Using The Inclusion-Exclusion Principle to Solve Some Problems

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Abstract

We use Inclusion-Exclusion Principle and rook polynomials to determine the number of different ways of playing certain games. Associated chessboards with darkened squares are used to determine the solution to these problems.

1 Introduction

While almost any text on Enumerative Combinatorics or even Discrete Mathematics has a chapter devoted to the Sieve method or the Inclusion-Exclusion Principle and Rook Polynomial, [1], [2] and [3] are excellent sources of material for this paper. Here we choose to use the notation used in [3]. Let $A_1, A_2, A_3, ..., A_n$ be sets in some universe U with N elements, and let S_k represent the sum of the sizes of all k-tuple intersections of the A_i's. Then by the Inclusion-Exclusion principle, we have

$$N(\overline{A}_1\overline{A}_2\overline{A}_3...\overline{A}_n) = N - S_1 + S_2 - S_3 + ... + (-1)^k S_k + ... + (-1)^n S_n$$

Let $r_k(B)$ represent the number of ways to place k mutually noncapturing rooks on board B. Likewise, let $r_k(B_1)$ and $r_k(B_2)$ represent the number of ways to place k mutually noncapturing rooks on subboards B_1 and B_2, respectively. $r_0(B) = 1$ for any board B. Let B be a board of darkened squares that decomposes into disjoint subboards B_1 and B_2. Then $r_k(B) = r_k(B_1)r_0(B_2)+r_{k-1}(B_1)r_1(B_2)+...+r_0(B_1)r_k(B_2)$. The **rook polynomial** $R(x, B)$ of the board B is defined as $R(x, B) = r_0(B)+r_1(B)x+r_2(B)x^2+ \ldots + r_n(B)x^n + \ldots$. If B is a board of darkened squares that decomposes into disjoint subboards B_1 and B_2. Then $R(x, B) = R(x, B_1)R(x, B_2)$. If B is a board with restricted positions and rook polynomial $R(x, B)$, then the number of ways to arrange n distinct objects given the restrictions on B is equivalent to $n!-r_1(B)(n-1)!+r_2(B)(n-2)!+...+(-1)^k r_k(B)(n-k)!+...+(-1)^n r_n(B)0!$. If we let $B_{m,n}$ denote an $m\times n$ chessboard with no restricted positions, and let $s = min\{m, n\}$, then $R(x, B_{m,n}) = \sum_{k=0}^{s}\binom{m}{k}P(n,k)x^k$.

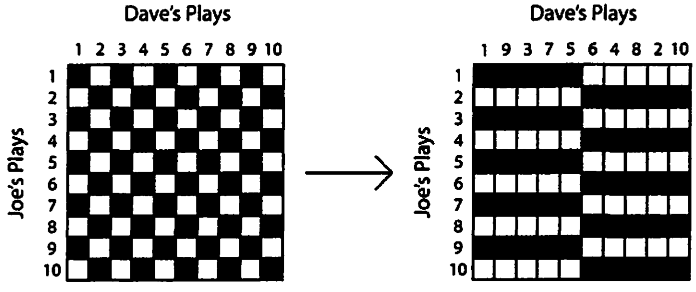
2 Odd-Even Game

Joe and Dave decide to play an extended game of Odds and Evens, where each person writes down a number from 1 to n on a piece of paper, n being even. If the sum of the two written numbers is odd, Joe wins the round; if the sum of the numbers is even, Dave wins. If the game consists of n rounds, over which each player must use every number once and only once, in how many ways can Joe win every round?

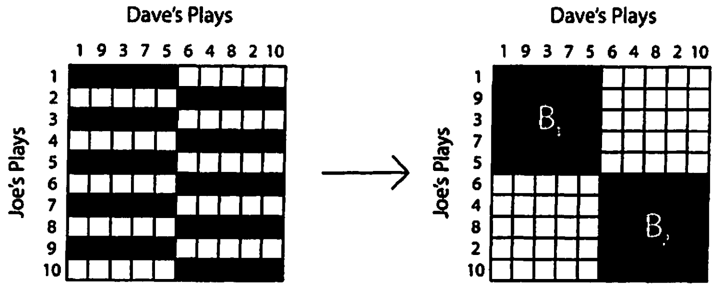
If n is an even number and $C_{n,n}$ denotes an $n\times n$ chessboard with restrictions on any position whose row and column values sum to an even number, then we claim that

$$R(x, C_{n,n}) = [R(x, B_{\frac{n}{2},\frac{n}{2}})]^2 = [\sum_{k=0}^{\frac{n}{2}}\binom{\frac{n}{2}}{k}P(\frac{n}{2},k)x^k]^2$$

Consider the chessboard that models the problem for the case where $n = 10$. Rearranging the columns so that all even-numbered columns are gathered on the right half of the board yields the following chessboard:



Rearranging the rows so that all odd-numbered rows are on the upper half of the board yields the final result:



In general, $C_{n,n}$ can be rearranged into a chessboard consisting of two disjoint subboards: twin copies of $B_{\frac{n}{2},\frac{n}{2}}$ With $n = 6$, we get

$$R(x, C_{6,6}) = [\sum_{k=0}^{3} \binom{3}{k} P(3,k) x^k]^2 =$$

$$1 + 18x + 117x^2 + 336x^3 + 432x^4 + 216x^5 + 36x^6$$

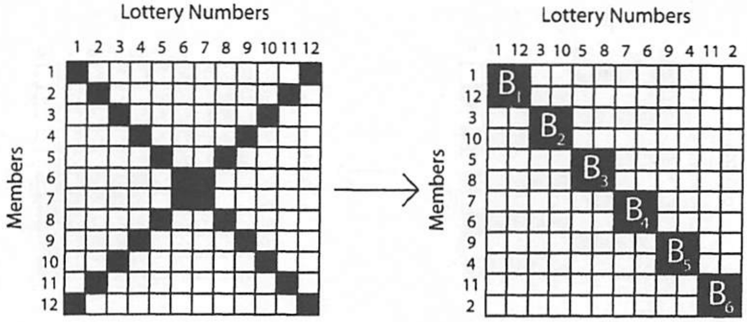
Therefore, if Joe and Dave play 6 rounds, there are 36 ways that Joe can win every round.

3 The Lottery

A casino holds a weekly lottery for its n members, n an even number. Upon joining, each member is assigned a number, based on the order in which they have joined, i.e., the i^{th} member to join is assigned the number i. In the weekly lottery drawing, each member is assigned randomly a number from 1 to n, such that each number is used once and only once, and winners are determined as follows: if Member i draws either i or $n + 1 - i$ for the week, he or she wins. In the case of multiple winners, the pot would be split among them. How many ways are there for no member to win during a week?

If n is an even number, and $S_{n,n}$ denotes an $n \times n$ chessboard with the specifications that for every row i, the i^{th} and $(n + 1 - i)^{th}$ positions are restricted, then we claim that $R(x, S_{n,n}) = (1 + 4x + 2x^2)^{\frac{n}{2}}$.

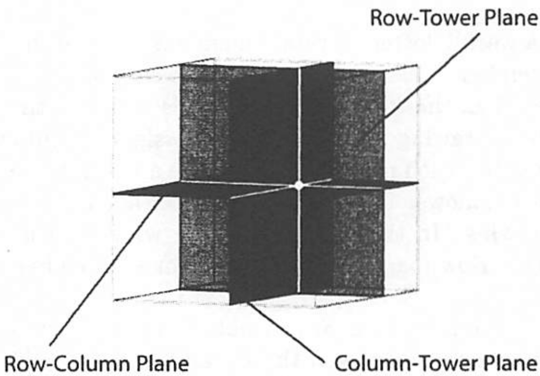
Consider $S_{n,n}$, the chessboard that models the lottery problem. Note that each pairing of the i^{th} and $(n + 1 - i)^{th}$ rows constitutes a disjoint subboard, which can be rearranged to yield $B_{2,2}$. In general, there are $\frac{n}{2}$ such pairings, we have that $S_{n,n}$ can be rearranged into $\frac{n}{2}$ disjoint copies of $B_{2,2}$.



$$R(x, S_{n,n}) = [R(x, B_{2,2})]^{\frac{n}{2}} = [\sum_{k=0}^{2} \tbinom{2}{k} P(2,k) x^k]^{\frac{n}{2}} = (1 + 4x + 2x^2)^{\frac{n}{2}}$$

4 Polynomials for Three-Dimensional Chessboards

Considering every two-dimensional level of a three-dimensional chessboard, we still have rows and columns in the traditional sense, each of which is a one-dimensional array used to describe position within the board.



We define a **tower** as a one-dimensional array used to describe position along the added third dimension. In dealing with rook polynomials of two-dimensional chessboards, we treated every rook in position {i,j} as the only object to occupy row i and column j. On a two-dimensional board, each rook placement prohibits any further rook placements in the union of that rook's row and column, i.e. two intersecting lines. On a three-dimensional board, however, each rook placement prohibits any further rook placements in the union of three intersecting planes formed by the $\binom{3}{2} = 3$ intersections of the dimensions, i.e. the plane formed using as a basis that rook's row and column, the plane formed using the rook's row and tower, and the plane formed using the rook's column and tower. It is easy to see that

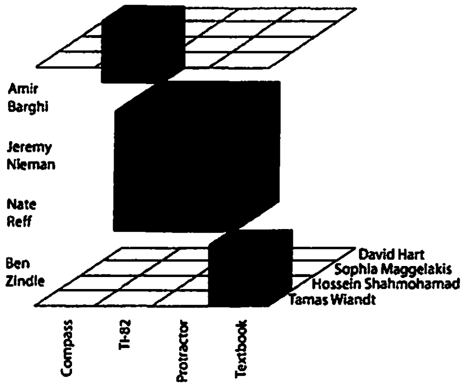
Theorem 4.1 *Let $B_{m,n,r}$ denote an $m \times n \times r$ chessboard with no restricted positions, and let $s = min\{m, n, r\}$. Then*

$$R(x, B_{m,n,r}) = \sum_{k=0}^{s} \binom{m}{k} P(n,k) P(r,k) x^k$$

Theorem 4.2 *Let B be a three-dimensional chessboard with restricted positions and rook polynomial $R(x, B)$. Then the number of ways to arrange n distinct objects given the restrictions on B is equivalent to $(n!)^2 - r_1(B)(n-1)!^2 + ... + (-1)^k r_k(B)(n-k)!^2 + ... + (-1)^n r_n(B)(0!)^2$.*

5 A Game of Clue in Building 8

Amir Barghi, Jeremy Nieman, Nate Reff and Ben Zindle each file reports with Campus Safety claiming to have been beaten simultaneously in different locations by a masked faculty member of the School of Mathematical Sciences. Four scuffed and bloodied objects were found in garbage cans around the campus: a compass, a TI-82 graphing calculator, a protractor, and a James Stewart calculus textbook. The only four faculty members unaccounted for during the hour in question are David Hart, Sophia Maggelakis, Hossein Shahmohamad and Tamas Wiandt, all of whom are known for their violent tempers. The following facts are established: David Hart could not have attacked Amir Barghi with the compass, Sophia Maggelakis and Hossein Shahmohamad could not have attacked Jeremy Nieman or Nate Reff with either the TI-82 calculator or the protractor, and Tamas Wiandt could not have attacked Ben Zindle with the calculus textbook. Given that Campus Safety personnel lack access to equipment for testing DNA or fingerprints, and are unable to tell the difference between trauma caused by a compass and trauma caused by a textbook, in how many ways can they randomly guess who was beaten by whom and with what so that they don't contradict the evidence and embarrass themselves? We start by modeling the problem with the following chessboard:



Note that the board is already decomposed into three disjoint subboards: two copies of $B_{1,1,1}$ and one copy of $B_{2,2,2}$. Therefore we see that the rook polynomial for this board B is: $R(x,B) = R(x,B_{2,2,2})[R(x,B_{1,1,1})]^2 =$

$$[\sum_{k=0}^{2}\binom{2}{k}[P(2,k)]^2x^k][\sum_{k=0}^{1}\binom{1}{k}[P(1,k)]^2x^k]^2 =$$

$$(1+8x+4x^2)(1+x)^2 = 1+10x+21x^2+16x^3+4x^4.$$

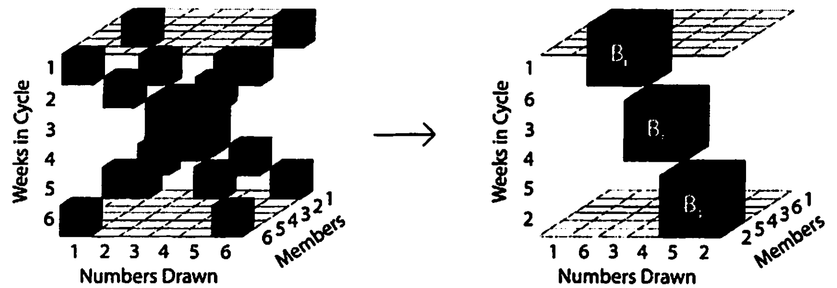
There are $(4!)^2 - 10\times(3!)^2 + 21\times(2!)^2 - 16\times(1!)^2 + 4\times(0!)^2 = 288$ ways to guess the outcome.

6 The Lottery Revisited

If a casino has n members, n an even number, and member i wins a share of the weekly lottery only if he or she draws either i or $n+1-i$, then in how many ways can an n-week cycle of lottery drawings pass in which no Member i wins a share of the pot during the i^{th} or $(n+1-i)^{th}$ weeks of the cycle for the following values of n? As with the earlier problems, we will develop an expression for any n, then plug in the values the problem specifies.

Theorem 6.1 *Let n be an even number, and let $S_{n,n,n}$ denote an $n\times n\times n$ chessboard where for every tower level i, the following positions are restricted: row i, column i; row i, column $n+1-i$; row $n+1-i$, column i; and 4.) row $n+1-i$, column $n+1-i$. Then $R(x,S_{n,n,n}) = (1+8x+4x^2)^{\frac{n}{2}}$.*

Proof: Consider the board $S_{n,n,n}$, and $n=6$ as an example. Note that for every pairing of tower levels i and $n+1-i$ the darkened squares on those levels combined constitute a disjoint subboard of $S_{n,n,n}$, as none of those squares, when considered as a grouping, share any row-column plane, row-tower plane or column-tower plane with any other darkened squares within the board. Thus through a brief series of row, column and tower exchanges, we arrive at the following decomposed representation of $S_{n,n,n}$:



$S_{n,n,n}$ can be decomposed into $\frac{n}{2}$ copies of $B_{2,2,2}$. Therefore,

$$R(x, S_{n,n,n}) = [R(x, B_{2,2,2})]^{\frac{n}{2}} = [\sum_{k=0}^{2}\binom{2}{k}[P(2,k)]^2 x^k]^{\frac{n}{2}} = (1+8x+4x^2)^{\frac{n}{2}} \quad \blacksquare$$

For $n = 6$, $R(x, S_{6,6,6}) = (1 + 8x + 4x^2)^3 = 1 + 24x + 204x^2 + 704x^3 + 816x^4 + 384x^5 + 64x^6$. Entering the coefficients of $R(x, S_{6,6,6})$ gives 267,904 ways for no member to win during a week whose number would give them a win if drawn in the lottery.

7 Odds and Evens Tournament

Joe and Dave decide to enter an Odds and Evens tournament. The tournament plays host to $2n$ players [$n = 2^m$, $m \in \mathbb{N}$] who are seated randomly at one of n numbered tables in the hall where the game is to be conducted. One player at each table will be designated Odd, meaning that that player wins a round if the sum of the numbers played at that table during the round is odd. The other player will be designated Even. Two bins each containing n folded slips of paper numbered 1 to n are placed in the center of the room, one marked Odds and the other marked Evens. At the beginning of each round, all Odd players draw a number from the Odds bin and all Even players draw a number from the Evens bin. The number each player draws will be the number he or she plays for that round. At the conclusion of the round the winner at each table proceeds, while the loser is eliminated. For the next round, play moves to the first $\frac{n}{2}$ tables, with the procedure of designating players Odd or Even repeated, and the Odds and Evens bins emptied of all but the folded slips of paper numbered from

1 to $\frac{n}{2}$. Play will continue in this fashion round to round until only two players remain, who will then participate in a final round at a single table, where each player has as an option the numbers 1 and 2. Whichever player wins the final round is dubbed the winner of the tournament. We consider the following problems:

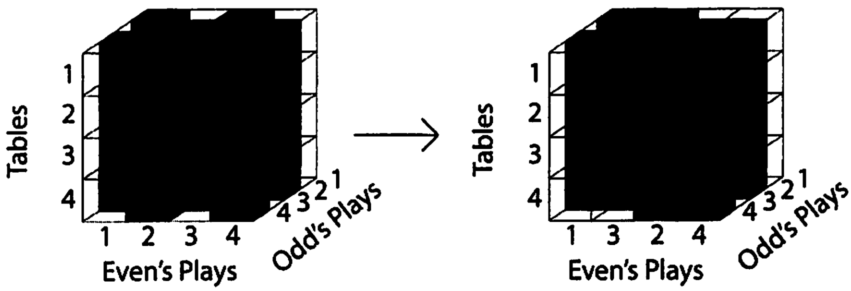
a.) If 16 players play in the tournament, in how many ways can the first round end with every Odd player a winner?

b.) In how many ways can the same tournament end with every round at every table won by an Odd player?

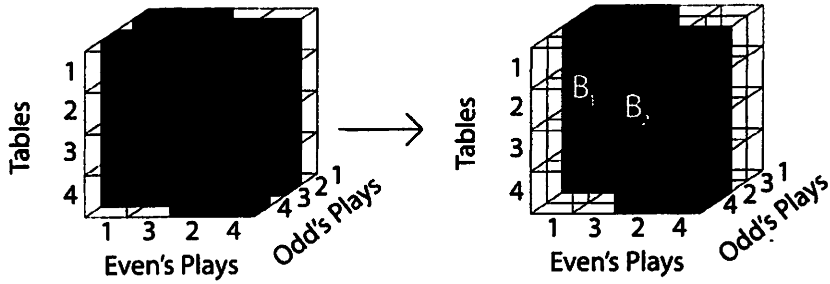
Given that with the conclusion of every round the size of the tournament is halved, it would be beneficial to develop an expression modeling a single round for any value of n, where $n = 2^m$, $m \in \mathbb{N}$.

Theorem 7.1 *Let n be an even number, and let $C_{n,n,n}$ denote an $n \times n \times n$ chessboard with restrictions on any position whose row and column values sum to an even number. Then $R(x, C_{n,n,n}) = \sum_{k=0}^{n} r_k(x, C_{n,n}) P(n,k) x^k$.*

Proof: Consider the case $n = 4$. Rearranging the columns so that all even-numbered columns are gathered on the right half of the board yields the following chessboard:



Rearranging the rows so that all odd-numbered rows are on the upper half of the board yields the final result:



Note that $C_{n,n,n}$ can be rearranged into two pillars, which, although they cannot simultaneously be touched by any row-tower or column-tower planes, are not disjoint because at every level they can both be touched by a row-column plane. However, this will not prove much of a setback.
If we were to place k rooks in the resultant three-dimensional board, the shadows of those rooks would be projected onto a two-dimensional copy of the decomposed form of $C_{n,n}$, and so the number of ways to arrange the k shadows on the floor would be $r_k(x, C_{n,n})$. Thus we obtain the number of ways to place k rooks in $C_{n,n,n}$ by multiplying the $r_k(x, C_{n,n})$ ways to determine the row-column positions of the rooks by the $P(n, k)$ ways to choose and permute the rooks' k tower positions, yielding every possible collection of rook placements within the board. Therefore, the rook polynomial for $C_{n,n,n}$ over all k is $R(x, C_{n,n,n}) = \sum_{k=0}^{n} r_k(x, C_{n,n})P(n, k)x^k$.
Since there are $2n = 16$ players, then $n = 8$ for the first round.

$$R(x, C_{8,8}) = [R(x, B_{4,4})]^2 = [1 + 16x + 72x^2 + 96x^3 + 24x^4]^2$$

$$R(x, C_{8,8,8}) = \sum_{k=0}^{8} r_k(x, C_{8,8})P(8, k)x^k = 1 + 256x + \ldots + 23,224,320x^8$$

Therefore there are 23,224,320 ways for every Odd player to win the first round of the tournament.
Determining the number of ways for every Odd player to win every round of an entire 16-player tournament requires we first determine $R(x, C_{4,4,4})$ and $R(x, C_{2,2,2})$, which will represent Rounds 2 and 3 of the tournament, respectively.

$$R(x, C_{4,4}) = [R(x, B_{2,2})]^2 = [\sum_{k=0}^{2} \tbinom{2}{k} P(2, k)x^k]^2 = (1 + 4x + 2x^2)^2$$

$$R(x, C_{2,2}) = [R(x, B_{1,1})]^2 = [\sum_{k=0}^{1} \tbinom{1}{k} P(1, k)x^k]^2 = (1 + x)^2 = 1 + 2x + x^2$$

Now that we have $R(x, C_{4,4})$ and $R(x, C_{2,2})$, we can determine

$$R(x, C_{4,4,4}) = 1 + (8 \times 4)x + (20 \times 12)x^2 + (16 \times 24)x^3 + (4 \times 24)x^4 =$$
$$1 + 32x + 240x^2 + 384x^3 + 96x^4$$

$$R(x, C_{2,2,2}) = 1 + (2 \times 2)x + (1 \times 2)x^2 = 1 + 4x + 2x^2$$

Thus there are 96 ways for an Odd player to win at every table in Round 2. Noticing that $(2!)^2 - 4 \times (1!)^2 + 2 \times (0!)^2 = 2$, we see that there are 2 ways for the Odd players in Round 3 to win. Round 4, the final round of the tournament, is played at a single table between the two remaining players, in which there are four possible combinations of plays: (1,1), (1,2), (2,1), and (2,2). Thus there are 2 ways for the Odd player to win the final round. To determine the number of ways that the tournament ends with every round at every table won by an Odd player, we simply multiply our results for each round, arriving at $23,224,320 \times 96 \times 2 \times 2 = 8,918,138,880$.

References

[1] I. Goulden; D. Jackson, *Combinatorial Enumeration*, Dover Publications, 1983

[2] R. Stanley, *enumerative Combinatorics*, Volume I, Cambridge University press, 1997

[3] A. Tucker, *Applied Combinatorics*, 5th Edition, Wiley, 2005