoto, E. Salehi, and P. Zhang, On multiset colorings of graphs. *Discuss. Math. Graph Theory.* To appear.