Dynamic Behavior of Perturbed Logistic Model

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ABSTRACT- ***A model that represents the rate of changes of the population with limited environmental resources can be described by,***

$$\frac{dp}{dt} = p(a - bp) + g(t, p), \quad p(t_0) = p_0$$

where a measures the growth rate in the absence of the restriction force b and a/b is called the carrying capacity of the environment. The random perturbation g(t, p) is generated by random change in the environment. The behavior of the solution of this model for continuous and discrete case when g(t,p)=w(t) is density independent with a constant random factor w in a short time interval $[t, t + \delta t)$ ***will be studied. The stability and the behavior of the equilibrium point will also be investigated. A computational approach to the solution using Excel spreadsheet and Maple will be presented.***

1- Introduction and History

The evolution of mathematical models used to describe population growth and balance is a great example of how this field has evolved over the years. The first mathematical model, referred to as the classical Malthusian scheme for population growth, is based on the work of Thomas R. Malthus (1766-1834). In *The Principle of Population* essay that he published in 1798, Malthus explained in fundamental and brilliantly simple terms his theories of human population growth and the connection between over-population and misery. One of the fundamental concepts that he brought up is that of unlimited population growth (see Bulaevsky, [2]). Tomas R Malthus was one of the first people to use math modeling to solve a population growth problem. His model stated that the size population for one generation depends on population size of the last generation. In a discrete model, the equation that he used was: $P_{t+1} = r.P_t$ (see *Interesting Facts about Population Growth Mathematical Models written by* Bulaevsky [3]). After Malthus, mathematicians and biologists used math modeling for different types of population problems. Particularly in the area of unlimited growth, limited growth, age Structure dependencies, competition, predator and prey model, natural equilibrium, dynamic behavior, and stability of the equilibrium points have been studied during the past centuries ([4], [5]).
Random logistic model were used in random drift / migration and directional changes in gene frequency (for details, see notes on **population structure and gene flow** and Futuyma 2005 [4]. The

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dynamic behavior of logistic model leading to stability, instability, and chaos has been studied in many different areas of science and mathematics during the past decades [7]. One of the characteristics of the random-looking behavior or deterministic chaos is its sensitivity and dependence on the initial conditions. This means that the separation between two nearby orbits of the system and the prediction of the long term behavior of the system can only be described in probabilistic language [5]. In computational approach a computer algebra systems like Maple can be used for simulation and control of chaos ([8],[12]).
The structure of this paper will be the following: First we begin to discuss the principles of the random perturbation in setting the model. In the next step we will study the solution of Random Perturbed Logistic Model (RPLM). The analytical and numerical solution will be presented. A computer algebra system will be used to demonstrate the exact solution.

2.1- Random Perturbations in the logistic Environment: We start with the logistic equation itself:

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right) \quad (2.1)$$

If the paradise of nature provides unlimited resources, then the population increases exponentially to $P(t) = P_0e^{rt}$. When the resources are limited, nature will impose a rule of restriction force of reproduction against the exponential increase.
The behavior of many populations cannot be modelled using (2.1), as it is affected by an additional perturbation term

$$\frac{dP(t)}{dt} = rP\left(1 - \frac{P(t)}{K}\right) + g(t, P(t)) \quad (2.2)$$

We next study what happens when we perturb a logistic population. How do we perturb it? One of the important principles of evolution is the variation of condition where the individual will learn how to adapt in a new environmental condition. The random perturbation w may represent the amount of individuals being recruited or harvested from the population, which **may not be independent from the density of the population.**
The equation (2.2) can be studied when w is a discrete constant recruitment or harvesting term **independent from the population density.**
This is a euphemism for killing some individuals or providing resources to flourish another individual. If we do so for food, we call it *harvesting*, whereas if we do it in order to save the underlying environment (and hence the population itself), we call it *culling*. If the environment does this

random change then we do not know when it is going to be killing one or flourishing another individual.
Small fluctuations in the perturbation w such as climate changes and food supply can have amplified effects on the population growth rate. The stochastic effects can be examined when a random noise term is added to the perturbation w.

To study these problems, we need to assume that a perturbation w imposed in the original equation (2.1) is density independent that is g(t,p(t))=w(t). Thus, in such a condition, the general form of random logistic can be describe by

$$\frac{dp}{dt} = a \cdot p\left(1 - \frac{p}{K}\right) + w(t) \quad (2.3)$$

This model is particularly useful to represent a random logistic model.
The random function w in the logistic equation was a subject of study in many areas of computation biology, population dynamics, and many theoretical aspects of stochastic differential equations (see Scheuring and Domokos 2007).
Ecological systems are inherently noisy and the data series are short and unreliable (see Scheuring et all [14]). **The main question is whether the noise can induce a chaotic behavior?**

3. Continuous Random Logistic Model: We will impose the following general conditions to determine the mathematical model.
3.1- Modeling with the Postulates for Open Environment: Naturally these parameters satisfy the following conditions:
1- in the absence of random effect (in a closed environment) the population may be changed with the exponential rate of a when there is no overcrowded restriction force factor (or equivalently b=0).
2- In the absence of the environment random change and presence of overcrowded restriction forces factor b, the rate of change of the population is proportional to the population rate = a-b.p(t) that is p'(t)=p(a-bp).
3- If the environment is not closed to random change during the time interval $[t, t + \Delta t)$, resources may be eliminated or added. Consequently, the growth rate of changes of the population may be increased or decreased and the population rate speeding up or slowing down. Assume that the perturbation function is density dependent and it is denoted by g(t,P (t)).
In the absence of the influences of the variation forces, that is when g(t,p(t)) =0, the model is called classical logistic equation. The impulsive

behavior of the model (2.3) has been studied by many authors (Heydar, et all, [6])

We can describe the population by the following differential equation, which is called the perturbed logistic differential equation.

$$\frac{dP}{dt} = P(a - bP) + g(t, P), \quad P(t_0) = P_0 \qquad (3.1)$$

In the original model of (3.1) the perturbation function g(t,P) is a function of time t and the population P(t). We would like to apply this model to a special case when

i) the random perturbed function is independent from the value P. In other words the perturbed function g(t,P(t))= w(t) is a function of t but it will stay constant in a short time interval $[t, t + \Delta t)$.

Rearranging the equation (3.1)

$$\frac{dP}{dt} = aP(1 - \frac{b}{a}P) + w(t), \quad P(t_0) = P_0 \qquad (3.2)$$

ii) The second form of the perturbed function may be introduced by g(t,P(t))=w(t).P(t). Thus the equation (3.2) will be affected by random environmental changes of external force w(t) such that:

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{K}\right) + w(t) \cdot P(t) = P \cdot [a(1 - \frac{P}{K}) + w(t)] \qquad (3.3)$$

The ratio $\frac{a}{b} = K$ is known a carrying capacity and the equation (3.2) is called Random Perturbed Logistic Model (RPLM).

3.2- Analytical Solutions:

The random perturbation on the logistic model represents the openness of the environment to the variations of resources. These variations can be observed in a small time interval Δt showing increases or decreases in environmental resources. Mathematically, this can be interpreted as how much this bio-energy will increase or decrease the rate of change of the reproduction or the population. This phenomenon can be described by two control functions; g(t) acts like pedal to increase the speed and h(t) acts like break to decrease the speed.

In this sense the random perturbed function can be written by w(t)=g(t)-h(t). Due to the uncertainty arising from the environmental variation, the value of w(t) may be negative, zero, or positive.

We say a perturbation has a positive effect during the time interval Δt if $\delta = a^2 + 4bw \geq 0$ and has a negative effect when $\delta = a^2 + 4bw < 0$.

Our goal is to investigate the solution of the random perturbed logistic equation with positive or negative effects.

Case (I): The random number w has a positive effect meaning that $\delta = a^2 + 4bw \geq 0$. Thus integrating (3.1) will provide

$$dt = \frac{dp}{-bp^2 + ap + w} = \frac{dp}{-b[(p - \frac{a}{2b})^2 - (\frac{\sqrt{a^2 + 4bw}}{2b})^2]} = \frac{-dp}{b[u^2 - m^2]}$$

Integrating both sides by introducing auxiliary variables u and m will produce the following:

$$-b(t + c_1) = \ln|\frac{u-m}{u+m}| \text{ where } u = p - \frac{a}{2b} \quad and \quad m = \frac{\sqrt{a^2 + 4bw}}{2b}$$

The general solution p(t) can be calculated:

$$p(t) = \frac{a}{2b} + \frac{\sqrt{a^2 + 4bw}}{2b} \frac{1 + ce^{-bt}}{1 - ce^{-bt}} \tag{3.4}$$

Assuming K=a/b is the carrying capacity of the environment the result (3.4) can be described by the following simplified version:

$$p(t) = \frac{K}{2} + \sqrt{(\frac{K}{2})^2 + \frac{w}{b}} \frac{1 + e^{\beta_0 + \beta_1 t}}{1 - e^{\beta_0 + \beta_1 t}} \tag{3.5}$$

where $e^{\beta_0} = c$ and $\beta_1 = -b$.

Case (II-a): When the random number perturbation w is a negative real number but has a positive effect: We define a positive effect when w=-k for some non-negative real number k (small k) and

$\delta = a^2 - 4bk > 0$ (or equivalently $k < \frac{a^2}{4b}$). Rearrange the equation to integrate as follows

$$dt = \frac{dp}{-bp^2 + ap - k} = \frac{dp}{-b[(p - \frac{a}{2b})^2 - (\frac{\sqrt{a^2 - 4bk}}{2b})^2]} = \frac{-dp}{b[u^2 - m^2]}$$

Integrating both sides will produce the same result as in case (I) by replacing the following:

$$-b(t+c_1) = \ln\left|\frac{u-m}{u+m}\right| \text{ where } u = p(t) - \frac{a}{2b} \quad and \quad m = \frac{\sqrt{a^2-4bk}}{2b}.$$

for some constant number c_1. The general solution p(t) can be calculated:

$$p(t) = \frac{a}{2b} + \frac{\sqrt{a^2-4bk}}{2b}\frac{1+ce^{-bt}}{1-ce^{-bt}} \tag{3.6}$$

for all nonnegative parameters a, b, and k. The constant number c can be determined through the initial information.

Case II-b: Let us assume that for w=-k for some non-negative real number k the value of

$\delta = a^2 - 4bk < 0$ (or equivalently $k > \frac{a^2}{4b}$)

$$dt = \frac{dp}{-bp^2+ap-k} = \frac{dp}{-b[(p-\frac{a}{2b})^2 + (\frac{\sqrt{-a^2+4bk}}{2b})^2]} = \frac{-dp}{b[u^2+m^2]}$$

Integrating both sides and use substitution method will produce the following result:

$-b(t+c_1) = \int \frac{du}{u^2+m^2} = \frac{1}{m}\tan^{-1}(\frac{u}{m})$ where

$u = p - \frac{a}{2b} \quad and \quad m = \frac{\sqrt{-a^2+4bk}}{2b}.$

Substitute m and u in $u(t) = m \cdot \tan(bmc_1 - bm \cdot t)$

$$\tan^{-1}(\frac{u}{m}) = -mb(t+c_1) \Rightarrow u = m\tan(-mbt - mbc_1)$$

This relation can be modified into

$$p(t) = \frac{a}{2b} - m\tan(\beta + \alpha t) \tag{3.7}$$

where $\beta = mbc_1 \quad and \quad \alpha = mb$. Substitute back K=a/b and small k=-w

$$p(t) = \frac{K}{2} - \sqrt{-(\frac{K}{2})^2 + \frac{k}{b}} \cdot \tan(\beta + \alpha t)$$

We will obtain the following solution after combining the results of part (I) and (II) together.

$$p(t) = \begin{cases} \frac{a}{2b} + m\frac{1+\exp(\alpha - bt)}{1-\exp(\alpha - bt)} & \text{if } \delta = a^2 - 4bk \geq 0 \\ \frac{a}{2b} + m\tan(\alpha + \beta t) & \text{if } \delta = a^2 - 4bk < 0 \end{cases} \quad \textit{(3.8)}$$

We can summarise our computation in the following theorems.
Theorem 3.1: Assume the environmental conditions for population p(t) satisfies all three postulates. If the perturbed function w(t) is a nonnegative constant with a positive effect $\delta = a^2 + 4bw > 0$. The original random perturbed logistic model

$$\frac{dp}{dt} = p(a - bp) + w, \quad p(0) = p_0$$

i) has a general solution on the small subinterval $[t, t + \Delta t)$ and the solution $p(t) = \frac{K}{2} + \sqrt{(\frac{K}{2})^2 + \frac{w}{b}}\frac{1 + e^{\beta_0 + \beta_1 t}}{1 - e^{\beta_0 + \beta_1 t}}$
where a and b are constant real numbers and K=a/b is the carrying capacity of the environment.

Theorem 3.2: Assume that the conditions for the population p(t) satisfies all three postulates of the open environment. If the perturbed function w(t) is a negative constant real number with a positive effect $\delta = a^2 + 4bw > 0$ (for dimensionless system $\delta = 1 + 4w > 0$). The original random perturbed logistic model

$$\frac{dp}{dt} = p(a - bp) + w, \quad p(0) = p_0$$

i) has a general solution on the small subinterval $[t, t + \Delta t)$ and

$$p(t) = \begin{cases} \frac{a}{2b} + m\frac{1+\exp(\alpha - bt)}{1-\exp(\alpha - bt)} & \text{if } \delta = a^2 - 4bk > 0 \textit{ has a positive effect} \\ \frac{a}{2b} + m\tan(\beta + \alpha t) & \text{if } \delta = a^2 - 4bk < 0 \textit{ has a negative effect} \end{cases}$$

where w=-k for some non-negative real number k.

Corollary 3.1: If w the periodic environmental contribution is nonnegative then

$$\lim_{t \to \infty} p(t) = \frac{K}{2} + \sqrt{(\frac{K}{2})^2 + \frac{w}{b}} \quad (3.9)$$

This result is consistent when the symbiotic contribution stops (w=0) and the value p(t) will approach the carrying capacity K.

4.1 – Equilibrium Points for Random Perturbed Continuous Logistic Model

First, we study the perturbed logistic model described in (3.2) where the perturbation is a constant real number w. To find the equilibrium points, set $\frac{dP}{dt} = P(a-bP)+w = 0$ and solve the quadratic equation to find the values of P.

$$P = \frac{a \pm \sqrt{\delta}}{2b}, \quad where \quad \delta = a^2 + 4bw \geq 0 \qquad (4.1)$$

Two equilibrium points may exist where we can label them

$$P_1 = \frac{K}{2} + \sqrt{(\frac{K}{2})^2 + \frac{w}{b}} \quad and \quad P_2 = \frac{K}{2} - \sqrt{(\frac{K}{2})^2 + \frac{w}{b}} \qquad (4.2)$$

where K is the carrying capacity. When the constant real number w is zero then two equilibrium points will be K and 0. Notice that $P_1 > P_2$. Let us evaluate $f'(P_1)$ and $f'(P_2)$:

$$f(P_1) = -\sqrt{a^2+4bw} < 0 \quad and \quad f(P_2) = +\sqrt{a^2+4bw} > 0$$

P(t)	P'(t)=f(P)	f'(P)=a-2bP
$P<P_2$	Negative	P(t) is decreasing
$P=P_2$	0	Unstable since $f(P_2)>0$
$P_2<P<P_1$	Positive	P(t) is increasing
$P=P_1$	0	Stable since $f(P_1)<0$
$P>P_1$	Negative	P(t) is decreasing

Fig. 4.1: stability analysis of the equilibrium points

4.2: Graphical Approach for Stability Discussion

The quadratic function f(P) can be demonstrated by a parabola where the constant value w will determine its position in the rectangular plane.

$$f(P) = P(a-bP) + w = -bP^2 + aP + w = -b[P^2 - \frac{a}{b}P] + w$$

$$= -b(P - \frac{a}{2b})^2 + b(\frac{a}{2b})^2 + w = -b(P - \frac{K}{2})^2 + \frac{a^2+4bw}{4b}$$

This parabola has vertex at $(\frac{K}{2}, \sqrt{(\frac{K}{2})^2 + \frac{w}{b}})$ and intersects the horizontal P axis at two points P_1 and P_2.

5. Introduction to the Discrete Model:

In the rescaled form of logistic model in (3.4)

$\frac{dP}{dt} = P(A - BP) + w(t), \qquad P(0) = P_0$, we can assume that there is a short time interval $[t_i, t_i + \Delta t) \quad where \quad i = 0,1,2,\ldots n$ the system will not be disturbed. Meaning that w(t) will stay constant. Thus, for sufficiently small interval we can approximate

$$\frac{P(t_{i+1}) - P(t_i)}{\Delta t} \approx P(t_i)(A - BP(t_i)) + w(t_i)$$

Let us use a notation $P(t_n) = P_n \quad and \quad w(t_n) = w_n$ then we will obtain the following discrete form

$$P_{n+1} = P_n + \Delta t[(A - BP_n)P_n + w_n] \quad where \quad P(t_0) = P_0 \qquad (5.1)$$

If the value of the increment is equal to a unit interval that is $\Delta t = 1$ the relation (5.1) will be

$$P_{n+1} = P_n[1 + (A - BP_n)] + w_n \quad where \quad P(t_0) = P_0 \qquad (5.2)$$

Section 5.2: Equilibrium Point of Discrete Random Perturbed Logistic Model:

Case (I)- Density Independent: We would like to find the equilibrium point of the density independent random perturbation in the Logistic Model (5.2). We assume for large n,

$\lim_{n\to\infty} P_n = \lim_{n\to\infty} P_{n+1} = P_E$, thus

$$P_E = P_E[1 + (A - BP_E) + w_n] \quad where \quad P(t_0) = P_0$$

We will produce and solve a quadratic equation: $BP_E - AP_E - w_n = 0$

$$\Rightarrow P_E = \frac{A \pm \sqrt{A^2 + 4Bw_n}}{2B}.$$

Thus, for every w_n there are two equilibrium points:

$$P_{E1} = \frac{A + \sqrt{A^2 + 4Bw_n}}{2B} \quad and \quad P_{E2} = \frac{A - \sqrt{A^2 + 4Bw_n}}{2B} \qquad (5.4)$$

Using the carrying capacity $K = \frac{A}{B}$ the relation (5.4) can be described by the following two equilibrium points

$$P_E = \frac{K}{2} \pm \sqrt{(\frac{K}{2})^2 + \frac{w_n}{B}} \qquad (5.5)$$

5.3- Stability of the Equilibrium Point for Random Logistic Model with Density Independent Perturbation g(t, P(t))=w(t)

To find the stability of this problem, we use the equation from when we found stability the first time:

$$P_{k+1} = P_k + (1+A-BP_k) + g(t_k, P(t_k)) \quad \text{where} \quad g(t, P(t)) = w(t) \qquad (5.6)$$

Where w is a random change of the environment that causes change in the population, so (5,5) is going to be

$$P_{k+1} = P_k[1 + A - B(P_k)] + w(t_k) \qquad (5.7)$$

Now let's assume that the population is near the equilibrium point P_E, that is $P_k = P_E + \varepsilon, \quad and \quad P_{k+1} = P_E + \varepsilon'$.

Then the relation (5.7) will be

$$(P_E + \varepsilon') = (P_E + \varepsilon)[1 + A - B(P_E + \varepsilon)] + w_k$$

The right hand side equals to

$$= (P_E + \varepsilon) + A(P_E + \varepsilon) - BP_E(P_E + \varepsilon) - B\varepsilon(P_E + \varepsilon) + w_k$$

This equals

$$P_E + \varepsilon' = P_E + \varepsilon + AP_E + A\varepsilon - BP_E^{\,2} - 2BP_E\varepsilon - B\varepsilon^2 + w_k$$

$$= P_E(1 + A - BP_E) + w_k + \varepsilon + A\varepsilon - 2BP_E\varepsilon - B\varepsilon^2$$

$$= P_E + \varepsilon + A\varepsilon - 2BP_E\varepsilon - B\varepsilon^2$$

Notice that we replace

$P_E + AP_E - BP_E^{\,2} + w_k \quad by \quad \Rightarrow P_E = P_E(1 + A - BP_E) + w_k$.

Simplifying and using the relation (5.7) implies

$$\varepsilon' = \varepsilon + A\varepsilon - 2B\varepsilon P_E$$

Factor ε on the right hand side:

$$\varepsilon' = \varepsilon(1 + A - 2BP_E - B\varepsilon)$$

For this converse it is required that

$$\frac{\varepsilon'}{\varepsilon} \to 0 \Leftrightarrow |1 + A - 2BP_E| < 1 \qquad (5.8)$$

Then solve for P_E, $\Rightarrow \frac{A}{2B} < P_E < \frac{A+2}{2B}$. This is the condition for stability.

$$\Rightarrow \frac{K}{2} < P_E < \frac{K}{2} + \frac{1}{B}. \qquad (5.9)$$

Checking the Stability of the Equilibrium Points:

Let us evaluate the relation (5.8) at two equilibrium points of (5.4). Thus

$$|1 + A - 2BP_{E1}| = |1 + A - 2B(\frac{A + \sqrt{\delta}}{2B})| = |1 - \sqrt{\delta}| < 1$$

This inequality implies that the equilibrium point P_{E1} is stable. We will evaluate the status of the P_{E2}.

$$|1 + A - 2BP_{E2}| = |1 + A - 2B(\frac{A - \sqrt{\delta}}{2B})| = |1 + \sqrt{\delta}| \geq 1$$

This result shows that the second equilibrium point P_{E2} is unstable.

Theorem 5.1: In a short time interval $[t, t + \Delta t)$, the density independent random perturbed logistic model (5.2) has two possible equilibrium points P_{E1} and P_{E2} described in (5.4).

If there is random harvesting or symbiotic contribution, in the environment resources to the logistic system w_k stay constant, then the equilibrium point P_{E1} is stable and P_{E2} is unstable.

Theorem 5.2- Quasi-equilibrium Point for Discrete Random Perturbed Logistic Model:

1- The unstable equilibrium point $P_E = 0$ will not be disturbed but the second equilibrium point $P_E = \dfrac{A - w}{B} = K - \dfrac{w}{B}$ is subject to or associated with a random change.

Let us call this kind of equilibrium a Quasi Equilibrium.

2- The quasi-equilibrium points are stable if

$$\Rightarrow \frac{A + r}{2B} < P_E < \frac{A + r + 2}{2B}.$$

The quasi-equilibrium point is stable if

$$\frac{K}{2} + \frac{w}{2B} < P_E < \frac{K}{2} + \frac{w}{2B} + \frac{1}{B} \qquad (5.10)$$

6. Computational Approach to the solution and the Stability of the Equilibrium Points in Discrete Random Perturbation Logistic Model

We have demonstrated the analytical solutions of a random perturbed logistic model in the past sections in the form of differential equations. The analytical solution and stability of the equilibrium points of the discrete systems of (5.2) and (5.3) were investigated in the form of difference equations.

6.1- Computational Approach using MAPLE: We developed Maple programs to solve random perturbed logistic differential equation with

given initial data. The following Maple algorithm uses "dsolve" command to find the solution in every consecutive time interval with a density independent random harvesting perturbation w=-rand(1..100)() (see Fig 6.1). It is important to notice that I) the final position of the solution in each loop will be the initial position of the consecutive stage. II) Due to the randomness of w, in each stage of computation $\delta = a^2 + 4bw$ may be positive, zero, or negative. According to (3.10) this may switch the solution from one kind to another. III) It is clear that the negative population does not make sense and part of the computation generates negative numbers, which kept in the graph to study the mathematical behavior and perhaps applications in other areas like economics.
The following Maple procedure is for Density Independent Random Perturbation of Logistic model:

```
> restart;
logisticpplm := proc (a, b, ic1, n)
local i, eq, s, c, ic, f, g; c[1] := 0; ic[1] := y(c[1]) = ic1;
for i to n do
eq := diff(y(x), x) = y(x)*(a-b*y(x))-(1/100)*(rand(1 .. 6))();
s[i] := rhs(dsolve({eq, ic[i]}, y(x)));
c[i+1] := 50*i/n; ic[i+1] := y(c[i+1]) = evalf(subs(x = c[i+1],
s[i])); f[i] := s[i]*Heaviside(x-c[i])*(1-Heaviside(x-c[i+1])) end do;
g := seq(f[i], i = 1 .. n)
end proc;
> plot([logisticpplm(.5, 0.1e-2, 5, 100)], x = 1 .. 50, discont = true);
> a[1] := .4; a[2] := .5; a[3] := .6;
> for i to 3 do f[i] := logisticpplm(a[i], 0.1e-2, 5, 50) end do;
> g := seq(f[i], i = 1 .. 3);
> plot([g], x = 1 .. 50, discont = true);
```

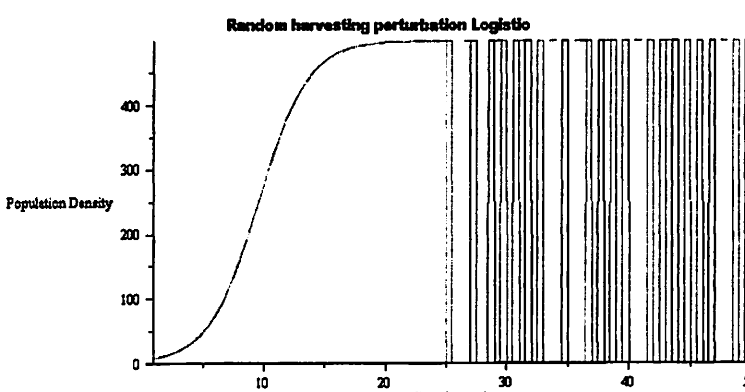

Fig.6.1: Random Harvesting Perturbation w=-randd(1..10)()/100 causes distrbances in the solution of the logistic model.

Negative Effect of Random Perturbed Logistic Model;

```
> restart;
> a := .5; b := 0.1e-1; c[1] := 0; ic[1] := y(c[1]) = 5; n := 100;
> for i to n do eq := diff(y(x), x) = y(x)*(a-b*y(x))-(rand(1 .. 100))();
s[i] := rhs(dsolve({eq, ic[i]}, y(x)));
c[i+1] := 50*i/n;
ic[i+1] := y(c[i+1]) = evalf(subs(x = c[i+1],
s[i])); f[i] := s[i]*Heaviside(x-c[i])*(1-Heaviside(x-c[i+1]))
end do;
> g := seq(f[i], i = 1 .. n);
> plot([g], x = 0 .. 50);
```

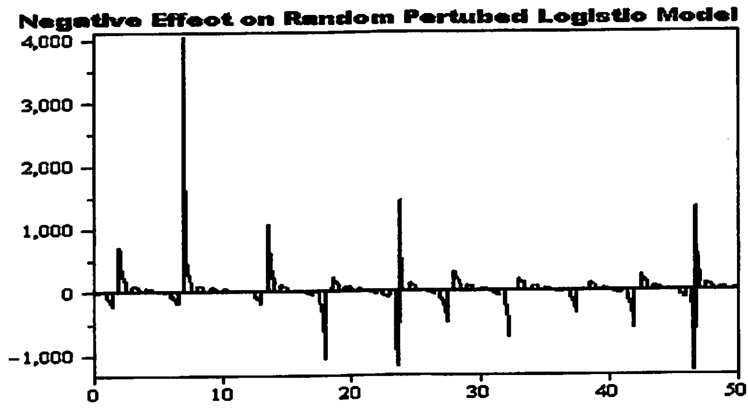

Fig.6.2: Negative effect by random disturbances in envirmental resources may change the logistic behavior.

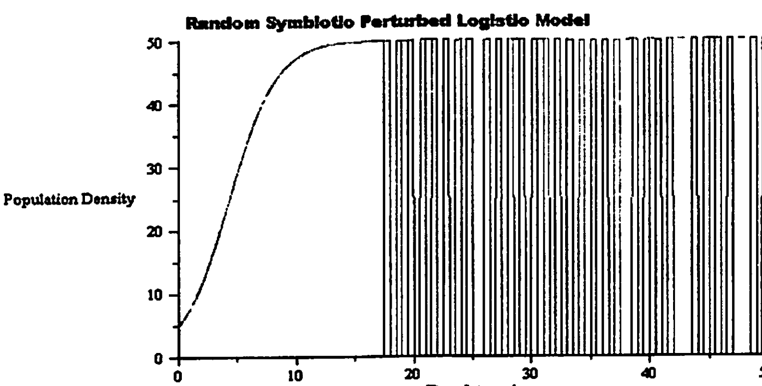

Fig. 6.3: The random symbiotic perturbation function w(n)=rand(1..100)() /1000 used in this algorithm is density independent.

6.2- Exact Solution-

We will assume some constant values for Random Perturbed Logistic Differential Equations and use Maple "dsolve" command to solve the problem. We will also simulate the solution based on the various values of the constant of integration.

Recall the package of differential equations tools (DEtools).

Example 1: The following is a random logistic differential equation with a random harvesting perturbation with a positive effect. This means delta is positive and Maple will help us to verify that the analytical solution is in the form of hyperbolic tangent.

Solution by Maple dsolve for negative Perturbation:

```
> a := .045 : b := .00003 : r := rand( -6 ..1 ) ( ) :
> K := a/b :
> delta := a^2 - 4·b·r;
```

$\delta := 0.002025$

```
> with (DEtools):
> f:=y(x)*(a-b*y(x))+r;
```

Solution by Maple dsolve for negative perturbation:

```
deq1 := diff(y(x), x) = f;
> dsolve( deq1, y(x) );
```

$$y(x) = 750 + \frac{50}{30}\sqrt{1785}\tanh(\frac{1}{2000}\sqrt{1785}x + \frac{1}{2000}\sqrt{1785}\cdot C)$$

Parameter C can be determined using the initial condition of the differential equations.

```
yourtoplot := [seq(L(x), C=-5 ..5)] :
plot(yourtoplot, x=-100 ..150);
```

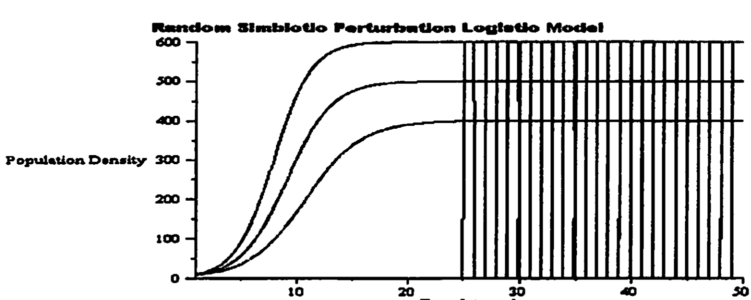

Fig.6.4- This is a simulation of random simbiotic perturbation w=rand(1..10) 0/100 in logistic model which causes disturbances in the solution.

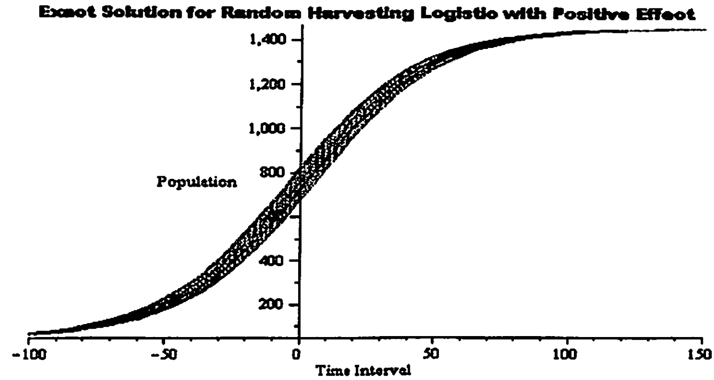

Fig. 6.5- Random Harvesting when delta is positive lead to hypeerbolic tangent solution.

Example 2: We will experience a perturbation, which has a negative effect on the resources carrying capacity:

a **:= 1 : *b* := 1 : *r1* := *rand*(-9 ..9) ( ) :**

$$M := \frac{a}{b}:$$

delta := $a^2 - 4 \cdot b \cdot r1$;

g:=z(x)*(a-b*z(x))-r1;

***deq2* := *diff*(*z*(*x*), *x*) = *g*;**

***dsolve*(*deq2*, *z*(*x*));**

$$L1 := x \rightarrow \frac{1}{2} - \frac{1}{2}\sqrt{3}\tan\left(\frac{1}{2}\sqrt{3}\,x + \frac{1}{2}\sqrt{3}\cdot C1\right);$$

***myplot* := [*seq*(*L1*(*x*), *C1* = -5 ..5)] :**

***plot*(*myplot*, *x* = -100 ..150);**

Example 3: With some arbitrary initial data, one can simulate a solution g(x) with a positive effect on the resources carrying capacity; that is:

$$g(x) = 2500 + 100\sqrt{626}\,\tanh\left(\frac{\sqrt{626}}{50}x - \frac{\sqrt{626}}{50}\right)$$

The following is a demonstration of the solution.

***mytoplot* := [*seq*(*G*(*x*), *C* = -5 ..5)] :**

***plot*(*mytoplot*, *x* = -10 ..10) :**

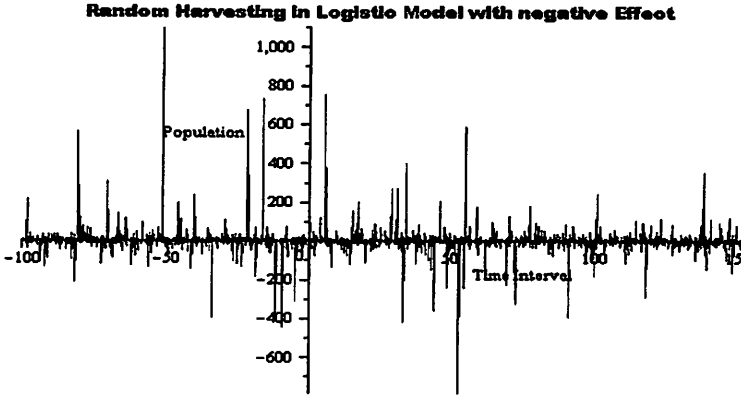

Fig. 6.6- Random Harvesting for Logistic Model with negative delta (negative effect) will lead the tangent solution.

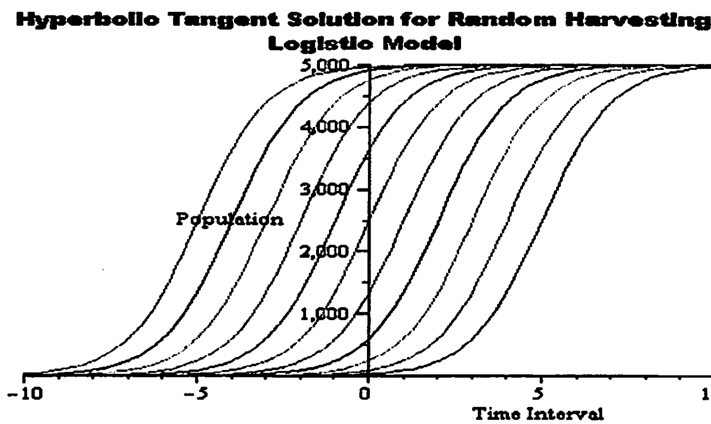

Fig.6.7- Simulation of the solution to a differential equation represents the random harvesting logistic with positive effect.

7- Discussion and Open Problems for Future Challenges:

We explored density independent types of random perturbation for a logistic model. In this model the general perturbation function g(t,P(t)) is independent from density function P(t) and is equal to w(t) which will stay constant in a short time interval $[t, t+\Delta t)$.

The dynamic behavior of the solution and the stability of the equilibrium points have been studied when the random perturbed function is positive or negative in any small time interval.
There are many existing challenges on these associated questions and they can be investigated in future works. Some of them are listed below.
- Demonstrate the result of discrete systems by a computational approach using Excel Spreadsheet.
- Explore in depth, more application, relation of this perturbation with noise, and stochastic form.
- Determine the bifurcations of random perturbed logistic models.
-Study the problem when the general perturbation g(t,P(t)) is some probability distribution.
-Determine when this dynamical system leads to a chaotic behavior.

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