LOCATING MOBILE INTRUDERS USING DOMINATING SETS

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ABSTRACT. We consider the placement of detection devices at the vertices of a graph G, where a detection device at vertex v has three possible outputs: there is an intruder at v; there is an intruder at one of the vertices in the open neighborhood $N(v)$, the set of vertices adjacent to v, but which vertex in $N(v)$ can not be determined; or there is no intruder in $N[v] = N(v) \cup \{v\}$. We introduce the 1-step locating-dominating problem of placing the minimum possible number of such detection devices in $V(G)$ so that the presence of an intruder in $V(G)$ can be detected, and the exact location of the intruder can be identified, either immediately or when the intruder has moved to an adjacent vertex. Some related problems are introduced.

1. INTRODUCTION

A graph $G = (V, E)$ might be used to model a facility with each vertex in $V(G)$ representing an area of the facility such as a room, hallway or ventilation duct. Edges of G could link vertices representing adjacent areas of the facility. A facility area will be identified with the vertex that represents it. These facilities are subject to having an "intruder" such as a thief, saboteur or fire that must be detected and have its location precisely identified. It is assumed here that the possible locations for the intruder are all the vertices in $V(G)$.

The neighborhood of a vertex $v \in V(G)$ is the set of vertices adjacent to it, $N(v) = \{x \in V(G) : vx \in E(G)\}$, and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. It is assumed that a detection device placed at a vertex v can detect the presence of an intruder precisely when the intruder is in $N[v]$. Thus, to be able to determine if there is an intruder in the system one needs to place detection devices at a dominating set $D \subseteq V(G)$, a

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set with $\cup_{v \in D} N[v] = V(G)$. When a detection device at vertex v can distinguish between there being an intruder at v or at a vertex in $N(v)$, but which vertex in $N(v)$ can not be pinpointed, then one is interested in having a locating-dominating set. Locating sets were introduced in Slater [44] and subsequently by Harary and Melter [18] where they were called metric bases. The concepts of locating and dominating were merged in [45, 46]. Further studies of locating-dominating sets include [6, 11, 13, 14, 15, 17, 23, 27, 32, 33, 41, 47, 48]. When only the presence of an intruder in $N[v]$ can be detected, with no information as to which vertex in $N[v]$ contains the intruder, one is interested in identifying-codes as introduced in [35] and further studied in [1, 2, 3, 4, 5, 7, 8, 9, 10, 12, 16, 26, 28, 29, 30, 31, 34, 36, 37, 38, 39, 40, 42, 43].

Our model involves locating-dominating sets. A detection device placed at a vertex v is assumed to transmit one of three possible outcomes in each time period: 0 if there is no intruder in $N[v]$; 1 if there is an intruder in $N(v)$; and 2 if the intruder is at v. (We assume here that there is at most one intruder.) As noted, a dominating set D has $\cup_{v \in D} N[v] = V(G)$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set $D \subseteq V(G)$. See Haynes, Hedetniemi and Slater [19, 20]. For example, for path $P_n = v_1, v_2, v_3, \ldots, v_n$ we can select every third vertex and let $D = \{v_2, v_5, v_8, \ldots\}$ (along with v_n if $n = 3k + 1$). Note, for example, that if the detection device at v_5 outputs a 1, then the intruder's location is not determined because it is either v_4 or v_6. A dominating set D is a locating-dominating set if our three-state detection devices placed at the vertices in D can precisely identify the location of any intruder. Equivalently, $\cup_{v \in D} N[v] = V(G)$ and if $u, x \notin D$ then $N(u) \cap D \neq N(x) \cap D$. The locating-dominating number $LD(G)$ or $\gamma_{LD}(G)$ is the minimum cardinality of a locating-dominating set $D \subseteq V(G)$. For path P_n (with $n = 5k$) let $D = \{v_2, v_4, v_7, v_9, v_{12}, v_{14}, \ldots, v_{5k-3}, v_{5k-1}\}$, and D is easily seen to be a $\gamma_{LD}(G)$-set. Thus, $\gamma(P_n) \approx \frac{n}{3}$ and $\gamma_{LD}(P_n) \approx \frac{2n}{5}$. Because every locating-dominating set must dominate, we always have $\gamma(G) \leq \gamma_{LD}(G)$.

In this paper we introduce the 1-step locating-dominating set problem of placing the minimum possible number of our three-state detection devices in $V(G)$ so that the presence of an intruder in $V(G)$ can be detected, and the exact location of the intruder can be identified either immediately (at time zero) or when the intruder has moved to an adjacent vertex (at time one). We will call a dominating set D with this capability a *1-step locating-dominating set*, and the *1-step-locating-domination number* $\gamma_{LD(1)}(G)$ is the minimum possible cardinality of such a set. In the last section, Section 4, we define 1-step identifying codes and several k-step parameters.

Because every locating-dominating set is a 1-step locating-dominating set (locating-dominating sets identify the exact location of an intruder immediately), we have the first theorem.

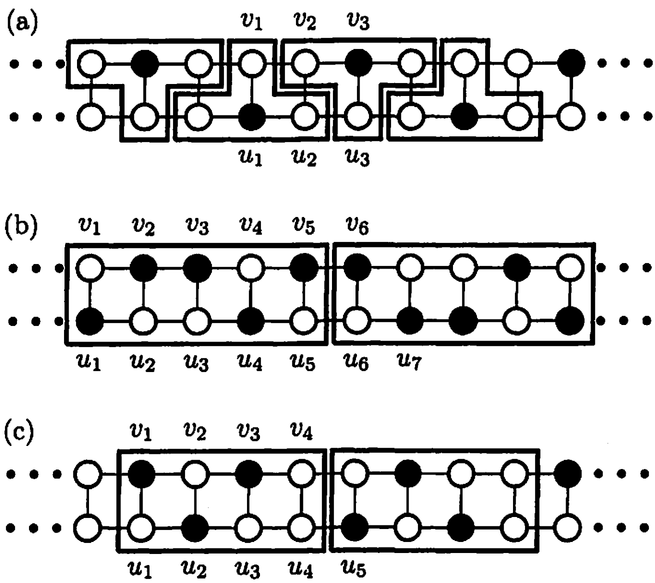

FIGURE 1. For the "long" ladder L of order $|V(L)| = n = 2k$ the dark shaded vertices show (a) a dominating set, (b) a 1-step locating-dominating set, and (c) a locating-dominating set. In each case, the repeating pattern is outlined.

Theorem 1.1. *For every graph G we have $\gamma(G) \leq \gamma_{LD(1)}(G) \leq \gamma_{LD}(G)$.*

Consider the "long" ladder L of order $|V(L)| = n = 2k$ illustrated in Figure 1. Each vertex can dominate at most four vertices (that is, $|N[v]| \leq 4$ for each $v \in V(L)$), so $\gamma(L) \geq \frac{n}{4}$, and Figure 1(a) shows that $\gamma(L) \approx \frac{n}{4}$.

Now consider a locating-dominating set of L. Let S be the set of shaded vertices in Figure 1(a). Notice that S does not form a locating-dominating set of L. In particular, each vertex x in $N(u_1)$ has $N[x] \cap S = \{u_1\}$. Now let S be the shaded vertices in Figure 1(c) and consider the eight vertices in the first boxed region. v_1, u_2, and v_3 are in S and so are located. Vertices v_4 and u_4 are the distinct vertices whose neighborhoods' intersections with S are $\{v_3\}$ and $\{u_5\}$, respectively. Likewise, u_1 and u_3 are the distinct vertices whose intersection with S are $\{v_1, u_2\}$ and $\{u_2, v_3\}$, respectively. Finally, v_2 is the distinct vertex whose intersection with S is $\{v_1, u_2, v_3\}$. Each boxed region has a similar characteristic. Hence, approximately $\frac{3n}{8}$ vertices suffice for 1-step locating-dominating.

To improve on this upper bound for $\gamma_{LD}(L)$, S must contain just two vertices of some group of eight vertices in $V(L)$. Suppose just two vertices of $\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ are in S. Since S must dominate, our choices are limited to four cases, depicted in Figure 1. If the dark vertices of Figures 1(a) or 1(b) are the only vertices in $S \cap \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$, then the gray vertices in these figures are not located. For Figures 1(c) and 1(d), we see that the two dark vertices are sufficient, but in each case at least three vertices are required to be in S for each group of eight vertices on either side of $\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$. To see this, consider Figure 1(c). Here, u_3 and v_4 are not located by just v_2 and v_3. So, v_5 must be in S to distinguish the locations of v_4 and u_3. Also, u_5 must be in S to dominate u_4. Given this, the locations of u_4 and u_6 are not distinguished. We can use u_7 to both distinguish the locations of u_4 and u_6, and to dominate v_7. A similar argument shows that this pattern of three vertices must repeat in every subsequent group of eight vertices on both sides of $\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ in Figure 1(c) either indefinitely or until we reach a group of eight vertices, some four of which are in S. Likewise, there must be three vertices in every group of eight vertices on both sides of $\{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ in Figure 1(d) extending either indefinitely or until a group of eight vertices is reached, some four of which are in S. Thus, we have $\gamma_{LD}(L) \approx \frac{3n}{8}$.

When shifting from dominating sets to 1-step locating-dominating sets for the "long" ladder, we can not use as few as $\frac{1}{4}|V(L)|$ vertices. This can be seen by considering the following three 'moves' in Figure 1(a). (i) An intruder is at v_1 at time zero and at v_2 at time one. (ii) An intruder is at u_2 at time zero and at v_2 at time one. (iii) an intruder is at u_2 at time zero and at u_3 at time one. With all three moves, the intruder starts at some vertex in $N(u_1)$ and ends at some vertex in $N(v_3)$. Hence, if S is the set of dark vertices in Figure 1(a), then S is not a 1-step locating-dominating set.

We can, however, use fewer than the $\frac{3}{8}|V(L)|$ vertices required for a locating-dominating set of L by relaxing the requirement that we immediately locate every intruder (locating-domination) to the requirement that we locate either immediately or after the intruder has moved to an adjacent vertex (1-step locating-domination). In particular, if $S \subseteq V(L)$ is the set of dark vertices in Figure 1(b), then S uses only $\frac{3}{10}|V(L)|$ vertices and is a 1-step locating-dominating set. To see this, note first that each vertex x in S is immediately located. Also, the gray vertices in Figure 1(b) are immediately located since, for example, v_3 and v_5 are the private neighbors of v_2 and v_6, respectively. While v_4 is not immediately located, an intruder at v_4 at time zero must move to a vertex that is immediately located by S. Hence, such an intruder is located at time one. Likewise, an intruder at u_2 at time zero is not immediately located since $N[u_2] \cap S = N[v_1] \cap S$.

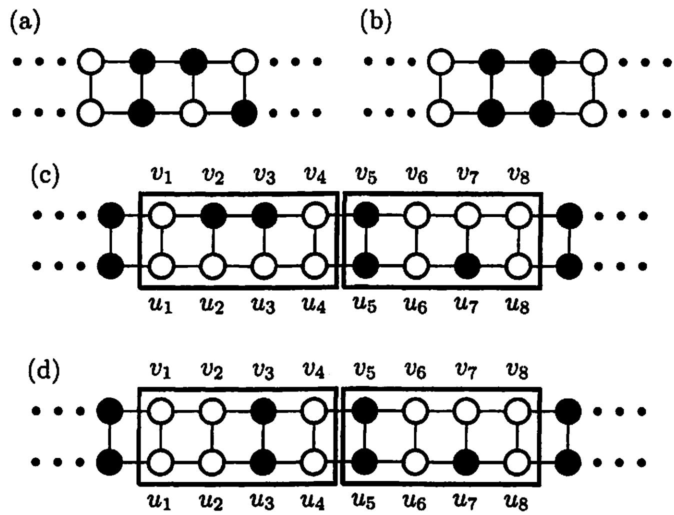

FIGURE 2. To improve on the upper bound of $\gamma_{LD}(L) \leq \frac{3n}{8}$, a locating-dominating set $S \subseteq V(L)$ must contain just two vertices of some group of eight vertices in $V(L)$. These four cases show the consequences of such.

However, such an intruder can only move to either a vertex that is immediately located by S or to u_3. Even though u_3 is not immediately located by S, we know that the intruder has moved from a vertex in $N(u_1) \cap N(v_2)$ to a vertex in $N(u_4)$. Thus, the intruder must be at u_3 after this move. That is, the intruder's location is known to be u_3 at time one. A similar argument can be used to show that a move from u_3 to u_2, u_5 to u_6, or u_6 to u_5 results in the intruder being located at time one.

To improve on this upper bound for $\gamma_{LD(1)}(L)$, S would have to contain just two vertices of some group of ten vertices in $V(L)$. Suppose just two vertices in some group of ten vertices in $V(L)$ are in S. Since S must dominate, our choices are limited to two cases, depicted in Figure 3. If the dark vertices of Figures 3(a) or 3(b) are the only vertices of the depicted group of ten vertices that are in S, then the moves indicated by directed arcs are not distinguished by S. Hence, S must contain at least $\frac{3}{10}|V(L)|$ vertices and $\gamma_{LD(1)}(L) \approx \frac{3n}{10}$.

Proposition 1.2. *For the "long" ladder of order $|V(L)| = n = 2k$, we have $\gamma(L) \approx \frac{n}{4}$, $\gamma_{LD(1)}(L) \approx \frac{3n}{10}$ and $\gamma_{LD}(L) \approx \frac{3n}{8}$.*

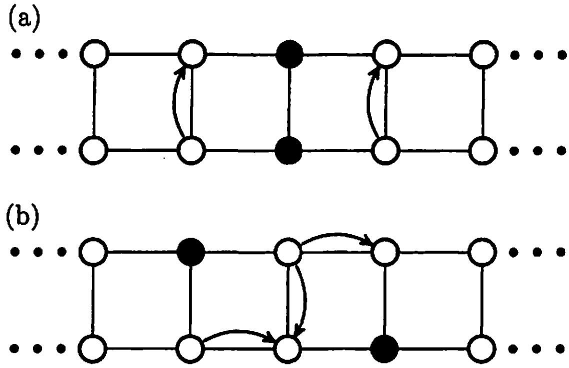

FIGURE 3. To improve on the upper bound of $\gamma_{LD(1)}(L) \leq \frac{3n}{10}$, a 1-step locating-dominating set $S \subseteq V(L)$ must contain just two vertices of some group of ten vertices in $V(L)$. These two cases show the impossibility of such.

The following two lemmas deal with twins in a graph and are generally useful when working with locating-dominating sets or 1-step locating-dominating sets. Two vertices u and v in $V(G)$ are twins if $N(u) = N(v)$ and are identical twins if $N[u] = N[v]$.

Lemma 1.3. *If $N[u] = N[v]$ and S is a $\gamma_{LD(1)}(G)$-set, then $S \cap \{u, v\} \neq \phi$.*

Lemma 1.4. *If $N[u] = N[v]$ or $N(u) = N(v)$, then S is locating-dominating implies $S \cap \{u, v\} \neq \phi$.*

Theorem 1.5. *If $n_1 \geq n_2$, then $\gamma_{LD(1)}(K_{n_1,n_2}) = n_2$. Also, given integers $n_1 \geq n_2 \geq \ldots \geq n_{t'} \geq 2$ with $n_2 \geq 2$, for the complete t-partite graph $G = K_{n_1,\ldots,n_{t'},1,\ldots,1}$ (where $t \geq t'$ and $t \geq 3$), $\gamma_{LD(1)}(G) = \left(\sum_{i=2}^{t} n_i\right) - (t' - 1)$.*

Proof. In the complete bipartite case, notice that if $S \subseteq V(K_{n_1,n_2})$ contains the n_2 vertices in the smaller partite set, then S 1-step locating-dominates. If there are vertices u and v in the same partite set, then by Lemma 1.4 either $\{u, v\} \cap S \neq \phi$ or u and v are not located at time 0. It follows that $\gamma_{LD(1)}(K_{n_1,n_2}) = n_2$.

For the complete t-partite case, construct set S as follows. For each partite set S_i of G corresponding to index n_i, $2 \leq i \leq t'$, place $n_i - 1$ vertices of S_i in S. For each singleton partite set S_i of G (corresponding to index n_i, $t' + 1 \leq i \leq t$), place the vertex of S_i in S.

We claim that S is a 1-step locating-dominating set of G. Consider an intruder at some vertex v in S_i, $2 \leq i \leq t$, at time 0. Either the intruder is located at time 0 because $v \in S$, or v is the only vertex in S_i such that $v \notin S$. In the latter case, the intruder is still located at time 0 since v is the only location in G such that $v \in N(s)$ for each $s \in S - S_i$ and $v \notin N(s)$ for each $s \in S \cap S_i$.

Now consider an intruder at some vertex v in S_1. Clearly, the intruder is not located at time 0 since $n_1 \geq 2$. But u is located at time 0 for each $u \in N(v)$, as shown above, so the intruder is located at time 1. Thus, $\gamma_{LD(1)}(G) \leq \sum_{i=2}^{t'}(n_i - 1) + (t - t')(1) = \left(\sum_{i=2}^{t} n_i\right) - (t' - 1)$.

To show the opposite inequality, let S be any $\gamma_{LD(1)}(G)$-set and take i, j such that $1 \leq i < j \leq t$. Suppose $(S_i \cup S_j) \cap S = \phi$. Given $v_i \in S_i$ and $v_j \in S_j$, the following are indistinguishable by S. (i) The intruder is at v_i at time 0 and at v_j at time 1. (ii) The intruder is at v_j at time 0 and at v_i at time 1. Hence, S is not a 1-step locating-dominating set of G, a contradiction. Hence, we have the following.

$$(1) \qquad (S_i \cup S_j) \cap S \neq \phi \text{ for } 1 \leq i < j \leq t$$

Now take S_i and S_j to be partite sets of G such that there are vertices v_1 and v_2 in $S_i \cap \bar{S}$ and there are vertices $u_1$and u_2 in $S_j \cap \bar{S}$. Note that none of v_1, v_2, u_1, u_2 are located at time 0. The following are indistinguishable by S. (i) The intruder is at v_1 at time 0 and at u_1 at time 1. (ii) The intruder is at v_2 at time 0 and at u_2 at time 1. Hence, S is not a 1-step locating-dominating set of G, a contradiction. Hence, we have (2).

$$(2) \qquad \left|S_i \cap \bar{S}\right| \geq 2 \quad \text{for at most one partite set of} \quad G.$$

From (1) and (2) we see that at most one S_i satisfies $S_i \cap S = \phi$, and at most one S_i can fail to have two or more of its vertices in S. It follows that $\gamma_{LD(1)}(G) \geq \left(\sum_{i=2}^{t} n_i\right) - (t' - 1)$. □

2. Bounds on $\gamma_{LD(1)}$

We first show that $(\gamma(G), \gamma_{LD(1)}(G), \gamma_{LD}(G))$ can be any triple of values satisfying Theorem 1.1.

Theorem 2.1. *Given any positive integers a, b and c with $a \leq b \leq c$, there exists a graph G such that $\gamma(G) = a$, $\gamma_{LD(1)}(G) = b$ and $\gamma_L(G) = c$.*

Proof. Given integers a, b and c with $a \leq b \leq c$, construct graph G as follows. Let $V(G) = \{u_1, u_2, \ldots, u_{a-1}, v_1, v_2, \ldots, v_{b+1}, w_1, w_2, \ldots, w_{c-b+1}\}$, and let $E(G) = \{u_i v_i, 1 \leq i \leq a-1\} \cup \{v_i v_j, 1 \leq i < j \leq b+1\} \cup \{v_a w_i, 1 \leq i \leq c - b + 1\}$.

Note that each dominating set for G must contain either u_i or v_i, $1 \leq i \leq a - 1$, and either w_1 or v_a. Hence, $\{v_1, v_2, \ldots, v_a\}$ is a $\gamma(G)$-set and $\gamma(G) = a$.

Let S be a $\gamma_{LD(1)}(G)$-set. Because S is a dominating set, we have $S \cap \{u_i, v_i\} \neq \phi$ for $1 \leq i \leq a-1$ and $S \cap \{w_1, v_a\} \neq \phi$. If $b+1 \geq a+2$ and $a+1 \leq i < j \leq b+1$, then $N[v_i] = N[v_j]$. Thus, S contains $b-a$ of the vertices in $\{v_{a+1}, \ldots, v_{b+1}\}$ by Lemma 1.3. In particular, $|S| \geq b$. Let $S = \{v_1, v_2, \ldots, v_b\}$. To see that S is a 1-step locating-dominating set, note that an intruder at any $v \in S$ is immediately located (at time zero). For any $v \in V(G) - S$ we have $N(v) \subseteq S$, so any intruder at vertex v at time zero will be located at a vertex in S at time one. Hence, $\{v_1, v_2, \ldots, v_b\}$ is a 1-step locating-dominating set and $\gamma_{LD(1)}(G) = b$.

For $a = b = 1$, we can let G be $K_{1,c}$. So, assume $b \geq 2$. $S = \{v_1, v_2, \ldots, v_a, v_{a+1}, \ldots, v_b, w_1, w_2, \ldots, w_{c-b}\}$ is easily seen to be a locating-dominating set. So, $\gamma_L(G) \leq c$. Let S be a $\gamma_L(G)$-set. Then $S \cap \{u_i, v_i\} \neq \phi$ for $1 \leq i \leq a-1$ because S dominates. Also, $|S \cap \{v_{a+1}, v_{a+2}, \ldots, v_{b+1}\}| \geq b-a$ by Lemma 1.4. If $v_a \notin S$, then $\{w_1, w_2, \ldots, w_{c-b+1}\} \subseteq S$ because S dominates, and $|S \cap \{w_1, w_2, \ldots, w_{c-b+1}\}| \geq c-b$ by Lemma 1.4. So, $|S \cap \{v_a, w_1, w_2, \ldots, w_{c-b+1}\}| \geq c-b+1$. Thus, $|S| \geq (a-1)+(b-a)+(c-b+1) = c$. Hence, $\gamma_L(G) = c$. □

We let $IL(0)$ and $IL(1)$ denote the intruder's vertex location at times 0 and 1, respectively. Note that $\{IL(0), IL(1)\}$ must be an edge in $E(G)$. For each $v \in D$ we let $v(0)$ and $v(1)$ denote the output (0, 1, or 2) transmitted from v in times 0 and 1, respectively. For example, $w(0) = 1$ and $w(1) = 0$ means that $IL(0) \in N(w)$ and $IL(1) \notin N[w]$, and $w(0) = 2$ means that $IL(0) = w$.

The girth of G, the length of a shortest cycle, is denoted by $g(G)$. Acyclic graphs are assumed to have infinite girth. The next theorem gives a sufficient condition for the "strong equality" of $\gamma(G)$ and $\gamma_{LD(1)}(G)$ in the sense of Haynes and Slater [25, 24] and Haynes, Henning and Slater [21, 22], namely, $\gamma(G)$ and $\gamma_{LD(1)}(G)$ are strongly equal, denoted $\gamma(G) \equiv \gamma_{LD(1)}(G)$, when not only is $\gamma(G) = \gamma_{LD(1)}(G)$ but every $\gamma(G)$-set is actually a $\gamma_{LD(1)}(G)$-set.

Theorem 2.2. *Let G be a graph with girth $g(G) \geq 7$. Then, $\gamma(G)$ and $\gamma_{LD(1)}(G)$ are strongly equal, $\gamma(G) \equiv \gamma_{LD(1)}(G)$.*

Proof. Let $D \subseteq V(G)$ be a dominating set of G. If $IL(0) = v \in D$, then D locates the intruder at time zero. If $IL(0) = v \notin D$, then there is a vertex w in $D \cap N(v)$. Suppose $IL(1) = x \in N(v)$. If $x \in D$, then D locates the intruder at time one. So, suppose $x \notin D$. Note that $x \notin N(w)$ since G does not contain any triangles. Let $u \in D \cap N(x)$. Then, $uv \notin E(G)$, again because G does not contain any triangles. We have $w(0) = 1, u(0) = 0, w(1) = 0$, and $u(1) = 1$. So, it is known that the intruder has moved from $N(w)$ to $N(u)$. Suppose the intruder moves from $z \in N(w)$ at time zero to $y \in N(u)$ at time one where $y \neq x$. Then $v \notin N(y)$ or else v, x, u, y is a

4-cycle. So, $z \neq v$. But, then w, v, x, u, y, z is a 6-cycle. Since, $g(G) \geq 7$, no such y exists. Thus, the intruder is located at x at time one. Hence, D is a 1-step locating-dominating set. In particular, every $\gamma(G)$-set D is a $\gamma_{LD(1)}(G)$-set, so $\gamma(G) \equiv \gamma_{LD(1)}(G)$. □

Corollary 2.3. *For every tree T, we have $\gamma(T) \equiv \gamma_{LD(1)}(T)$. In particular, $\gamma_{LD(1)}(P_n) = \lceil \frac{n}{3} \rceil$.*

For $(\gamma(G), \gamma_{LD(1)}(G), \gamma_{LD}(G)) = (a, b, c)$, does having girth $g(G) \geq 7$ bound either $c - b$ or $\frac{b}{c}$ in any way? To see that the answer is no, consider the following construction. Attach c vertices to a path P_a on $a = b$ vertices such that each new vertex is attached to just one vertex on the path and each vertex on the path has at least one new vertex attached to it. Note that for $a = b = 1$ we have a star and for $a = b = 2$ we have a double star. This graph is a tree and so has girth greater than 7. Thus, even when restricted to trees, $c - b$ can be arbitrarily large or small and $\frac{b}{c}$ can be any rational number in $(0, 1]$.

Vertex set $S \subseteq V(G)$ is independent if no two vertices in S are adjacent, and the independence number $\beta(G)$ is the maximum cardinality of an independent set. Note that S is independent if and only if its complement $V(G) - S = R$ is a cover, that is, for every edge $\{u, v\} \in E(G)$ we have $\{u, v\} \cap R \neq \phi$. The cover number of G, denoted $\alpha(G)$, is the minimum cardinality of a cover. A theorem of Gallai states, in part, that $\alpha(G) + \beta(G) = n = |V(G)|$.

Theorem 2.4. *For graph G with minimum degree $\delta(G) \geq 1$, $\gamma_{LD(1)}(G) \leq \alpha(G) = |V(G)| - \beta(G)$.*

Proof. Let S be a $\beta(G)$-set and $R = V(G) - S$. We claim that R is a 1-step locating-dominating set. If $IL(0) = v \in R$, then the intruder is located at time zero with $v(0) = 2$. If $IL(0) = v \notin R$, then $N(v) \subseteq R$, and so $IL(1) \in R$ and the intruder is located at time one. In particular, $\gamma_{LD(1)}(G) \leq |R| = \alpha(G)$. □

Recall that distinct vertices u and v are twins if $N(u) = N(v)$. The final result in this section tells us how large $n = |V(G)|$ can be, for twin-free graph G, given the size of a minimum 1-step locating-dominating set for G.

Theorem 2.5. *If $\gamma_{LD(1)}(G) = h$, $n = |V(G)|$ and G is twin-free, then $n \leq h + 2^h - 1 + (2^h - 1)(2^h - 2)$.*

Proof. Let S be a $\gamma_{LD(1)}(G)$-set with $|S| = h$ and $R = \{x \in V(G) - S$ such that $N(x) \subseteq S\}$. Since G is twin-free, each $x \in R$ has a distinct, nonempty $N(x) \cap S$, so $|R| \leq 2^h - 1$. Now, consider $x \in T = V(G) - (S \cup R)$. Define $S_x = N(x) \cap S$ and let $N(x) - S = \{y_1, y_2, \ldots, y_t\}$. Also define $S_{x(i)} = N(y_i) \cap S, 1 \leq i \leq t$. Since S is a $\gamma_{LD(1)}(G)$-set, $S_x \neq S_{x(i)}$

for $1 \le i \le t$, and $S_{x(i)} \ne S_{x(j)}$ for $1 \le i < j \le t$. If $\Delta(< T >) \ge 2$, select x such that $deg(x) \ge 2$ and construct G^* from G by replacing $< \{x, y_1, y_2, \ldots, y_t\} >$ with the vertices $\{x_1, x_2, \ldots, x_t, y_1, y_2, \ldots, y_t\}$ and edges $\{x_i y_i, 1 \le i \le t\}$. Then connect each x_i to every vertex in S_x and each y_i to every vertex in $S_{x(i)}$. Then $T^* = V(G^*) - (S \cup R)$, $\Delta(< T^* >) \ge 1$ and $|V(G)| < |V(G^*)|$. If $\Delta(< T^* >) \ge 2$, construct G^{**} from G^* in the same way. Continue this process until $\Delta(< T^{**\cdots*} >) = 1$. Then $|T| \le |T^{**\cdots*}|$ and $|T^{**\cdots*}| \le (2^h - 1)(2^h - 2)$ - the number of ways to choose two distinct subsets of S. Thus, $n \le |S| + |R| + |T| \le h + 2^h - 1 + (2^h - 1)(2^h - 2)$ □

3. NP-COMPLETENESS

The decision problem of deciding if $\gamma_{LD}(G) \le k$ is shown to be NP-complete by Colbourn, Slater and Stewart in [11]. We conclude by showing that the following decision problem associated with parameter $\gamma_{LD(1)}$ is NP-complete.

1-Step Locating-Dominating (1SLD)
INSTANCE: Graph $G = (V(G), E(G))$, positive integer $k \le |V(G)|$.
QUESTION: Is there a 1-step locating-dominating set of size k or less for G, that is, a subset $S \subseteq V(G)$ with $|S| \le k$ such that S is a 1-step locating-dominating set?

To do so we reduce the Vertex Cover problem to 1SLD.

Vertex Cover (VC)
INSTANCE: Graph $H = (V(H), E(H))$, positive integer $j \le |V(H)|$.
QUESTION: Is there a vertex cover of size j or less for H, that is, a subset $T \subseteq V(H)$ with $|T| \le j$ such that for each edge uv in $E(H)$ at least one of u and v belongs to T?

Theorem 3.1. *The decision problem 1SLD is NP-complete.*

Proof. Given a graph H and integer j, an instance of VC, we construct an instance of 1SLD as follows. Take $k = j + |V(H)|$, $V(G) = \{u_i, u_i', u_i'' : u_i \in V(H)\} \cup \{u_{i,j} = u_{j,i} : u_i u_j = u_j u_i \in E(H)\}$ and $E(G) = \{u_i u_{i,j}, u_{i,j} u_j : u_i u_j \in E(H)\} \cup \{u_i u_i', u_i' u_i'' : u_i \in V(H)\}$. This construction is equivalent to first subdividing each edge of H and then attaching a P_2 to each vertex of H in the resulting graph. For example, if H is the House graph shown in Figure 4(a), then G is the graph shown in Figure 4(b).

Suppose we have a vertex cover $T \subseteq V(H)$ of H with $|T| \le j$. Our first claim is that $S = T \cup \{u_i' : u_i \in V(H)\}$ is a 1-step locating-dominating set of G with $|S| \le k$. Clearly, $|S| = |T| + |\{u_i' : u_i \in V(H)\}| \le j + |V(H)| = k$. To see the rest, first note that an intruder at any vertex in S is immediately

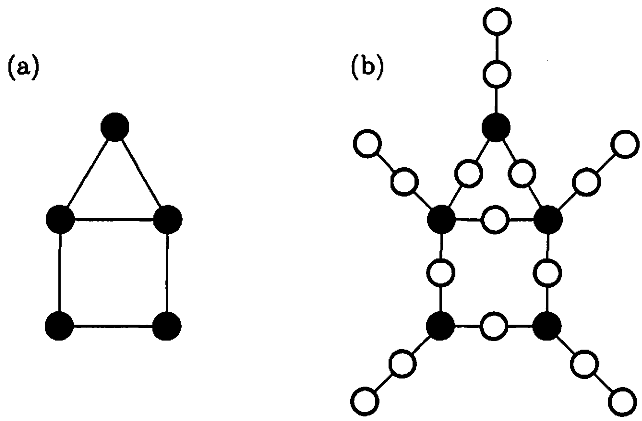

FIGURE 4. Starting with (a) the House graph H, we construct (b) G by first subdividing each edge of H and then attaching a P_2 to each vertex of H in the resulting graph.

located. However, an intruder at some $u_i'' \in V(G)$ may or may not be immediately located. If not, such an intruder can only move to $u_i' \in S$ at which point the intruder is located at time one. An intruder at some $u_i \in V(G) - S$ is in $N(u_i')$. Moving to u_i' locates the intruder at time one since $u_i' \in S$, while moving to some $u_{i,j}$ also locates the intruder at time one. To see this, recall that T is a vertex cover of H. Thus, for each $u_{i,j} \in V(G)$ one of u_i and u_j is in $T \subseteq S$. So, our intruder is now known to have moved from $N(u_i')$ to $N(u_j)$. Since there can only be one edge between any two vertices in $V(H)$, this can only happen if $IL(0) = u_i$ and $IL(1) = u_{i,j}$. Hence, the intruder is located at time one. Finally, consider an intruder at some $u_{i,j}$ at time zero. If both u_i and u_j are in S, then the intruder is immediately located. Otherwise, the intruder can only move to u_i or u_j, one of which is in S. WLOG, suppose u_i is in S. Moving to u_i, the intruder is located at time one since $u_i \in S$. Moving to u_j, the intruder is also located at time one. To see this, note that we know the intruder has moved from $N(u_i)$ to $N(u_j')$. Since $IL(1)$ cannot be u_j'' in this situation, we know $IL(1) = u_j$.

Now suppose we have a 1-step locating-dominating set $S \subseteq V(G)$ of G with $|S| \leq k$. We claim that there is a vertex cover $T \subseteq V(H)$ of H with with $|T| \leq j$. Since S dominates G, at least one of u_i'' and u_i' must be in S for each $u_i \in V(H)$. Excluding these vertices, there are $|S| - |V(H)| \leq k - |V(H)| = j$ vertices in S that must dominate, in particular, $\{u_{i,j} : u_iu_j \in E(H)\}$. Take T_1 to be $S \cap \{u_i : u_i \in V(H)\}$ and construct

T_2 as follows. For each $u_{i,j} = u_{j,i} \in S$ place one of u_i and u_j in T_2. Then, $T = T_1 \cup T_2$ must dominate $\{u_{i,j} : u_i u_j \in E(H)\}$. Hence, T is a vertex cover of H with $|T| \leq j$.

Thus, there is a "yes" answer to the instance of VC if and only if there is a "yes" answer to the constructed instance of 1SLD. Since 1SLD is clearly in NP, this shows that 1SLD is NP-complete. □

4. Related Parameters

In addition to further work on $\gamma_{LD(1)}$ we have started to consider the following related parameters.

When a detection device at vertex v can detect the presence of an intruder in $N[v]$ but which vertex in $N[v]$ can not be pinpointed, then one is interested in having an identifying-code. Here, instead of using our three-state detection devices, we use two-state detection devices with the following properties. A detection device placed at a vertex v is assumed to transmit one of two possible outcomes in each time period: 0 if there is no intruder in $N[v]$; 1 if there is an intruder in $N[v]$. We assume that there is at most one intruder, and $IC(G)$ or $\gamma_{IC}(G)$ denotes the minimum cardinality of an identifying code C for G, that is, a dominating set $C \subseteq V(G)$ such that $N[x] \cap C \neq N[y] \cap C$ for all vertices $x \neq y$ in $V(G)$. One can consider the 1-step identifying-code problem of placing the minimum possible number of two-state detection devices in $V(G)$ so that the presence of an intruder in $V(G)$ can be detected, and the exact location of the intruder can be identified either immediately (at time zero) or when the intruder has moved to an adjacent vertex (at time one). We will call a dominating set D with this capability a *1-step identifying-code*, and the *1-step identifying-code number* $\gamma_{IC(1)}(G)$ is the minimum possible cardinality of such a set.

With details to appear elsewhere, we have the next result.

Proposition 4.1. *For the (long) ladder L of order $|V(L)| = n = 2k$, we have $\gamma_{IC(1)}(L) \approx \frac{3n}{7}$.*

One can easily see that $LD(G) \leq IC(G)$ and $\gamma_{LD(1)}(G) \leq \gamma_{IC(1)}(G)$ when G has an identifying code.

Proposition 4.2. *When G has an identifying code, $\gamma(G) \leq \gamma_{LD(1)}(G) \leq \gamma_{LD}(G) \leq \gamma_{IC}(G)$ and $\gamma_{LD(1)}(G) \leq \gamma_{IC(1)}(G) \leq \gamma_{IC}(G)$.*

Parameters γ_{LD} and $\gamma_{IC(1)}$ are incomparable. We have $\gamma_{LD}(P_n) \approx \frac{2n}{5} < \gamma_{IC(1)}(P_n) = \gamma_{IC}(P_n) = \lceil \frac{n}{2} \rceil$, and for the tree on $n = k+3$ vertices in Figure 5 we have $\gamma_{IC(1)}(T_n) = 3 < \gamma_{LD}(T_n) = k + 1 = n - 2$.

A 1-step parameter can be extended to a k-step parameter in different ways. For example, we can consider the problem of precisely identifying the location of an intruder either immediately or while the intruder moves a distance of k, along a path of length k, along a trail of length k, or along

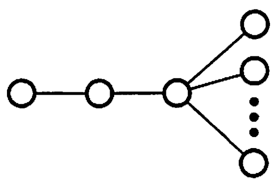

FIGURE 5. Tree T_n is a path on three vertices with k vertices attached to the same endpoint.

a walk of length k. These ideas lead to multiple k-step locating-dominating and identifying code parameters. We let $LD_{d(k)}$, $LD_{p(k)}$, $LD_{t(k)}$, $LD_{w(k)}$, and $IC_{d(k)}$, $IC_{p(k)}$, $IC_{t(k)}$, $IC_{w(k)}$ denote the parameters when an intruder can be located/identified while traveling a distance of k, a path of length k, a trail of length k, or a walk of length k, respectively.

A 1-step (or k-step) parameter can also be extended to consider multiple intruders that may, or may not, be able to simultaneously occupy the same location or to consider various kinds of detection device faults, be they faults in detection, faults in reporting or faults of another nature.

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