The Tree Connectivity of Regular Complete Bipartite Graphs

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Abstract

For a set S of two or more vertices in a nontrivial connected graph G of order n, a collection $\{T_1, T_2, \ldots, T_\ell\}$ of trees in G is said to be an internally disjoint set of trees connecting S if these trees are pairwise edge-disjoint and $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \le i, j \le \ell$. For an integer k with $2 \le k \le n$, the tree k-connectivity $\kappa_k(G)$ of G is the greatest positive integer ℓ for which G contains at least ℓ internally disjoint trees connecting S for every set S of k vertices of G. It is shown for every two integers k and r with $3 \le k \le 2r$ that $\kappa_k(K_{r,r}) = r - \lceil (k-1)/4 \rceil$.

Key Words: connectivity, internally disjoint set of trees, tree connectivity.
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1 Introduction

The *connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal from G results in a disconnected or trivial graph. If $\kappa(G) \ge t$ for some positive integer t, then G is t-connected. By a well-known theorem of Whitney [3], a graph G is t-connected if and only if G contains t internally disjoint $u - v$ paths for every two distinct vertices u and v of G. That is, G contains $u - v$ paths $P_1, P_2, \ldots, P_t$ such that $V(P_i) \cap V(P_j) = \{u, v\}$ and $E(P_i) \cap E(P_j) = \emptyset$ for all distinct integers i and j with $1 \le i, j \le t$. In particular, if $\kappa(G) = \ell$, then G contains ℓ internally disjoint $u - v$ paths for every pair u, v of vertices of G, but G does not contain $\ell + 1$ internally disjoint $x - y$ paths for some pair x, y of vertices of G.

In [1] a generalized connectivity was introduced for the purpose of studying rainbow edge colorings of graphs. Let G be a nontrivial connected graph of order n. For a set S of two or more vertices of G, a collection $\{T_1, T_2, \ldots, T_\ell\}$ of ℓ trees in G is called an *internally disjoint set of trees connecting* S if $V(T_i) \cap V(T_j) = S$ and $E(T_i) \cap E(T_j) = \emptyset$ for every two distinct integers i and j with $1 \leq i, j \leq \ell$. The *tree connectivity* $\kappa(S)$ of S is the maximum number of internally disjoint trees connecting S. For an integer k with $2 \leq k \leq n$, the *tree k-connectivity* (or, more simply, the *k-connectivity*) $\kappa_k(G)$ of G is defined by

$$\kappa_k(G) = \min\{\kappa(S)\},$$

where the minimum is taken over all sets S of k vertices of G. Thus $\kappa_2(G) = \kappa(G)$.

In [1] the k-connectivity of complete graphs of order n was determined for each integer k with $2 \leq k \leq n$.

Proposition 1.1 [1] *For every two integers k and n with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

By Proposition 1.1, $\kappa_4(K_6) = 4$. For the complete graph K_6 in Figure 1 and the set $S = \{u, v, w, x\}$ of vertices of K_6, four internally disjoint trees connecting S are shown in this figure.

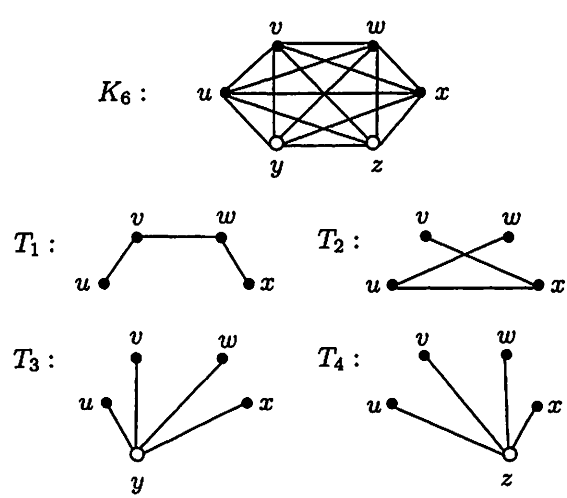

Figure 1: Four internally disjoint trees connecting the set $S = \{u, v, w, x\}$ in K_6

For every positive integer r, it is well known that the connectivity of the regular complete bipartite graph $K_{r,r}$ is $\kappa_2(K_{r,r}) = \kappa(K_{r,r}) = r$. In this paper, we determine $\kappa_k(K_{r,r})$ for every integer k with $3 \le k \le 2r$. We refer to the book [2] for graph theory notation and terminology not described in this paper.

2 The Tree Connectivity of $K_{r,r}$

By a theorem of Whitney [3],

$$\kappa(G) \le \delta(G)$$

for a graph G, where $\delta(G)$ is the minimum degree of G. Thus if G is r-regular, then $\kappa(G) \le r$. There are many regular graphs for which $\kappa_2(G) = \kappa(G) = \delta(G)$, including the complete graph K_{r+1} and the complete bipartite graph $K_{r,r}$, as we noted above. By Proposition 1.1,

$$\kappa_k(K_{r+1}) = r + 1 - \lceil k/2 \rceil$$

for every integer k with $2 \le k \le r+1$. Thus, $\kappa_k(K_{r+1}) \le r-1$ for every integer k with $3 \le k \le r+1$. In fact, $\kappa_k(G) \le r-1$ for every r-regular graph G of order n and every integer k with $3 \le k \le n$.

Proposition 2.1 *If G is an r-regular graph of order n and k is an integer with $3 \le k \le n$, then $\kappa_k(G) \le r-1$.*

Proof. Assume, to the contrary, that $\kappa_k(G) \ge r$. Let S be a k-element subset of $V(G)$, where $k \ge 3$, and suppose that $v \in S$. Let $N(v) = \{u_1, u_2, \ldots, u_r\}$ be the neighborhood of v. By assumption, there is a collection $\mathcal{T} = \{T_1, T_2, \ldots, T_r\}$ of r internally disjoint trees connecting S. Since these r trees are edge-disjoint, we may assume that $vu_i \in E(T_i)$ for $1 \le i \le r$. Therefore, every vertex in S is an end-vertex in each tree T_i for $1 \le i \le r$. Now consider a k-element subset $S' = \{w_1, w_2, \ldots, w_k\}$ of $V(G)$, where w_1 and w_2 are adjacent. Let $\mathcal{T}' = \{T_1', T_2', \ldots, T_r'\}$ be an internally disjoint set of trees connecting S' and suppose that the edge w_1w_2 belongs to T_1'. Since both w_1 and w_2 are end-vertices in T_1', it follows that $T_1' = K_2$. However, this is impossible since T_1' is a tree of order $k \ge 3$. ■

For the r-regular complete bipartite graph $K_{r,r}$, $r \ge 2$, we show that

$$\kappa_k(K_{r,r}) = r - \lceil (k-1)/4 \rceil \qquad (1)$$

for every integer k with $3 \le k \le 2r$. For this purpose, the following lemma will be useful.

Lemma 2.2 *For a positive integer p, the maximum number of pairwise edge-disjoint spanning trees in $G \in \{K_{p,p+1}, K_{p+1,p+1}\}$ is $\lfloor \frac{p+1}{2} \rfloor$.*

Proof. We first consider the graph $K_{p,p+1}$. Suppose that

$$U = \{u_1, u_2, \ldots, u_p\} \text{ and } W = \{w_1, w_2, \ldots, w_{p+1}\}$$

are the partite sets. Since the size of $K_{p,p+1}$ is $p(p+1)$ and each spanning tree contains $2p$ edges, the maximum number of pairwise edge-disjoint spanning trees is at most $\left\lfloor \frac{p(p+1)}{2p} \right\rfloor = \lfloor \frac{p+1}{2} \rfloor$. For $1 \le i \le \lfloor \frac{p+1}{2} \rfloor$,

$$T_i : w_{2i-1}, u_1, w_{2i}, u_2, w_{2i+1}, u_3, \ldots, w_{2i-2+p}, u_p, w_{2i-1+p},$$

where the subscript $2i-2+j$ $(1 \le j \le p+1)$ is expressed as one of the integers $1, 2, \ldots, p+1$ modulo $p+1$, is a Hamiltonian path in $K_{p,p+1}$. Then $\left\{T_1, T_2, \ldots, T_{\lfloor \frac{p+1}{2} \rfloor}\right\}$ is a set of $\lfloor \frac{p+1}{2} \rfloor$ pairwise edge-disjoint spanning trees in $K_{p,p+1}$.

For $K_{p+1,p+1}$, let

$$U = \{u_1, u_2, \ldots, u_{p+1}\} \text{ and } W = \{w_1, w_2, \ldots, w_{p+1}\}$$

be the partite sets. The maximum number of pairwise edge-disjoint spanning trees is at most $\left\lfloor \frac{(p+1)^2}{2p+1} \right\rfloor = \lfloor \frac{p+1}{2} \rfloor$. For $1 \le i \le \lfloor \frac{p+1}{2} \rfloor$,

$$T_i : w_{2i-1}, u_1, w_{2i}, u_2, w_{2i+1}, u_3, \ldots, w_{2i-1+p}, u_{p+1},$$

where again the subscript $2i-2+j$ $(1 \le j \le p+1)$ is expressed as one of the integers $1, 2, \ldots, p+1$ modulo $p+1$, is a Hamiltonian path in $K_{p+1,p+1}$. Therefore,

$$\left\{T_1, T_2, \ldots, T_{\lfloor \frac{p+1}{2} \rfloor}\right\}$$

is a set of $\lfloor \frac{p+1}{2} \rfloor$ pairwise edge-disjoint spanning trees in $K_{p+1,p+1}$. ∎

We are now prepared to verify (1).

Theorem 2.3 *For $r \ge 1$ and $3 \le k \le 2r$,*

$$\kappa_k(K_{r,r}) = r - \lceil (k-1)/4 \rceil .$$

Proof. First, we show that

$$\kappa_k(K_{r,r}) \le r - \lceil (k-1)/4 \rceil .$$

Let $U = \{u_1, u_2, \ldots, u_r\}$ and $W = \{w_1, w_2, \ldots, w_r\}$ be the partite sets of $K_{r,r}$. We consider two cases according to whether k is even or k is odd.

Case 1. *k is even.* Then $k = 2a$ for some integer $a \ge 2$. Consider the set

$$S = \{u_1, u_2, \ldots, u_a, w_1, w_2, \ldots, w_a\}$$

of $2a$ vertices. A vertex not belonging to S is referred to as an *external vertex*. An S*-tree* is a tree T with $V(T) = S$, an S'*-tree* is a tree T with $S \subseteq V(T)$ such that T contains exactly one external vertex, while an S''*-tree* is a tree T with $S \subseteq V(T)$ such that T contains two or more external vertices. Let $\mathcal{A} = \mathcal{A}^\circ \cup \mathcal{A}' \cup \mathcal{A}''$ be a set of pairwise edge-disjoint trees connecting S, where $\mathcal{A}^\circ$ is the set of S-trees, $\mathcal{A}'$ is the set of S'-trees, and $\mathcal{A}''$ is the set of S''-trees. Suppose that $|\mathcal{A}'| = p \geq 0$.

Let $H \cong K_{a,a}$ be the subgraph of G induced by S and $X = E(H)$. Hence $|X| = a^2$. Observe that if $T \in \mathcal{A}'$, then T contains at least a edges belonging to X. Therefore, $0 \leq p \leq a$ and furthermore,

$$|\mathcal{A}^\circ| \leq \left\lfloor \frac{a^2 - ap}{2a-1} \right\rfloor \text{ and } |\mathcal{A}''| \leq \left\lfloor \frac{2r - (2a+p)}{2} \right\rfloor.$$

Therefore,

$$\begin{aligned} |\mathcal{A}| &\leq \left\lfloor \frac{a^2 - ap}{2a-1} \right\rfloor + p + \left\lfloor \frac{2r - (2a+p)}{2} \right\rfloor \\ &\leq \frac{a^2 - ap}{2a-1} + p + \frac{2r - 2a - p}{2} \\ &= r - \frac{2a^2 - 2a + p}{2(2a-1)} \leq r - \frac{a^2 - a}{2a-1}. \end{aligned}$$

If $a = 2b$ for some $b \geq 1$, then $\frac{a^2-a}{2a-1} > b - 1$ and so

$$|\mathcal{A}| \leq r - b = r - \left\lceil \frac{4b-1}{4} \right\rceil = r - \left\lceil \frac{k-1}{4} \right\rceil.$$

If $a = 2b+1$ for some $b \geq 1$, then $\frac{a^2-a}{2a-1} > b$ and so

$$|\mathcal{A}| \leq r - (b+1) = r - \left\lceil \frac{(4b+2)-1}{4} \right\rceil = r - \left\lceil \frac{k-1}{4} \right\rceil.$$

Case 2. k is odd. We consider two subcases.

Subcase 2.1. $k = 4b+1$ for some integer $b \geq 1$. Consider the set

$$S = \{u_1, u_2, \ldots, u_{2b}, w_1, w_2, \ldots, w_{2b+1}\}$$

of $4b+1$ vertices and let $\mathcal{A} = \mathcal{A}^\circ \cup \mathcal{A}' \cup \mathcal{A}''$ be a set of pairwise edge-disjoint trees connecting S, where these sets are defined as before. Also, let $|\mathcal{A}'| = p \geq 0$.

Let $H \cong K_{2b,2b+1}$ be the subgraph of G induced by S and $X = E(H)$. Hence $|X| = (2b)(2b+1)$. Observe that if $T \in \mathcal{A}'$, then T contains at least $2b$ edges belonging to X. Therefore, $0 \leq p \leq 2b+1$ and furthermore,

$$|\mathcal{A}^\circ| \le \left\lfloor \frac{(2b)(2b+1)-(2b)p}{4b} \right\rfloor \text{ and } |\mathcal{A}''| \le \left\lfloor \frac{2r-(4b+1+p)}{2} \right\rfloor.$$

Therefore,

$$\begin{aligned} |\mathcal{A}| &\le \left\lfloor \frac{(2b)(2b+1)-(2b)p}{4b} \right\rfloor + p + \left\lfloor \frac{2r-(4b+1+p)}{2} \right\rfloor \\ &\le \frac{2b+1-p}{2} + p + \frac{2r-4b-1-p}{2} = r-b \\ &= r - \left\lceil \frac{(4b+1)-1}{4} \right\rceil = r - \left\lceil \frac{k-1}{4} \right\rceil. \end{aligned}$$

Subcase 2.2. *$k = 4b+3$ for some integer $b \ge 0$.* Consider the set

$$S = \{u_1, u_2, \ldots, u_{2b+1}, w_1, w_2, \ldots, w_{2b+2}\}$$

of $4b+3$ vertices and let $\mathcal{A} = \mathcal{A}^\circ \cup \mathcal{A}' \cup \mathcal{A}''$ be a set of pairwise edge-disjoint trees connecting S, where again these sets are defined as before. Also, suppose that $|\mathcal{A}'| = p \ge 0$.

Let $H \cong K_{2b+1,2b+2}$ be the subgraph of G induced by S and $X = E(H)$. Hence $|X| = (2b+1)(2b+2)$. Observe that if $T \in \mathcal{A}'$, then T contains at least $2b+1$ edges belonging to X. Therefore, $0 \le p \le 2b+2$ and furthermore,

$$|\mathcal{A}^\circ| \le \left\lfloor \frac{(2b+1)(2b+2)-(2b+1)p}{4b+2} \right\rfloor \text{ and } |\mathcal{A}''| \le \left\lfloor \frac{2r-(4b+3+p)}{2} \right\rfloor.$$

Therefore,

$$\begin{aligned} |\mathcal{A}| &\le \left\lfloor \frac{(2b+1)(2b+2)-(2b+1)p}{4b+2} \right\rfloor + p + \left\lfloor \frac{2r-(4b+3+p)}{2} \right\rfloor \\ &\le \frac{2b+2-p}{2} + p + \frac{2r-4b-3-p}{2} = r-b-\frac{1}{2} \end{aligned}$$

and so

$$|\mathcal{A}| \le r-(b+1) = r - \left\lceil \frac{(4b+3)-1}{4} \right\rceil = r - \left\lceil \frac{k-1}{4} \right\rceil.$$

Therefore, if S is the set of k vertices in $K_{r,r}$ described in each case, then the number of pairwise edge-disjoint trees connecting S is at most $r - \lceil \frac{k-1}{4} \rceil$. Consequently, $\kappa_k(K_{r,r}) \le r - \lceil \frac{k-1}{4} \rceil$.

It now remains to show that

$$\kappa_k(K_{r,r}) \ge r - \lceil (k-1)/4 \rceil.$$

As before, we denote the partite sets of $K_{r,r}$ by U and W. Let S be a set of k vertices of $K_{r,r}$, where $S \cap U = S_U$ and $S \cap W = S_W$, where say $|S_U| \le |S_W|$.

Thus $S = S_U \cup S_W$. We show that there are $r - \lceil \frac{k-1}{4} \rceil$ internally disjoint trees connecting S. We now consider four cases, according to whether k is congruent to 0, 1, 2, or 3 modulo 4.

Case 1. $k = 4a$ *for some positive integer* a. Then $|S_U| = 2a - b$ and $|S_W| = 2a + b$ for some integer b with $0 \le b \le 2a$. In this case, we show that there are $r - a$ internally disjoint trees connecting S.

The result holds for $b = 2a$, for in this case $S = S_W$ and there are r stars $T_u \cong K_{1,4a}$, one for each $u \in U$, where u is the center of T_u. Hence we may assume that $0 \le b \le 2a - 1$. Let

$$S_U = \{u_1, u_2, \ldots, u_{2a-b}\} \text{ and } S_W = \{w_1, w_2, \ldots, w_{2a+b}\}.$$

If $2a + b < r$, then let

$$X = \{x_1, x_2, \ldots, x_{r-2a-b}\} = W - S_W \text{ and}$$
$$Y = \{y_1, y_2, \ldots, y_{r-2a-b}\} \subseteq U - S_U,$$

while $X = Y = \emptyset$ if $r = 2a + b$. Also, if $b > 0$, then let

$$Z = \{z_1, z_2, \ldots, z_{2b}\} = U - [S_U \cup Y].$$

Let $\mathcal{B}'' = \emptyset$ if $r = 2a+b$. Otherwise, let $\mathcal{B}'' = \{S_1'', S_2'', \ldots, S_{r-2a-b}''\}$ be the set of $r - 2a - b$ double stars with $V(S_i'') = S \cup \{x_i, y_i\}$ (and x_i and y_i are the central vertices of S_i'') for $1 \le i \le r - 2a - b$.

If $b = 0$, then by Lemma 2.2 there are a pairwise edge-disjoint S-trees $T_1, T_2, \ldots, T_a$, and

$$\{T_1, T_2, \ldots, T_a\} \cup \mathcal{B}''$$

is a collection of $r-a$ internally disjoint trees connecting S. If $a \le b \le 2a-1$, then let $T_1', T_2', \ldots, T_{a+b}'$ be the S'-trees such that $V(T_i') = S \cup \{z_i\}$ and

$$E(T_i') = \{u_j w_{j+i-1} : 1 \le j \le 2a - b\} \cup \{w_j z_i : 1 \le j \le 2a + b\}.$$

Then

$$\{T_1', T_2', \ldots, T_{a+b}'\} \cup \mathcal{B}''$$

is a collection of $r - a$ internally disjoint trees connecting S.

If $1 \le b \le a - 1$, then we first construct $2b$ internally disjoint S'-trees $S_1', S_2', \ldots, S_{2b}'$. For $1 \le i \le b + 1$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : 1 \le j \le 2a - b\},$$

where the subscript $j + i + (2a - 2b - 1)$ is expressed as one of the integers $1, 2, \ldots, 2a - b + 1$ modulo $2a - b + 1$, while for $b \ge 2$ and $b + 2 \le i \le 2b$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : 1 \le j \le 2a - b\},$$

where the subscript $j+i+(2a-2b-1)$ is expressed as one of the integers $2a-b+2, 2a-b+3, \ldots, 2a+b$ modulo $2b-1$. Then let S_i' be the tree such that $V(S_i') = S \cup \{z_i\}$ and

$$E(S_i') = E_i \cup \{w_j z_i : 1 \le j \le 2a+b\}.$$

Let $\mathcal{B}' = \{S_1', S_2', \ldots, S_{2b}'\}$. Next let q and r be the unique nonnegative integers such that $2a-b = (2b-1)q + r$ with $0 \le r < 2b-1$ and consider a vertex $w \in \{w_{2a-b+2}, w_{2a-b+3}, \ldots, w_{2a+b}\}$. Observe that the number of vertices u in S_U such that the edge uw belongs to some tree in $\mathcal{B}' \cup \mathcal{B}''$ is at most

$$(b-1)q + \min\{b-1, r\} < a.$$

Therefore, for each w_α with $2a-b+2 \le \alpha \le 2a+b$, there exists a set

$$\{u_{1,\alpha}, u_{2,\alpha}, \ldots, u_{a-b,\alpha}\} \subseteq S_U$$

of $a-b$ vertices such that the edge $u_{\beta,\alpha}w_\alpha$, $1 \le \beta \le a-b$, does not belong to any tree in $\mathcal{B}' \cup \mathcal{B}''$. For $1 \le i \le a-b$, let

$$F_i = \{u_j w_{j+2i-2}, u_j w_{j+2i-1} : 1 \le j \le 2a-b\},$$

where each of the subscripts $j+2i-2$ and $j+2i-1$ is expressed as one of the integers $1, 2, \ldots, 2a-b+1$ modulo $2a-b+1$. Construct the S-tree S_i by taking

$$E(S_i) = F_i \cup \{u_{i,j} w_j : 2a-b+2 \le j \le 2a+b\}.$$

Then

$$\{S_1, S_2, \ldots, S_{a-b}\} \cup \mathcal{B}' \cup \mathcal{B}''$$

is a collection of $r-a$ internally disjoint trees connecting S. Hence

$$\kappa_{4a}(K_{r,r}) \ge r-a.$$

Case 2. $k = 4a+1$ for some positive integer a. Then $|S_U| = 2a-b$ and $|S_W| = 2a+b+1$ for some integer b with $0 \le b \le 2a$. We show that there are $r-a$ internally disjoint trees connecting S.

The result holds for $b = 2a$, for in this case $S = S_W$ and there are r stars $T_u \cong K_{1,4a+1}$, one for each $u \in U$, where u is the center of T_u. Hence we may assume that $0 \le b \le 2a-1$. Let

$$S_U = \{u_1, u_2, \ldots, u_{2a-b}\} \text{ and } S_W = \{w_1, w_2, \ldots, w_{2a+b+1}\}.$$

If $2a+b+1 < r$, then let

$$X = \{x_1, x_2, \ldots, x_{r-2a-b-1}\} = W - S_W \text{ and}$$
$$Y = \{y_1, y_2, \ldots, y_{r-2a-b-1}\} \subseteq U - S_U,$$

while $X = Y = \emptyset$ if $r = 2a + b + 1$. Also, let

$$Z = \{z_1, z_2, \ldots, z_{2b+1}\} = U - [S_U \cup Y].$$

Let $\mathcal{B}'' = \emptyset$ if $r = 2a + b + 1$. Otherwise, let $\mathcal{B}'' = \{S_1'', S_2'', \ldots, S_{r-2a-b-1}''\}$ be the set of $r - 2a - b - 1$ double stars with $V(S_i'') = S \cup \{x_i, y_i\}$ (and x_i and y_i are the central vertices of S_i'') for $1 \le i \le r - 2a - b - 1$.

If $a \le b \le 2a - 1$, then let $T_1', T_2', \ldots, T_{a+b+1}'$ be the S'-trees such that $V(T_i') = S \cup \{z_i\}$ and

$$E(T_i') = \{u_j w_{j+i-1} : \ 1 \le j \le 2a - b\} \cup \{w_j z_i : \ 1 \le j \le 2a + b + 1\}.$$

Then

$$\{T_1', T_2', \ldots, T_{a+b+1}'\} \cup \mathcal{B}''$$

is a collection of $r - a$ internally disjoint trees connecting S.

If $0 \le b \le a - 1$, then we first construct $2b+1$ internally disjoint S'-trees $S_1', S_2', \ldots, S_{2b+1}'$. For $1 \le i \le b + 1$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : \ 1 \le j \le 2a - b\},$$

where the subscript $j + i + (2a - 2b - 1)$ is expressed as one of the integers $1, 2, \ldots, 2a - b + 1$ modulo $2a - b + 1$, while for $b \ge 1$ and $b + 2 \le i \le 2b + 1$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : \ 1 \le j \le 2a - b\},$$

where the subscript $j + i + (2a - 2b - 1)$ is expressed as one of the integers $2a - b + 2, 2a - b + 3, \ldots, 2a + b + 1$ modulo $2b$. Then let S_i' be the tree such that $V(S_i') = S \cup \{z_i\}$ and

$$E(S_i') = E_i \cup \{w_j z_i : \ 1 \le j \le 2a + b + 1\}.$$

Let $\mathcal{B}' = \{S_1', S_2', \ldots, S_{2b+1}'\}$. Next let q and r be the unique nonnegative integers such that $2a - b = (2b)q + r$ with $0 \le r < 2b$ and consider a vertex $w \in \{w_{2a-b+2}, w_{2a-b+3}, \ldots, w_{2a+b+1}\}$. Observe that the number of vertices u in S_U such that the edge uw belongs to some tree in $\mathcal{B}' \cup \mathcal{B}''$ is at most

$$bq + \min\{b, r\} \le a.$$

Therefore, for each w_α with $2a - b + 2 \le \alpha \le 2a + b + 1$, there exists a set

$$\{u_{1,\alpha}, u_{2,\alpha}, \ldots, u_{a-b,\alpha}\} \subseteq S_U$$

of $a - b$ vertices such that the edge $u_{\beta,\alpha} w_\alpha$, $1 \le \beta \le a - b$, does not belong to any tree in $\mathcal{B}' \cup \mathcal{B}''$. For $1 \le i \le a - b$, let

$$F_i = \{u_j w_{j+2i-2}, u_j w_{j+2i-1} : \ 1 \le j \le 2a - b\},$$

where each of the subscripts $j+2i-2$ and $j+2i-1$ is expressed as one of the integers $1, 2, \ldots, 2a-b+1$ modulo $2a-b+1$. Construct the S-tree S_i by taking

$$E(S_i) = F_i \cup \{u_{i,j}w_j :\ 2a-b+2 \le j \le 2a+b+1\}.$$

Then

$$\{S_1, S_2, \ldots, S_{a-b}\} \cup \mathcal{B}' \cup \mathcal{B}''$$

is a collection of $r-a$ internally disjoint trees connecting S. Hence

$$\kappa_{4a+1}(K_{r,r}) \ge r-a.$$

Case 3. $k = 4a+2$ *for some positive integer* a. Then $|S_U| = 2a-b+1$ and $|S_W| = 2a+b+1$ for some integer b with $0 \le b \le 2a+1$. We show that there are $r-a-1$ internally disjoint trees connecting S.

The result holds for $b = 2a+1$, for in this case $S = S_W$ and there are r stars $T_u \cong K_{1,4a+2}$, one for each $u \in U$, where u is the center of T_u. Hence we may assume that $0 \le b \le 2a$. Let

$$S_U = \{u_1, u_2, \ldots, u_{2a-b+1}\} \ \text{ and } \ S_W = \{w_1, w_2, \ldots, w_{2a+b+1}\}.$$

If $2a+b+1 < r$, then let

$$X = \{x_1, x_2, \ldots, x_{r-2a-b-1}\} = W - S_W \ \text{ and}$$
$$Y = \{y_1, y_2, \ldots, y_{r-2a-b-1}\} \subseteq U - S_U,$$

while $X = Y = \emptyset$ if $r = 2a+b+1$. Also, if $b > 0$, then let

$$Z = \{z_1, z_2, \ldots, z_{2b}\} = U - [S_U \cup Y].$$

Let $\mathcal{B}'' = \emptyset$ if $r = 2a+b+1$. Otherwise, let $\mathcal{B}'' = \{S_1'', S_2'', \ldots, S_{r-2a-b-1}''\}$ be the set of $r-2a-b-1$ double stars with $V(S_i'') = S \cup \{x_i, y_i\}$ (and x_i and y_i are the central vertices of S_i'') for $1 \le i \le r-2a-b-1$.

If $b = 0$, then by Lemma 2.2 there are a pairwise edge-disjoint S-trees $T_1, T_2, \ldots, T_a$, and

$$\{T_1, T_2, \ldots, T_a\} \cup \mathcal{B}''$$

is a collection of $r-a-1$ internally disjoint trees connecting S. If $a \le b \le 2a$, then let $T_1', T_2', \ldots, T_{a+b}'$ be the S'-trees such that $V(T_i') = S \cup \{z_i\}$ and

$$E(T_i') = \{u_j w_{j+i-1} :\ 1 \le j \le 2a-b+1\} \cup \{w_j z_i :\ 1 \le j \le 2a+b+1\}.$$

Then

$$\{T_1', T_2', \ldots, T_{a+b}'\} \cup \mathcal{B}''$$

is a collection of $r-a-1$ internally disjoint trees connecting S.

If $1 \le b \le a-1$, then we first construct $2b$ internally disjoint S'-trees $S'_1, S'_2, \ldots, S'_{2b}$. If $b = 1$, then for $i = 1, 2$ let

$$E_i = \{u_j w_{j+i+(2a-3)} : \ 1 \le j \le 2a\},$$

where the subscript $j + i + (2a - 3)$ is expressed as one of the integers $1, 2, \ldots, 2a+1$ modulo $2a+1$. If $b \ge 2$, then for $1 \le i \le b+2$ let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : \ 1 \le j \le 2a-b+1\},$$

where the subscript $j + i + (2a - 2b - 1)$ is expressed as one of the integers $1, 2, \ldots, 2a-b+2$ modulo $2a-b+2$. Also, for $b \ge 3$ and $b+3 \le i \le 2b$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : \ 1 \le j \le 2a-b+1\},$$

where the subscript $j + i + (2a - 2b - 1)$ is expressed as one of the integers $2a-b+3, 2a-b+4, \ldots, 2a+b+1$ modulo $2b-1$. Then let S'_i be the tree such that $V(S'_i) = S \cup \{z_i\}$ and

$$E(S'_i) = E_i \cup \{w_j z_i : \ 1 \le j \le 2a+b+1\}.$$

Let $\mathcal{B}' = \{S'_1, S'_2, \ldots, S'_{2b}\}$. Next let q and r be the unique nonnegative integers such that $2a - b + 1 = (2b - 1)q + r$ with $0 \le r < 2b - 1$ and consider a vertex $w \in \{w_{2a-b+3}, w_{2a-b+4}, \ldots, w_{2a+b+1}\}$. Observe that the number of vertices u in S_U such that the edge uw belongs to some tree in $\mathcal{B}' \cup \mathcal{B}''$ is at most

$$(b-2)q + \min\{b-2, r\} < a.$$

Therefore, for each w_α with $2a - b + 3 \le \alpha \le 2a + b + 1$, there exists a set

$$\{u_{1,\alpha}, u_{2,\alpha}, \ldots, u_{a-b,\alpha}\} \subseteq S_U$$

of $a - b$ vertices such that the edge $u_{\beta,\alpha} w_\alpha$, $1 \le \beta \le a - b$, does not belong to any tree in $\mathcal{B}' \cup \mathcal{B}''$. For $1 \le i \le a - b$, let

$$F_i = \{u_j w_{j+2i-2}, u_j w_{j+2i-1} : \ 1 \le j \le 2a-b+1\},$$

where each of the subscripts $j + 2i - 2$ and $j + 2i - 1$ is expressed as one of the integers $1, 2, \ldots, 2a - b + 2$ modulo $2a - b + 2$. Construct the S-tree S_i by taking

$$E(S_i) = F_i \cup \{u_{i,j} w_j : \ 2a-b+3 \le j \le 2a+b+1\}.$$

Then

$$\{S_1, S_2, \ldots, S_{a-b}\} \cup \mathcal{B}' \cup \mathcal{B}''$$

is a collection of $r - a - 1$ internally disjoint trees connecting S. Hence

$$\kappa_{4a+2}(K_{r,r}) \geq r - a - 1.$$

Case 4. $k = 4a+3$ *for some nonnegative integer* a. Then $|S_U| = 2a-b+1$ and $|S_W| = 2a+b+2$ for some integer b with $0 \leq b \leq 2a+1$. We show that there are $r-a-1$ internally disjoint trees connecting S.

The result holds for $b = 2a+1$, for in this case $S = S_W$ and there are r stars $T_u \cong K_{1,4a+3}$, one for each $u \in U$, where u is the center of T_u. Hence we may assume that $0 \leq b \leq 2a$. Let

$$S_U = \{u_1, u_2, \ldots, u_{2a-b+1}\} \text{ and } S_W = \{w_1, w_2, \ldots, w_{2a+b+2}\}.$$

If $2a+b+2 < r$, then let

$$X = \{x_1, x_2, \ldots, x_{r-2a-b-2}\} = W - S_W \text{ and}$$
$$Y = \{y_1, y_2, \ldots, y_{r-2a-b-2}\} \subseteq U - S_U,$$

while $X = Y = \emptyset$ if $r = 2a+b+2$. Also, let

$$Z = \{z_1, z_2, \ldots, z_{2b+1}\} = U - [S_U \cup Y].$$

Let $\mathcal{B}'' = \emptyset$ if $r = 2a+b+2$. Otherwise, let $\mathcal{B}'' = \{S_1'', S_2'', \ldots, S_{r-2a-b-2}''\}$ be the set of $r-2a-b-2$ double stars with $V(S_i'') = S \cup \{x_i, y_i\}$ (and x_i and y_i are the central vertices of S_i'') for $1 \leq i \leq r-2a-b-2$.

If $b = 0$, then by Lemma 2.2 there are $a+1$ pairwise edge-disjoint S-trees $T_1, T_2, \ldots, T_{a+1}$, and

$$\{T_1, T_2, \ldots, T_{a+1}\} \cup \mathcal{B}''$$

is a collection of $r-a-1$ internally disjoint trees connecting S. If $a \leq b \leq 2a$, then let $T_1', T_2', \ldots, T_{a+b+1}'$ be the S'-trees such that $V(T_i') = S \cup \{z_i\}$ and

$$E(T_i') = \{u_j w_{j+i-1} : 1 \leq j \leq 2a-b+1\} \cup \{w_j z_i : 1 \leq j \leq 2a+b+2\}.$$

Then

$$\{T_1', T_2', \ldots, T_{a+b+1}'\} \cup \mathcal{B}''$$

is a collection of $r-a-1$ internally disjoint trees connecting S.

If $1 \leq b \leq a-1$, then we first construct $2b+1$ internally disjoint S'-trees $S_1', S_2', \ldots, S_{2b+1}'$. For $1 \leq i \leq b+2$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : 1 \leq j \leq 2a-b+1\},$$

where the subscript $j+i+(2a-2b-1)$ is expressed as one of the integers $1, 2, \ldots, 2a-b+2$ modulo $2a-b+2$, while for $b \geq 2$ and $b+3 \leq i \leq 2b+1$, let

$$E_i = \{u_j w_{j+i+(2a-2b-1)} : 1 \leq j \leq 2a-b+1\},$$

where the subscript $j+i+(2a-2b-1)$ is expressed as one of the integers $2a-b+3, 2a-b+4, \ldots, 2a+b+2$ modulo $2b$. Then let S_i' be the tree such that $V(S_i') = S \cup \{z_i\}$ and

$$E(S_i') = E_i \cup \{w_j z_i : \ 1 \le j \le 2a+b+2\}.$$

Let $\mathcal{B}' = \{S_1', S_2', \ldots, S_{2b+1}'\}$. Next let q and r be the unique nonnegative integers such that $2a-b+1 = (2b)q + r$ with $0 \le r < 2b$ and consider a vertex $w \in \{w_{2a-b+3}, w_{2a-b+4}, \ldots, w_{2a+b+2}\}$. Observe that the number of vertices u in S_U such that the edge uw belongs to some tree in $\mathcal{B}' \cup \mathcal{B}''$ is at most

$$(b-1)q + \min\{b-1, r\} < a.$$

Therefore, for each w_α with $2a-b+3 \le \alpha \le 2a+b+2$, there exists a set

$$\{u_{1,\alpha}, u_{2,\alpha}, \ldots, u_{a-b,\alpha}\} \subseteq S_U$$

of $a-b$ vertices such that the edge $u_{\beta,\alpha}w_\alpha$, $1 \le \beta \le a-b$, does not belong to any tree in $\mathcal{B}' \cup \mathcal{B}''$. For $1 \le i \le a-b$, let

$$F_i = \{u_j w_{j+2i-2}, u_j w_{j+2i-1} : \ 1 \le j \le 2a-b+1\},$$

where each of the subscripts $j+2i-2$ and $j+2i-1$ is expressed as one of the integers $1, 2, \ldots, 2a-b+2$ modulo $2a-b+2$. Construct the S-tree S_i by taking

$$E(S_i) = F_i \cup \{u_{i,j} w_j : \ 2a-b+3 \le j \le 2a+b+2\}.$$

Then

$$\{S_1, S_2, \ldots, S_{a-b}\} \cup \mathcal{B}' \cup \mathcal{B}''$$

is a collection of $r-a-1$ internally disjoint trees connecting S. Hence

$$\kappa_{4a+3}(K_{r,r}) \ge r-a-1.$$

This completes the proof. ■

As an example, we now consider the graph $G = K_{12,12}$ and let S be a set of 15 vertices in G. According to Theorem 2.3, there is a collection $\mathcal{A}$ of 8 internally disjoint trees connecting S. To illustrate how such 8 trees are constructed in the proof, we consider a set S with $|S_U| = 7-b$ and $|S_W| = 8+b$, where $b \in \{0, 2, 4\}$.

- If $b = 0$, then $S = \{u_1, u_2, \ldots, u_7\} \cup \{w_1, w_2, \ldots, w_8\}$. Let

$$\mathcal{A} = \{T_1, T_2, T_3, T_4\} \cup \{S_1'', S_2'', S_3'', S_4''\}$$

be the set of 8 trees, where T_1, T_2, T_3, T_4 are the paths defined by

$$\begin{aligned}
T_1 &: \ w_1, u_1, w_2, u_2, w_3, \ldots, w_7, u_7, w_8 \\
T_2 &: \ w_3, u_1, w_4, u_2, w_5, \ldots, w_1, u_7, w_2 \\
T_3 &: \ w_5, u_1, w_6, u_2, w_7, \ldots, w_3, u_7, w_4 \\
T_4 &: \ w_7, u_1, w_8, u_2, w_1, \ldots, w_5, u_7, w_6
\end{aligned}$$

and S_i'' $(1 \leq i \leq 4)$ is a double star whose central vertices are x_i and y_i such that

$$N(x_i) = \{u_1, u_2, \ldots, u_7, y_i\} \text{ and } N(y_i) = \{w_1, w_2, \ldots, w_8, x_i\}.$$

- If $b = 2$, then $S = \{u_1, u_2, \ldots, u_5\} \cup \{w_1, w_2, \ldots, w_{10}\}$. Let

$$\mathcal{A} = \{S_1\} \cup \{S_1', S_2', \ldots, S_5'\} \cup \{S_1'', S_2''\}$$

be the set of 8 trees, where S_1 and S_j' $(1 \leq j \leq 5)$ are shown in Figure 2 and S_i'' $(i = 1, 2)$ is a double star whose central vertices are x_i and y_i such that $N(x_i) = \{u_1, u_2, \ldots, u_5, y_i\}$ and $N(y_i) = \{w_1, w_2, \ldots, w_{10}, x_i\}$.

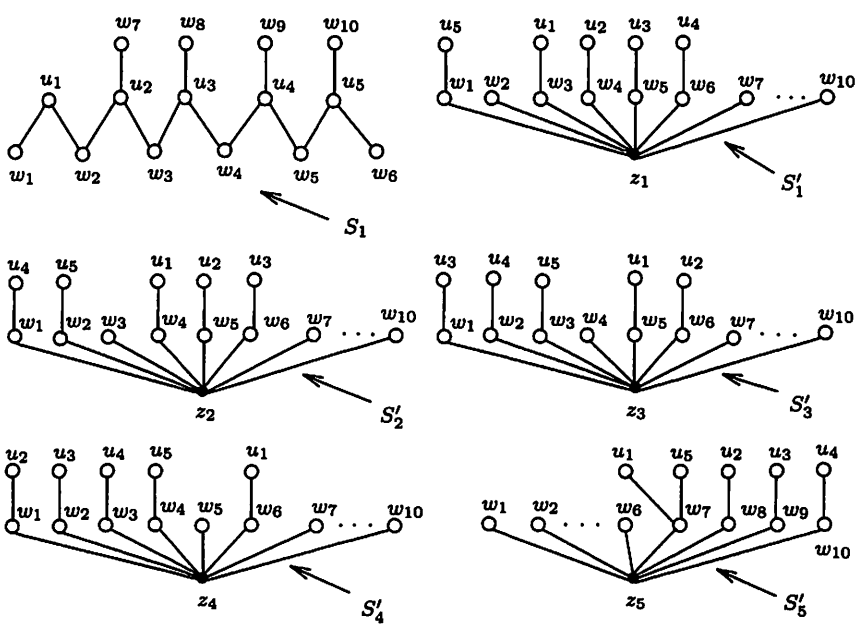

Figure 2: Internally disjoint trees connecting S for $b = 2$

- If $b = 4$, then $S = \{u_1, u_2, u_3\} \cup \{w_1, w_2, \ldots, w_{12}\}$. Let

$$\mathcal{A} = \{T'_1, T'_2, \ldots, T'_8\}$$

be the set of 8 trees, where T'_i ($1 \leq i \leq 8$) is obtained from the star $K_{1,12}$, whose vertex set is $V(K_{1,12}) = \{z_i\} \cup \{w_1, w_2 \ldots, w_{12}\}$ and whose central vertex is z_i, by adding the vertices u_1, u_2, u_3 and joining u_j to w_{j+i-1} for $j = 1, 2, 3$. The trees T'_1, T'_2, and T'_8 are shown in Figure 3.

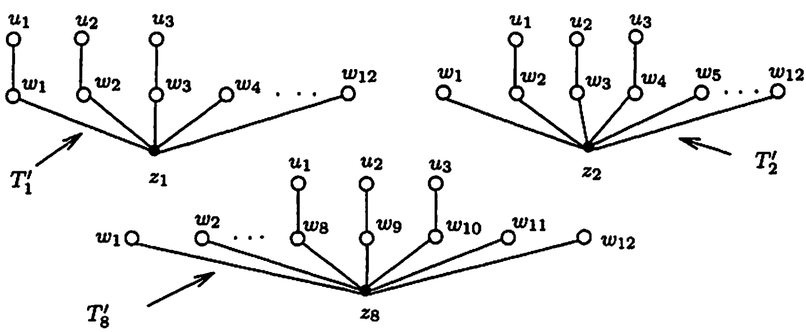

Figure 3: Internally disjoint trees connecting S for $b = 4$

3 Acknowledgments

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