Monadic Balanced Ternary Designs

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Abstract

This paper investigates the existence of monadic balanced ternary designs (BTDs). A monadic BTD is a BTD where each size K block contains one element that appears doubly and $K-2$ elements that appear singly. The authors show that the conditions (1) $\rho_1 = 2\rho_2$, (2) $\Lambda(V-1) = 10\rho_2$, and (3) $\Lambda \neq 3$ are sufficient for the existence of monadic BTD$(V; B; \rho_1, \rho_2, R; 4; \Lambda)$s. The authors also give necessary and sufficient conditions for the existence of monadic BTDs where the block size is five and Λ is 3 or 6.

Keywords and phrases: BIBD, BTD, balanced ternary design, nested design, balanced incomplete block design.

1 Introduction

A balanced ternary design, or BTD, with parameters (V, B, R, K, Λ) is a collection of B blocks on V elements such that (1) each element occurs R times in the design, (2) each pair of distinct elements occurs Λ times in the design, where Λ is called the index of the design, and (3) each block contains K elements, where an element may occur 0, 1, or 2 times in a block (i.e., each block is a multiset).

An example of a BTD with parameters $(5; 10; 4, 2, 8; 4; 5)$ appears in Figure 1. In the figure, each column represents a block of the design.

1	0	2	3	4	0	1	4	3	2
1	0	2	3	4	0	1	4	3	2
0	1	1	2	0	2	3	0	0	1
4	2	3	4	3	3	4	2	1	4

Figure 1: A BTD$(5; 10; 4, 2, 8; 4; 5)$.

BTDs are regular in the sense that each point occurs singly in ρ_1 blocks and doubly in ρ_2 blocks [5]. Because of this regularity, BTD parameters are most often given as $(V; B; \rho_1, \rho_2, R; K; \Lambda)$.

The purpose of this paper is to investigate a class of BTDs which we call monadic. A BTD$(V; B; \rho_1, \rho_2, R; K; \Lambda)$ is said to be monadic if every block in the design contains one element which appears doubly and $K-2$ elements which appear singly. The design shown in Figure 1 is a monadic BTD(5; 10; 4, 2, 8; 4; 5)

Counting arguments can be used to establish the necessity of the following well-known relationships among a BTD's parameters:

$$VR = BK,$$
$$\Lambda(V-1) = \rho_1(K-1) + 2\rho_2(K-2), \text{ and}$$
$$R = \rho_1 + 2\rho_2.$$

Since each block in a monadic BTD contains exactly one doubleton, in monadic BTDs it is further true that $B = V\rho_2$. Thus, in monadic BTDs, the parametric relationships reduce to:

$$R = \rho_2 K,$$
$$\rho_1 = (K-2)\rho_2, \text{ and}$$
$$\Lambda(V-1) = (K+1)(K-2)\rho_2.$$

These reduced relationships imply that V, K, and Λ are sufficient to specify the other parameters of a monadic BTD. We take advantage of this fact throughout the paper by using the notation BTD(V, K, Λ) for monadic BTDs.

In Section 2 we investigate monadic BTDs with block size four. By showing the non-existence of BTD$(V, 4, 3)$ and the existence of (1) monadic BTD$(5t+1, 4, 2s)$ for $t \geq 1$ and $s \geq 1$, (2) monadic BTD$(V, 4, 5s)$ for $V \geq 3$ and $s \geq 1$, and (3) monadic BTD$(V, 4, 10s)$ for $V \geq 3$ and $s \geq 1$ the sufficiency of the conditions (1) $\rho_1 = 2\rho_2$, (2) $\Lambda(V-1) = 10\rho_2$, and (3) $\Lambda \neq 3$ for the existence of monadic BTD$(V, 4, \Lambda)$ is established.

In Section 3 we investigate monadic BTDs with block size five. Necessary and sufficient conditions are established for BTD$(V, 5, 3)$s and BTD$(V, 5, 6)$s.

For the interested reader, more detail about BTDs and their generalizations appear in [4], [5], [9], [12], [13] and [15], and about nested designs in [10], [11], and [14]. In [8], the present authors considered nested BTDs with odd block size in which each block has exactly one singleton and $(k-1)/2$ doubletons. Since, for block size three, the previous study and this would overlap, we begin this study with block size four.

2 Monadic BTDs with Block Size Four

In this section we restrict our investigation to monadic BTDs with block size four.

Proposition 1 *If a BTD$(V; B; \rho_1, \rho_2, R; 4; \Lambda)$ is monadic, then (1) $R = 4\rho_2$, (2) $\rho_1 = 2\rho_2$, and (3) $\Lambda(V-1) = 10\rho_2$.*

The necessary conditions for the existence of monadic BTD$(V; B; \rho_1, \rho_2, R; 4; \Lambda)$ shown in Proposition 1 severely restrict the set of possible indices for monadic BTD$(V, 4, \Lambda)$s. Indices are further restricted by the extra regularity imposed on the blocks of monadic designs. This is illustrated by the following non-existence result.

Proposition 2 *No monadic BTD*$(V,4,3)$ *exists.*

Proof. Suppose that D is a monadic BTD$(V,4,3)$. Since the index of D is odd, each element x must appear with every other element in at least one block where both are singletons. This implies $\rho_1 \geq V-1$. But from Proposition 1 we know $\rho_1 = 2\rho_2$ and $3(V-1) = 10\rho_2$ in D. Hence, we have $2\rho_2 = \rho_1 \geq V-1 = 10\rho_2/3$, a contradiction. Thus, no monadic BTD$(V,4,3)$ can exist. ∎

Throughout the remainder of the section we turn our attention to existence results for monadic BTD$(V,4,\Lambda)$. In sub-Section 2.1 we show how to construct BTD$(V,4,\Lambda)$s for even Λ, in sub-Section 2.2 we show how to construct BTD$(V,4,\Lambda)$s for Λ divisible by five, and in Section 2.3 we show how to construct BTD$(V,4,\Lambda)$s for odd Λ not divisible by 5.

2.1 Monadic BTD(V, 4, 2t)

Proposition 3 *For any monadic BTD*$(V,4,2t)$, $V \equiv 1 \pmod 5$ *or* $t \equiv 0 \pmod 5$.

Proof. Follows from Proposition 1 part (3). ∎

In the special case where $\Lambda = 2$, Proposition 1 implies that $V \equiv 1 \pmod 5$. The smallest such BTD$(V,4,2)$ would have $V = 6$ elements.

Example 4 *[6] A monadic BTD*$(6,4,2)$ *exists. The blocks of the BTD*$(6,4,2)$ *are given by blocks* $\{0+i, 0+i, 1+i, 4+i\}$ *where* $i = 0, 1, ..., 5$, *and the addition is done mod* 6. *The block* $\{0,0,1,4\}$ *is called a starter block, and the design is said to be cyclically or additively generated.*

A BIBD(v,k,λ) can be defined as a BTD$(v;b;\rho_1,0,\rho_1;k;\lambda)$ (i.e., a BTD in which no element appears doubly in a block). The necessary conditions for BIBDs are (1) $v\rho_1 = bk$ and (2) $\lambda(v-1) = \rho_1(k-1)$.

Proposition 5 *If a BIBD*(v,k_1,λ_1) *and a monadic BTD*(k_1,k_2,λ_2)*, exist, then a monadic BTD*$(v,k_2,\lambda_1\lambda_2)$ *exists.*

Proof. Suppose B_i is a block of a BIBD(v,k_1,λ_1). Identify the k_1 elements of B_i arbitrarily with the k_1 elements of a monadic BTD(k_1,k_2,λ_2). Now create blocks as in the monadic BTD. Do this for each B_i. ∎

Note that Proposition 5 used with the monadic BTD$(6,4,2)$ of Example 4 implies that a BTD$(v,4,2\lambda)$ exists whenever a BIBD$(v,6,\lambda)$ exists.

For details on the existence and non-existence of BIBD$(v,6,\lambda)$ see Table 3.3 on page 72 in [2].

D. Donovan in [7] proved a key existence result for BTDs with block size four. We incorporate her result into Proposition 6 using the terminology of this paper.

Proposition 6 *A cyclic monadic BTD*$(5t+1, 4, 2s)$ *exists for all positive integers* t *and* s.

Proof. In [7] Donovan constructed cyclic monadic BTD$(5t+1, 4, 2)$ for all $t \geq 1$. To construct monadic BTD$(5t+1, 4, 2s)$ for $s > 1$ simply combine s copies of a BTD$(5t+1, 4, 2)$. ■

2.2 Monadic BTD(V, 4, 5t)

Proposition 2 implies that the smallest odd index for a monadic BTD$(V, 4, \Lambda)$ is five. Moreover, Proposition 1 tells us that when the index of a monadic BTD$(V, 4, \Lambda)$ is five, V is odd.

Example 7 *A cyclic monadic BTD*$(5, 4, 5)$ *is generated mod* 5 *by* $\{0, 0, 2, 3\}$ *and* $\{0, 0, 1, 4\}$.

Proposition 8 *There exists a monadic BTD*$(V, 4, \Lambda)$ *with:*
(a) $\Lambda = 5$ *when* $V \equiv 1, 3 \pmod 6$;
(b) $\Lambda = 15$ *when* $V \equiv 5 \pmod 6$;
(c) $\Lambda = 10$ *when* $V \equiv 0, 4 \pmod 6$; *and*
(d) $\Lambda = 30$ *when* $V \equiv 2 \pmod 6$.

Proof. Suppose $\{a, b, c\}$ is a block of a BIBD$(V, 3, \lambda)$. Replace the block with the three blocks $\{a, a, b, c\}$, $\{a, b, b, c\}$, and $\{a, b, c, c\}$. Doing this for each block of the BIBD creates the desired monadic BTD. The conditions in (a)..(d) correspond to the necessary conditions for the existence of a BIBD with the stated Λ. ■

Part (b) of the theorem can be lowered to index five by a variant of a construction from [8]. As we show below, part (d) can be improved to index 10. Before we present this result, we give two definitions and state the structure theorem of H. Agrawal's that is used in the construction.

One may modify a monadic BTD(V, B, R, K, Λ) by replacing each doubleton in a block with a single appearance of the same element. When this is done, the resulting structure consists of blocks of size $K-1$. Sometimes the resulting structure is a BIBD, sometimes it is not (the reduction of the design in Figure 1 is a BIBD while the reduction of the BTD in Proposition 4 is not). When the resulting structure is a BIBD we say it is nested in the monadic BTD. (The concept of nested designs appear in a more general form in the literature, see for example [10] or [11]. However, for our purposes the restricted definition given above suffices.)

The Agrawal result that we use in our construction is stated for binary equi-replicate designs. A binary equi-replicate design is a collection of b size k blocks (i.e., each block is a set) over a v-set of elements such that each element appears in r blocks.

Proposition 9 *[3] The elements of every binary equi-replicate design with $bk = vr$ and $b = mv$, can be arranged in a $k \times b$ array such that each column represents a block of the design and each row contains m copies of each element.*

We are now in a position to prove the main results of the section.

Proposition 10 *Every BIBD$(v, 3, 3)$ is nested within a monadic BTD$(v, 4, 5)$ with $b = v(v-1)/2$, $\rho_1 = (v-1)$, and $\rho_2 = (v-1)/2$. Moreover, for every odd v, there exists a monadic BTD$(v, 4, 5)$ which has a BIBD$(v, 3, 3)$ nested within it.*

Proof. Suppose D is a BIBD$(v, 3, 3)$. Each block $\{x, y, z\}$ in D corresponds to a set of three unordered pairs $\{xy, xz, yz\}$. If we regard each block of D as a block of pairs, we get an equi-replicate design in which each pair occurs three times and m, as defined in Proposition 9, is one. Thus by Agrawal's theorem, we can create a $3 \times b$ array where each column represents a block of the equi-replicate design and every pair appears exactly once in each row. Use this array to build a new structure on the elements of D as follows. If a column in the array corresponds to the block $\{xy, xz, yz\}$ with xy appearing in row one, xz appearing in row two and yz appearing in row three, define a block $\{x, x, y, z\}$ in the new structure. The two singletons in the new block are the two elements from the pair in row three. The doublteton in the new block is the element of D that does not appear in the pair of row three of the array but which appears in both rows one and two.

Since each pair of elements x and y of D appear together once in each row of the array, they appear together, counting multiplicities, five times in the new structure. Each is a singleton in the block that correponds to the column where they appear as a pair in row three, and one is a doubleton and the other a singleton in the two blocks that correspond to the two columns where they appear as a pair in either row one or row two.

Since each element x of D appears $3(v-1)/2$ times in D, it appears in $3(v-1)/2$ columns of the array. Of these $3(v-1)/2$ columns, $v-1$ contain an appearance of x in row three and one other row, and the reamining $3(v-1)/2 - (v-1) = (v-1)/2$ contain an appearance of x in rows one and two. Thus, each x appears singly in $v-1$ blocks of the new structure and doubly in $(v-1)/2$ blocks. It now follows that the new struture is a monadic BTD with the indicated parameters. The second sentence of the theorem follows since there exists a BIBD$(v, 3, 3)$ for every odd $v \geq 3$. ∎

The above construction can also be used with a BIBD$(v, 3, 6)$. Here the corresponding Agrawal array will have each element pair xy occurring twice in each row. Thus, the same construction will generate two blocks where x and y are both singletons and four blocks where exactly one of x and y is a singleton and the other is a doubleton. Since BIBD$(v, 3, 6)$ exist for all $v \geq 3$, this yields the following result:

Proposition 11 *Let $v \geq 3$, then there exists a BIBD$(v,3,6)$ which is nested within a monadic BTD$(v,4,10)$, and conversely, there exists a BTD$(v,4,10)$ which has a BIBD$(v,3,6)$ nested within.*

Using repeated copies of a BTD$(V,4,5)$ we can conclude:

Proposition 12 *There exists a monadic BTD$(V,4,5s)$ for all odd $V \geq 3$ and $s \geq 1$.*

Using repeated copies of a BTD$(V,4,10)$ we can conclude:

Proposition 13 *There exists a monadic BTD$(V,4,10s)$ for all $V \geq 3$ and $s \geq 1$.*

2.3 Monadic BTD(V, 4, Λ) with Λ odd and $\Lambda \geq 5$

Proposition 14 *A BTD$(V,4,5)$ and a BTD$(V,4,2)$ both exist, if and only if $V = 10t+1$ for $t \geq 1$.*

Proof. Combine Propositions 6 and 12. ∎

Proposition 15 *There exists a monadic BTD$(V,4,\Lambda)$ for all $V = 10t+1$ and odd Λ greater than three.*

Proof. Suppose Λ is odd and greater than three. Then Λ can be written as $2s+5$ for some $s \geq 0$. Construct a monadic BTD$(V,4,2s+5)$ by combining the blocks of a BTD$(V,4,5)$ and s-copies of a BTD$(V,4,2)$. Note that a BTD(V,4,5) and a BTD(V,4,2) both exist in this case. ∎

3 BTDs with Block Size Five

In this section we restrict our investigation to monadic BTDs with block size five. Unlike the case where the monadic designs had block size four, we do not give a complete picture. However, we do present necessary and sufficient conditions for the existence of monadic BTD$(V,5,\Lambda)$s when Λ is 3 or 6. The section also includes several other isolated examples of monadic BTD$(V,5,\Lambda)$s.

Proposition 16 *If a BTD$(V;B;\rho_1,\rho_2,R;5;\Lambda)$ is monadic, then (1) $R = 5\rho_2$, (2) $\rho_1 = 3\rho_2$, and (3)$\Lambda(V-1) = 18\rho_2$.*

As was true in the block size four case, the necessary parameter relationships are not sufficient for the existence of a monadic design.

A BTD is said to be symmetric if the number of elements in the design equals the number of blocks in the design. If a monadic BTD$(10,5,2)$ exists it is necessarily symmetric since Proposition 16 implies that $b = 10$, the number of elements in the design.

Proposition 17 *There does not exist a monadic BTD$(10,5,2)$.*

Proof. Follows from the Bruck-Ryser-Chowla theorem for symmetric ternary designs since $5^2 - 2 * 10 = 5$ is not a perfect square. The Bruck-Ryser-Chowla Theorem for ternary designs [5] states: In a symmetric balanced ternary design with parameters v, k, and λ, (i) if v is even, then $k^2 - \lambda v$ is a perfect square, and (ii) if v is odd, then $z^2 = (k^2 - \lambda v)x^2 + (-1)^{(v-1)/2}\lambda y^2$ has a solution in integers x, y, z not all zero. ■

Although a BTD(10, 5, 2) does not exist, a monadic BTD(10, 5, 4) does exist. A cyclic monadic BTD(10, 5, 4) can be generated *mod* 10 by base blocks $\{0, 0, 5, 6, 8\}$ and $\{0, 0, 1, 3, 4\}$.

Proposition 18 *If a* $BTD(V; B; \rho_1, \rho_2, R; 5; \Lambda)$ *is monadic then:*
(a) if $\Lambda = 2$*, then* $V \geq 19$ *and* $V = 1 + 9\rho_2$*;*
(b) if $\Lambda = 3$*, then* $V = 1 + 6\rho_2$*;*
(c) if $\Lambda = 4$*, then* $V = 1 + 9t$ *for some* t*, and* $2(V - 1) = 9\rho_2$*;*
(d) if $\Lambda = 5$*, then* $V \equiv 1 \pmod{18}$*; and*
(e) if $\Lambda = 6$*, then* $V = 1 + 3\rho_2$*.*

Proof. Part (a) follows from Propositions 16 and 17. The other parts follow directly from Proposition 16 part (3). ■

We close by showing that the conditions in Proposition 18 parts (b) and (e) are sufficient as well as necessary.

A near-resolvable BIBD is a BIBD with the property that the blocks of the design can be partitioned into classes such that (1) no element appears more than once in a class, (2) each class is missing a single element, and (3) each element is missing from exactly one class. Near-resolvable BIBD($v, k, k - 1$) exist for all $v \equiv 1 \pmod{k}$, see [1].

Proposition 19 *A monadic* $BTD(v, 5, 6)$ *exists for all* $v \equiv 1 \pmod{3}$*.*

Proof. Assume A_x is the class of blocks missing element x in a near-resolvable BIBD($v, 3, 2$). Augment each of the blocks in the class with a doubleton of the element x. Do this for each of the near-resolvable classes. Now each pair of distinct elements x and y will appear together six times in the new collection of blocks. Four times in blocks created from A_x and A_y and twice more from the blocks created from the two original blocks where the pair x and y appear together. The result follows. ■

A BIBD($v, 3, 1$) over an element set X is said to be nested in a BIBD($v, 4, 2$) over the same element set if there exists a way to add one element of X to each block in the BIBD($v, 3, 1$) to produce the blocks of the BIBD($v, 4, 2$). Stinson proved the following:

Proposition 20 *[16] For* $v = 1 + 6t$*, there exists a* $BIBD(v, 3, 1)$ *which can be nested in a* $BIBD(v, 4, 2)$*.*

The Stinson constructions can be used to generate BTD$(v,5,3)$s.

Proposition 21 *A monadic BTD$(v,5,3)$ exists if and only if $v = 1 + 6t$.*

Proof. To construct the monadic BTD$(1+6t,5,3)$ use Stinson's constructions from Proposition 19. Instead of augmenting each block $\{a,b,c\}$ to $\{a,b,c,x\}$, augment it to $\{a,b,c,x,x\}$.

Proposition 18 tells us that $v = 6t + 1$ is necessary. ■

The final proposition of the paper further describes the designs of Proposition 21.

Proposition 22 *Every monadic BTD$(v,5,3)$ has a BIBD$(v,3,1)$ nested in it*

Proof. Let D be a monadic BTD$(v,5,3)$ with element set X and block set Y, and let $i \in X$. Define: $A(i) = \{y \in X : \{i,i,y,z,w\}$ is a block of Y for some z, w in $X\}$, $B(i) = \{y \in X : \{y,y,i,z,w\}$ is a block of Y for some z, w in $X\}$. $C(i) = \{y \in X : i$ and y appear together in three blocks of Y in which each is a singleton.$\}$. Since the index of D is 3, it is straightforward to show that $A(i)$, $B(i)$, and $C(i)$ are mutually disjoint and that $|A(i)| + |B(i)| + |C(i)| = V - 1$.

Since D is a monadic BTD$(v,5,3)$, Proposition 16 tells us that $\rho_1 = 3\rho_2$ and $(v-1) = 6\rho_2$. But clearly $|A(i)| = 3\rho_2$. Also $|B(i)| = \rho_1 = 3\rho_2$ since no two elements a and b can appear in a pair of blocks $\{a,a,b,c_1,d_1\}$, $\{a,a,b,c_2,d_2\}$. Thus $|A(i)| + |B(i)| = 6\rho_2$ which implies that $|C(i)| = 0$. From this we can conclude that every pair of elements in X appear together in one block of Y both as singletons and in one block where one is a singleton and the other is a doubleton. Thus, deleting the doubleton from each block will create a BIBD$(v,3,1)$. ■

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