Beautifully Ordered Balanced Incomplete Block Designs

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Abstract. Beautifully Ordered Balanced Incomplete Block Designs, BOBIBD$(v, k, \lambda, k_1, \lambda_1)$, are defined and the proof is given to show that necessary conditions are sufficient for the existence of BOBIBD with block size $k = 3$ and $k = 4$ for $k_1 = 2$ except possibly for eleven exceptions. Existence of BOBIBDs with block size $k = 4$ and $k_1 = 3$ is demonstrated for all but one infinite family and the non-existence of BOBIBD$(7, 4, 2, 3, 1)$, the first member of the unknown series, is shown.

1. Introduction

A *Balanced Incomplete Block design*, BIBD(v, k, λ), is a collection of k-subsets (called blocks) of a v-set such that each pair of distinct points occurs in exactly λ blocks where $k < v$. The definition a BIBD(v, k, λ) requires $k < v$, but sometimes the notation BIBD(v, v, λ) is used to denote λ copies of the complete block $\{1, 2, \cdots, v\}$. A *Nested Balanced Incomplete Block Design*, (NBIBD), is a BIBD(v, k, λ) in which it is possible to subdivide each block of the design into $\frac{k}{k_1}$ sub-blocks of size k_1 such that the sub-blocks themselves form a BIBD, here k and k_1 are positive integers such that k_1 divides k. For example, consider the following collection of five blocks of a BIBD(5,4,3) on five points {1,2,3,4,5}:

$$\{\{1,2,3,4\}, \{1,2,3,5\}, \{1,2,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}\}.$$

Now consider the following subdivision of these five blocks in two specific subblocks of size two:

block {1,2,3,4} into blocks {1,4}, and {2,3},
block {1,2,3,5} into blocks {3,5}, and {1,2},

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block {1,2,4,5} into blocks {4,2}, and {1,5},
block {1,3,4,5} into blocks {3,1}, and {4,5},
block {2,3,4,5} into blocks {2,5}, and {3,4}.

Notice that these ten subsets of size two form a BIBD(5,2,1) and hence above BIBD(5,4,3) is an NBIBD.
In the above example one can not arbitrarily partition each block of the original design into two blocks to get a BIBD, of course for an NBIBD this condition is not even required. Nested designs have been studied extensively [8]. The BIBDs with ordered blocks are also studied extensively in different context, for example see [7] and [2].

We are proposing to order the elements of the blocks of a BIBD in such a way that for any fixed set of k_1 locations, the collection of sub-blocks with entries from the fixed set of locations from all blocks gives a BIBD. For example, order the elements of the blocks of a BIBD$(5,4,6)$, obtained by taking two copies of a BIBD$(5,4,3)$, as follows:

{{1,2,3,4},{2,3,4,5},{3,4,5,1},{4,5,1,2},{5,1,2,3},
{2,4,1,3},{3,5,2,4},{4,1,3,5},{5,2,4,1},{1,3,5,2}}.

As the block size is four, each block has four locations, first, second, third and fourth. Choose ANY two locations, say first and fourth, and construct blocks from the entries at these locations of each block:

{{1, 4}, {2, 5}, {3,1}, {4, 2}, {5, 3}, {1, 2}, {3, 4},{5, 1},{2, 3}, {4, 5}}.

The blocks above give a BIBD$(5,2,1)$ as every pair has occurred exactly once in these (unordered) smaller blocks. Note that one may choose any other two distinct locations, viz., first and second, first and third, second and third, second and fourth or third and fourth and construct sub-blocks from the entries at these locations of the ordered blocks and still gets a BIBD$(5,2,1)$. We call such a BIBD with ordered blocks a *Beautifully Ordered Balanced Incomplete Block Design.* Formally,

DEFINITION 1. *If each of the blocks of a* BIBD(v,k,λ) *is ordered such that for any* k_1 *indices* $i_1, i_2, \cdots, i_{k_1}$ *the sub-blocks* $\{a_{i_1}, a_{i_2}, \cdots, a_{i_{k_1}}\}$ *of all ordered blocks* $\{a_1, a_2, \cdots, a_k\}$ *of the* BIBD(v,k,λ) *form a* BIBD(v,k_1,λ_1) *then we say that the collection of ordered blocks gives a Beautifully Ordered Balanced Incomplete Block Design,* BOBIBD$(v,k,\lambda,k_1,\lambda_1)$ *where* $2 \leq k_1 \leq k\text{-}1$.

Clearly when k_1 divides k, a BOBIBD gives a nested BIBD with (super) block size k and sub-block size k_1 but for a BOBIBD there is no restriction on k_1, hence BOBIBDs can be constructed even when k_1 is not a factor of k. Note that small BOBIBDs may be given as a $b \times k$ array where the rows are the ordered blocks.

EXAMPLE 1. *The following is a* BOBIBD(5, 5, 10, 2, 1).

1	3	5	2	4
3	5	2	4	1
5	2	4	1	3
2	4	1	3	5
4	1	3	5	2
1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1.1. Perpendicular and Beautiful Arrays. There is another combinatorial object which is quite similar to what we have defined. It is called Perpendicular Array [3]. The formal definition is:

DEFINITION 2. *A perpendicular array $PA_\lambda(t,k,v)$ is a $k \times \lambda\binom{v}{t}$ array with v entries such that*

(1) *each column has k distinct entries, and*
(2) *each set of t rows contains each set of t distinct entries as a column precisely λ times.*

Clearly when $t = 2$, the perpendicular array gives a BOBIBD with $k_1 = 2$ when we consider the columns of the array as the blocks of the BIBD(v, k, λ). Of course, for $k_1 \geq 3$, BOBIBD and perpendicular arrays are different combinatorial structures.

There are many existence results on perpendicular arrays as given in [3], for example, using our terminology, it is given in [3] that:

- BOBIBD$(v, 3, 3, 2, 1)$ exists for odd $v \geq 3$, [10]
- BOBIBD$(v, 4, 6, 2, 1)$ exists for odd $v \geq 5$ [9], [5] and
- BOBIBD$(v, 4, 12, 2, 2)$ exists for $v \geq 4$ [9]

In fact, the results proven in this paper for $k_1 = 2$ can be deduced from these results, though we give straight-forward independent proofs for general λ with usual design theory techniques. The results for $k_1 = 3$ may not be obtained from the results in [3]. Given such close relation, one may tempt to rewrite our definition and introduce BOBIBD as an array

DEFINITION 3. *A Beautiful Array $BA(v, k, \lambda, k_1, \lambda_1)$ is a $b \times k$ array ($k > 2$), where $b = \frac{\lambda v(v-1)}{k(k-1)} = \frac{\lambda_1 v(v-1)}{k_1(k_1-1)}$, the entries of which are drawn from a set of v symbols and are disposed so that* (a) *the rows of the array constitute the blocks of a* BIBND(v, k, λ), *and* (b) *if we form a $b \times k_1$ sub-array from any k_1 columns of the array, $1 < k_1 < k$, then the rows of the sub-array constitute the blocks of a* BIBD(v, k_1, λ_1).

We feel that having the word BIBD in the definition is more revealing than the word array as the underlying structure is a BIBD and substructures are also BIBDs.

1.2. Latin Square. We will need the following well known results about Latin squares. For basic definition and notation, please see [**12**]. A Latin square L of side n on symbols $Q = \{1, 2, \cdots, n\}$ can be considered as a Quasigroup $(Q, \circ)$, the rows and columns of L are labeled by the symbols in Q and $i \circ j$ is the $(i, j)^{th}$ element of L. When $(i, i)^{th}$ element of L is i for all $i = 1, 2, \cdots, n$, L is called an idempotent Latin square. Let $N(n)$ denote the number of Latin squares in the largest possible set of mutually orthogonal Latin squares of side n.

LEMMA 1. *([**12**], page 126) There exists a set of $N(n)$ - 1 mutually orthogonal idempotent Latin squares of side n.*

THEOREM 1. *([**12**], page 143) There exist three mutually orthogonal Latin squares of every side except 2, 3, 6, and possibly 10.*

COROLLARY 1. *([**12**], page 145) There is a pair of orthogonal idempotent Latin squares of every side except 2, 3 and 6.*

2. Necessary Conditions for BOBIBDs

From the definition, if a BOBIBD(v,k,λ,k_1,λ_1) exists, then

(1) BIBD(v,k,λ) exists, and
(2) BIBD(v,k_1,λ_1) exists.

Hence:

THEOREM 2. *Every necessary condition for the existence of* BIBD(v,k,λ) *is a necessary condition for the existence of* BOBIBD(v,k,λ,k_1,λ_1) *and every necessary condition for the existence of* BIBD(v,k_1,λ_1) *is a necessary condition for the existence of* BOBIBD(v,k,λ,k_1,λ_1).

For ease of reference the well known necessary conditions for BIBD(v,3,λ) and BIBD(v,4,λ), for $v \geq k$, are given below:

Block size 3:

λ	spectrum of λ-fold triple systems
$\lambda \equiv 0 \pmod 6$	all $v \neq 2$
$\lambda \equiv 1,5 \pmod 6$	all $v \equiv 1,3 \pmod 6$
$\lambda \equiv 2,4 \pmod 6$	all $v \equiv 0,1 \pmod 3$
$\lambda \equiv 3 \pmod 6$	all odd v

Block size 4:

λ	spectrum of λ-fold quadruple systems
$\lambda \equiv 0 (\mathrm{mod}\ 6)$	all v
$\lambda \equiv 1,5 (\mathrm{mod}\ 6)$	all $v \equiv 1,4 (\mathrm{mod}\ 12)$
$\lambda \equiv 2,4 (\mathrm{mod}\ 6)$	all $v \equiv 1 (\mathrm{mod}\ 3)$
$\lambda \equiv 3 (\mathrm{mod}\ 6)$	all $v \equiv 0,1 (\mathrm{mod}\ 4)$

Simple counting arguments give

THEOREM 3. *In a* BOBIBD(v,k,λ,k_1,λ_1), $\lambda = \frac{\binom{k}{k_1}\lambda_1}{\binom{k-2}{k_1-2}}$.

If we want to construct BOBIBD($v, k, \lambda, 2, \lambda_1$), there are $\binom{k}{2}$ ways we can pick up two locations in a block of BIBD(v, k, λ), hence λ is a multiple of $\binom{k}{2}$ and $\lambda = \binom{k}{2}\lambda_1$. This fact is included in the following corollary.

COROLLARY 2. *For any BOBIBD,*

(1) *if* $k_1 = 2$, *then* $\lambda = \binom{k}{2}\lambda_1$ *and the number of blocks must be a multiple of* $\binom{v}{2}$.

(2) *if* $k_1 = 3$, *then* $\lambda = \frac{(\binom{k}{3}\lambda_1)}{(k-2)} = \frac{\binom{k}{2}\lambda_1}{3}$.

THEOREM 4. *If a* BOBIBD(v,k,λ,2,λ_1) *exists, then a* BOBIBD(v,k,λ, k_1,$\binom{k_1}{2}\lambda_1$) *exists for* $2 \le k_1 \le k$.

In view of the above theorem, all results obtained for $k = 4$ and $k_1 = 2$ extend for $k = 4$ and $k_1 = 3$ as well and all examples constructed for BOBIBD($v, 4, \lambda, 2, \lambda_1$) are also the examples for BOBIBD($v, 4, \lambda, 3, 3\lambda_1$).

THEOREM 5. *If a* BOBIBD(v,k,λ,2,λ_1) *exists, then a* BOBIBD(v,k_1, $\binom{k_1}{2}\lambda_1$,2,λ_1) *exists, where* $2 \le k_1 \le k$.

PROOF. Fix any k_1 locations of the blocks of BOBIBD($v, k, \lambda, 2, \lambda_1$) and construct the ordered subblocks of size k_1 with elements from the fixed k_1 locations, we get a BOBIBD(v, k_1, $\binom{k_1}{2}\lambda_1, 2, \lambda_1$). □

EXAMPLE 2. *The* BOBIBD(5, 5, 10, 2, 1) *given in the introduction is also a* BOBIBD(5, 5, 10, 3, 3).

The above example has an easy generalization:

THEOREM 6. *A* BOBIBD($v, v, \lambda, k_1, \lambda_1$) *is also a* BOBIBD($v, v, \lambda, v$-$k_1, \lambda_2$), *where* $\lambda_2 = \frac{\lambda(v-k_1)(v-k_1-1)}{v(v-1)}$ *if* v-$k_1 \ge 2$.

PROOF. As we know that entries from any k_1 locations is a BIBD(v, k_1, λ_1), and hence the compliments of the blocks is a BIBD($v, v - k_1, \lambda_2$) for some λ_2. Note that the number of blocks and the replication number for the design is λ and as for a BOBIBD, at each location every element occurs $\frac{\lambda}{v}$, the replication number for BIBD(v, v-k_1, λ_2) is $(v - k_1)(\frac{\lambda}{v})$. Using the usual parametric relationships between design parameters, λ_2 is as given in the statement of the theorem. □

THEOREM 7. *If a* BOBIBD$(v, k, \lambda, 2, \lambda_1)$ *exists then* k *divides* r *and in the (ordered) blocks of* BOBIBD *each element occurs exactly* $\frac{r}{k}$ *times at each location of the blocks.*

PROOF. Let c_i denote the number of times an element a appears at the i^{th} location in the collection of ordered blocks of a BOBIBD(v,k,λ,2,λ_1). Consider any two locations i and j, as we have a BOBIBD with $k_1 = 2$, $c_i + c_j = \lambda_1(v-1)$. Similarly for locations i and k, $c_i + c_k = \lambda_1(v-1)$, hence for all $k \neq j$, $c_k = c_j$. As $c_1 + c_2 + \ldots c_k =$ r, $kc_j = r$, and hence k divides r. □

EXAMPLE 3. *For the* BIBD(4,4,6), $r = 6$ and $\frac{r}{k} = \frac{6}{4}$, which is not an integer. Therefore BOBIBD(4,4,6,2,1) does not exist .

The above theorem can be generalized easily as follows.

THEOREM 8. *If a* BOBIBD$(v, k, \lambda, k_1, \lambda_1)$ *exists then* k *divides* r *and in the (ordered) blocks of* BOBIBD *each element occurs exactly* $\frac{r}{k}$ *times at each location of the blocks.*

3. Block sizes $k = 3$, $k_1 = 2$

For any BOBIBD, if $k = 3$ then the only possible value of k_1 is 2. With these two values, Corollary 2 gives $\lambda = 3\lambda_1$ and hence λ has to be a multiple of 3. We therefore consider two cases: $\lambda = 6t+3$ and $\lambda = 6t$. It is well known that for block size $k = 3$ and $\lambda = 6t + 3$, v has to be odd. On the other hand, BIBD$(v, 3, 6t)$ exists for any v. In other words, the necessary conditions for the existence of a BOBIBD(v,3,λ,2,λ_1) are:

λ	spectrum of BOBIBD(v,3,λ,2,λ_1)'s
$\lambda \equiv 1, 2 \pmod 3$	none
$\lambda \equiv 3 \pmod 6$	odd v
$\lambda \equiv 0 \pmod 6$	all v

Table 1

We might write a block {a,b,c} as abc and the context will indicate when the block is ordered.

<u>Subcase $\lambda = 6t + 3$</u>

For this case, v must be odd, so $v \equiv 1, 3, 5 \pmod 6$.

For $v \equiv$ 1,3(mod 6), a BIBD$(v, 3, 1)$ exists. Arrange 3 copies of each block {a,b,c} of the BIBD$(v, 3, 1)$ as {a,b,c}, {c,a,b}, and {b,c,a} to get a BOBIBD$(v, 3, 3, 2, 1)$. A BOBIBD$(v, 3, 6t+3, 2, 2t+1)$ can be obtained by taking $(2t + 1)$ copies of a BOBIBD$(v, 3, 3, 2, 1)$.

For $v \equiv 5 \pmod 6$. Recall one can construct a PBD on $v = 6t+5$ points with exactly one block, say {1,2,3,4,5}, and all other blocks of size 3 [6]. Order 3 copies of each of the triples as in the above paragraph. Combining with triples 123,412,251,314,531,154,235,342,425,543 of a BOBIBD$(5,3,2,2,1)$ we get a BOBIBD$(v,3,3,2,1)$. Here again, $(2t+1)$ copies yield the required BOBIBD$(v,3,6t+3,2,2t+1)$.

Subcase $\lambda = 6t$

There is no restriction on v for $\lambda = 6t$.

Even though one can use similar arguments again for v odd, a general construction gives the required designs for $\lambda = 6t$ automatically.

Recall that one can construct a BIBD$(v,3,6)$ by an idempotent Quasigroup $(Q,\circ)$ of order v which exists for all order $v \geq 3$ where the collection of triples of the BIBD$(v,3,6)$ is $\{\{a,b,a\circ b\}$ where $a \neq b \in Q\}$. Keeping the ordering of the elements in triples as it is, the properties of the Latin square guarantee that each pair {a,b} occurs at the location i,j, $1 \leq i < j \leq 3$ in the triples exactly twice as required. Taking t copies of the design gives BOBIBD$(v,3,6t,2,2t)$. Hence we have:

THEOREM 9. *The necessary conditions given in Table* 1 *for the existence of* BOBIBD *with* $k = 3$ *and* $k_1 = 2$ *are sufficient.*

4. Block sizes $k = 4$, $k_1 = 2$

In this section, the BOBIBDs will have $k = 4$ and $k_1 = 2$. We begin with an easy application of Theorem 3 and Theorem 7.

THEOREM 10. *For any* BOBIBD$(v,4,\lambda,2,\lambda_1)$, $\lambda = 6\lambda_1$. Moreover if λ_1 is odd then v is odd, and if λ_1 is even then there is no condition on v.

In other words, the necessary condition for the existence of BOBIBD$(v,4,\lambda,2,\lambda_1)$ are:

λ	spectrum of BOBIBD$(v,4,\lambda,2,\lambda_1)$'s
$\lambda \equiv 6 \pmod{12}$	all odd $v \geq 5$
$\lambda \equiv 0 \pmod{12}$	no condition on v

It follows from this that we only need to consider two cases, $\lambda = 6$ and $\lambda = 12$. We do this in the next two subsections.

4.1. BOBIBD$(v,4,6,2,1)$ for odd $v \geq 5$.

EXAMPLE 4. *One can construct a* BOBIBD$(5,4,6,2,1)$ *by deleting the first entries of all the blocks of Example* 1, BOBIBD$(5,5,10,2,1)$.

Using the above example and the fact that BIBD$(v,5,1)$ exist for all $v \equiv 1,5 \pmod{20}$, we have

THEOREM 11. *A* BOBIBD$(v,4,6,2,1)$ *exists for all* $v \equiv$ 1,5(mod 20).

EXAMPLE 5. *A* BOBIBD$(7,4,6,2,1)$ *can be constructed with ordered difference sets* {7,1,2,4}, {7,2,4,1}, {7,4,1,2}.

EXAMPLE 6. *A* BOBIBD$(9,4,6,2,1)$ *is constructed below:*

7	1	2	3
8	2	3	1
9	3	1	2
1	4	5	6
2	5	6	4
3	6	4	5
4	7	8	9
5	8	9	7
6	9	7	8

6	1	4	7
9	4	7	1
3	7	1	4
4	2	5	8
7	5	8	2
1	8	2	5
5	3	6	9
8	6	9	3
2	9	3	6

3	1	5	9
4	5	9	1
8	9	1	5
1	2	6	7
5	6	7	2
9	7	2	6
2	3	4	8
6	4	8	3
7	8	3	4

9	1	6	8
2	6	8	1
4	8	1	6
7	2	4	9
3	4	9	2
5	9	2	4
8	3	5	7
1	5	7	3
6	7	3	5

.

THEOREM 12. *Necessary conditions ($v \geq 5$ and v odd) are sufficient for the existence of a* BOBIBD$(v,4,6t,2,t)$ *except possibly for* 15, 27, 33, 39, 51, 75, 87, 95, 99, 111, *and* 115.

PROOF. A BOBIBD$(v,4,6,2,1)$ exists for {5,7,9} and hence for $v \equiv$ 1(mod 2) except possibly for (11-19), 23, (27-33), 39, 43, 51, 59, 71, 75, 83, 87, 95, 99, 107, 111, 113, 115, 119, 139, 179 [1]. Excluding the eleven exceptions listed in the theorem, one can construct BOBIBDs using Theorem 13 given below. Take t copies of BOBIBD$(v,4,6,2,1)$ to construct BOBIBD$(v,4,6t,2,t)$ □

THEOREM 13. *For any prime p, ordered difference sets* {0,*i*,*p*-*i*,2*i*}, $i = 1, 2, \cdots, \frac{p-1}{2}$ *give* BOBIBD$(v,4,6,2,1)$.

PROOF. Differences from the ordered difference set {0,*i*,*p*-*i*,2*i*} are i, $2i$, $3i$, i, i, $2i$ and as i runs through 1 to $\frac{p-1}{2}$ every difference from 1 to $\frac{p-1}{2}$ occurs exactly once for each pair of locations. □

4.2. BOBIBD(v,4,12,2,2) for all $v \geq 4$.

EXAMPLE 7. BOBIBD(4,4,12,2,2)

1	2	3	4
1	4	2	3
1	3	4	2
2	1	3	4
2	4	1	3
2	3	4	1

3	1	2	4
3	4	1	2
3	2	4	1
4	1	2	3
4	3	1	2
4	2	3	1

.

Note for $v \equiv$ 1,4(mod 12), a BOBIBD can be constructed by rearranging the blocks of a BIBD$(v,4,1)$ according to the above example. Hence we have:

THEOREM 14. *A* BOBIBD$(v,4,12,2,2)$ *exists for all* $v \equiv 1,4 (mod\ 12)$.

We need a BOBIBD$(6,4,12,2,2)$ which is given below:

1	2	6	4
1	3	2	6
1	4	5	2
1	5	3	4
1	6	4	2
2	1	4	5
2	3	6	5
2	4	3	6
2	5	1	3
2	6	5	3

3	1	2	4
3	2	5	1
3	4	6	1
3	5	4	6
3	6	1	4
4	1	5	6
4	2	3	5
4	3	1	2
4	5	6	2
4	6	2	5

5	1	6	3
5	2	1	6
5	3	4	1
5	4	2	3
5	6	3	1
6	1	3	2
6	2	4	3
6	3	5	4
6	4	1	5
6	5	2	1

.

To construct BOBIBD$(v,4,12,2,2)$ for all values of $v \geq 4$, we can extend the construction for BIBD$(v,3,6)$ by an idempotent Quasigroup of order v which exist for all required values of v's.

THEOREM 15. *Let* $L_1 = (Q, \circ_1)$, $L_2 = (Q, \circ_2)$ *be two mutually orthogonal idempotent Latin squares of order* v. *Then the set of blocks* $T = \{\{a,b,a\circ_1 b,a\circ_2 b\} : a \neq b\ a,b \in Q\}$ *gives a* BOBIBD$(v,4,12,2,2)$.

PROOF. Let $L_1 = (Q, \circ_1)$, $L_2 = (Q, \circ_2)$ be two mutually orthogonal idempotent Latin squares of order v which exist for all values of v except 2,3, and 6 (see Theorem 1). Note that this construction generates $2\binom{n}{2} = n(n\text{-}1)$ blocks of size four which is the required number of blocks for a BIBD$(v,4,12)$. For any $a{\neq}b$, we know that pair $\{a,b\}$ and pair $\{b,a\}$ occurs at the first two locations of the blocks at least twice. Now consider the occurrences of the pair $\{a,b\}$ at the first and third location or second and third location. The third location entry is $a\circ_1 b$. It is clear that for some $x, y \in Q$, $a\circ_1 x = b$ and $y\circ_1 a = b$. Similarly, for some $w, z \in Q$, $b\circ_1 w = a$ and $z\circ_1 b = a$. Therefore the count of occurrences of the pair $\{a,b\}$ until now is at least 2+4 = 6. Next we consider the occurrences of the pair $\{a,b\}$ at first and fourth or second and fourth locations. The fourth location entry is $a\circ_2 b$. Same argument can be used again in this case. Hence $\{a,b\}$ occurs at least 6+4 = 10 times. Now since L_1 and L_2 are idempotent MOLS, there exists $p, q \in Q$, such that $p\circ_1 q = a$ and $p\circ_2 q = b$, and for some $r,s \in Q$ such that $r\circ_1 s = b$, $r\circ_2 s = a$. Hence $\{a,b\}$ occurs at least 10+2 = 12 times. The number of blocks is exactly the number of blocks needed for the design, $\lambda = 12$. This counting for the index λ also shows that the construction produces BOBIBD$(v,4,12,2,2)$. □

Theorem 15 gives the construction of BOBIBD$(v,4,12t,2,2t)$ except for v = 2,3, and 6. However, a BOBIBD$(6,4,12,2,2)$ is given above and as $k = 4$ is bigger than 2 and 3, we have

THEOREM 16. *The necessary condition, that $v \geq 4$, is sufficient for the existence of a* BOBIBD$(v, 4, 12t, 2, 2t)$.

5. Block sizes $k = 4$, $k_1 = 3$

THEOREM 17. *Necessary conditions for the existence of* BOBIBD$(v, 4, \lambda, 3, \lambda_1)$ *are* $\lambda = 2\lambda_1$ *(hence λ is even), and*

λ	λ spectrum
$\lambda \equiv 0 \pmod{12}$	all v
$\lambda \equiv 2,10 \pmod{12}$	$v \equiv 1 \pmod 6$
$\lambda \equiv 6 \pmod{12}$	all odd v
$\lambda \equiv 4,8 \pmod{12}$	$v \equiv 1 \pmod 3$

.

PROOF. Necesscary conditions for BIBD$(v, 4, \lambda)$ imply $\lambda(v-1) = 3r$ and $\frac{\lambda(v)(v-1)}{12} = b$. □

5.1. $\lambda \equiv 0,6 \pmod{12}$. We have proved for BOBIBD$(v, 4, 6, 12t + 6, 2, 2t + 1)$ exists for all odd v and BOBIBD$(v, 4, 12t, 2, 2t)$ exists for any $v \geq 4$ and hence we have the following result.

THEOREM 18.

(1) *The necessary conditions given in Theorem 17 are sufficient for* BOBIBD$(v, 4, 12t{+}6, 3, 6t{+}3)$.

(2) *The necessary conditions given in Theorem 17 are sufficient for* BOBIBD$(v, 4, 12t, 3, 6t)$.

EXAMPLE 8. *Using the Self-Orthogonal Latin squares of order* 7 *given in* [4] , *we can construct the following* BOBIBD(7,4,6,3,3)*:*

1	2	7	6
1	3	6	4
1	4	5	2
2	3	1	7
2	4	7	5
2	5	6	3
3	4	2	1
3	5	1	6
3	6	7	4
4	5	3	2
4	6	2	7

4	7	1	5
5	6	4	3
5	7	3	1
5	1	2	6
6	7	5	4
6	1	4	2
6	2	3	7
7	1	6	5
7	2	5	3
7	3	4	1

.

5.2. $\lambda = 2, 10$ and $\lambda_1 = 1$. In this subsection we consider the existence of BOBIBD$(v, 4, 2, 3, 1)$. For $v = 7$, the underlying BIBDs exist, and the necessary conditions are satisfied, therefore we might expect a BOBIBD$(7, 4, 2, 3, 1)$ to exist. However, the next two theorems indicate some of the difficulties for $\lambda_1 = 1$

THEOREM 19. BOBIBD(7,4,2,3,1) *does not exist.*

PROOF. Assume the design exists and is expressed as an array. Without loss of generality, let the last column and first row of the design be:

2	3	4	1
			2
			3
			4
			5
			6
			7

Since 1 is already paired with $2, 3, 4$ once and due to the facts that one already appears in the fourth column, without loss of generality, assume one is distributed on the diagonal in row 5 to row 7 as shown below,

2	3	4	1
			2
			3
			4
1			5
	1		6
		1	7

Note that the element 2 cannot be placed in row 5 and column 1. In addition, 5 has to occur in row 6 or row 7, but 5 cannot be in the first column. Hence we have two cases to consider because we can place 2 into row 6 and place 5 into row 7 or place 5 into row 6 and place 2 into row 7, for both cases, 2 or 5 has to be placed directly next to 1 in the 3^{rd} or the 2^{nd} column

2	3	4	1
			2
			3
			4
1			5
	1	2/5	6
	5/2	1	7

Consider Case 1: where 2 and 5 are placed in row 6 and row 7 respectively,

2	3	4	1
			2
			3
			4
1			5
	1	2	6
	5	1	7

.

As we can see from above, 7 is forced to be placed into row 6, 4 and 6 are placed into row 5 and finally 3 is placed into row 7,

2	3	4	1
			2
			3
			4
1	4	6	5
7	1	2	6
3	5	1	7

.

Going back to row 2, 7 and 3 are forced to be placed as follows,

2	3	4	1
	7	3	2
			3
			4
1	4	6	5
7	1	2	6
3	5	1	7

.

In row 3, 6, 2, and 5 are forced to be placed as shown below,

2	3	4	1
	7	3	2
			3
6	2	5	4
1	4	6	5
7	1	2	6
3	5	1	7

.

In the final configuration shown below, zero denotes the locations where the conflict occurs,

2	3	4	1
5	7	3	2
4	0	0	3
6	2	5	4
1	4	6	5
7	1	2	6
3	5	1	7

.

Case 1

The final configuration with conflict for case 2 is shown below as in Case 1,

2	3	4	1
5	4	6	2
7	5	2	3
0	0	0	4
1	7	3	5
4	1	5	6
6	2	1	7

.

Case 2

□

THEOREM 20. *The blocks of a* BIBD(v,4,2) *can not be ordered to contruct a* BOBIBD(v,4,2,3,1) *if there exist two identical blocks or two blocks with* 3 *common points.*

PROOF. Suppose the intersection number of two blocks is 4, i.e. two blocks are identical. Let b_1 = {a,b,c,d} = b_2 be two blocks of the BIBD(v,4,2). Without loss of generality, we only rearrange b_2, and hence we have the following four cases to consider:

(1) Consider configuration below:

a	b	c	d
a			

.

where we placed a in the first location, no matter which way we rearrange b,c,d, for some locations (i_1,i_2,i_3) a pair appears more than once.

(2) Consider b in the first location of second block, the same argument we can use as in Case 1.

(3) Consider c in the first location as displayed below:

a	b	c	d
c	d		

.

The only possible entry at the second location is d, and no matter how we place a and b, there exist three locations of b_1 and b_2 where a pair appears twice.

(4) Consider d in the first location and c in the second location. The same argument can be made as in Case 3.

Similarly one can show that if two blocks have 3 common element then it is impossible to order the blocks to get a BOBIBD$(v, 4, 2, 3, 1)$. □

5.3. Applying the $(3n+1)$ Construction to $\lambda_1 = 2$. The $3n+1$ Construction from [6] can be usefully employed for our purposes. Let $(Q, \circ)$ be an idempotent (not necessarily commutative) quasigroup of order n and set $S = \{\infty\} \cup (Q \times \{1,2,3\})$. Define a collection of triples T as follows:

Type 1: The four triples $\{\infty,(x,1),(x,2)\}$,$\{\infty,(x,2),(x,3)\}$,$\{\infty,(x,1),(x,3)\}$, $\{(x,1),(x,2),(x,3)\}$ belong to T for every $x \in Q$ (note: these 4 triples form a 2-fold triple system of order 4) and

Type 2: If $x \neq y$, the six triples $\{(x,1),(y,1),(x \circ y,2)\}$, $\{(y,1),(x,1),(y \circ x,2)\}$,$\{(x,2),(y,2),(x \circ y,3)\}$, $\{(y,2),(x,2),(y \circ x,3)\}$, $\{(x,3),(y,3),(x \circ y,1)\}$, $\{(y,3),(x,3),(y \circ x,2)\}$ belong to T.

Then (S,T) is a 2-fold triple system of order $3n+1$.

The above construction can be generalized to obtain a BOBIBD$(v, 4, 4, 3, 2)$.

THEOREM 21. *If two idempotent* MOLS *of order n exist, then* BOBIBD$(v = 3n+1,4,4,3,2)$ *exists.*

PROOF. Let $L_1 = (X, \circ_1)$ and $L_2 = (X, \circ_2)$ be two idempotent MOLS, and set $S = \{\infty\} \cup (X \times \{1,2,3\})$. Define a collection T of quadruples as follows:

Type 1: Four copies of the quadruple $\{\infty,(x,1),(x,2),(x,3)\}$ belong to T for every $x \in X$, and

Type 2: If $x \neq y$, the quadruples $\{(x,1),(y,1),(x \circ_1 y,2),(x \circ_2 y,2)\}$, $\{(y,1),(x,1),(y \circ_1 x,2),(y \circ_2 x,2)\}$, $\{(x,2),(y,2),(x \circ_1 y,3),(x \circ_2 y,3)\}$, $\{(y,2),(x,2),(y \circ_1 x,3),(y \circ_2 x,3)\}$, $\{(x,3),(y,3),(x \circ_1 y,1),(x \circ_2 y,1)\}$, $\{(y,3),(x,3),(y \circ_1 x,1),(y \circ_2 x,1)\}$ belong to T.

First we want to show that T gives the blocks of a BIBD$(3n+1, 4, 4)$ on S. It is easy to see that $k = 4$, since all the blocks are quadruple. Moreover, there are $4n$ blocks of Type 1 and $6\binom{n}{2}$ blocks of Type 2. Therefore we have the required number, $(3n+1)n$, of blocks, for a BIBD$(3n+1, 4, 4)$. Hence it is enough to show that each pair occurs at least four times. Let (x, i), (y, j) be any pairs. There are three cases to consider.

- Suppose that $x = y$, $i \neq j$, in this case, the four copies of Type 1 quadruples $\{\infty,(x,1),(x,2),(x,3)\}$ together contain (x, i) and (x, j) four times.

- Suppose that $i = j$. Then $x \neq y$. The blocks $\{(x,i),(y,i),(x \circ_1 y,(i+1) \pmod 3),(x \circ_2 y,(i+1) \pmod 3)\}$, $\{(y,i),(x,i),(y \circ_1 x,(i+1) \pmod 3),(y \circ_2 x,(i+1) \pmod 3)\}$ contain (x,i) and (y,i). Now using the orthogonality of latin squares, there are r, s, u, w in X such that $r \circ_1 s = x$, $r \circ_2 s = y$, $u \circ_1 w = y$, and $u \circ_2 w = x$. Therefore the other two blocks containing (x,i) and (y,i) are $\{(r,(i\text{-}1) \pmod 3)$, $(s,(i\text{-}1) \pmod 3))$, (x,i), $(y,i)\}$, and $\{(u,(i\text{-}1) \pmod 3)$, $(w,(i\text{-}1) \pmod 3))$, (y,i), $(x,i)\}$.

- Finally suppose that $x \neq y$ and $i \neq j$. Without loss of generality, assume that $i = 1$ and $j = 2$. since $(L_1, \circ_1)$ and $(L_2, \circ_2)$ are Latin square, $x \circ_1 a = y$ and $x \circ_2 b = y$ for some $a,b \in X$. Since L_1 and L_2 are idempotent MOLS and $x \neq y$, it must be that $a \neq x$ and $b \neq x$. Therefore

  $$\{(x,1),(a,1),(x \circ_1 a = y,2),(x \circ_2 a,2)\}$$
  and
  $$\{(x,1),(b,1),(x \circ_1 b,2),(x \circ_2 b = y,2)\}$$

  are Type 2 quadruples in T which contain $(x,1)$ and $(y,2)$. Other two blocks containing $(x,1)$ and $(y,2)$ are obtained similarly when $(x,1)$ occurs at the second location.

Next we show that the blocks can be ordered to get a BOBIBD. Each Type 1 quadruple appears four times, hence we can order the Type 1 blocks as in BOBIBD(4,4,4,3,2).
For Type 2 quadruple, we keep the order as is in the definition of Type 2 blocks. Here we have four cases to consider. Also, as the pairs $\{\infty, (x, i)\}$ and $\{(x, i), (x, j)\}$ $i \neq j$ occur twice at the required locations from Type I blocks, for all these four cases, we assume that we need to check the number of occurrences of the pairs of type $\{(x, i), (y, i)\}$ where $x \neq y$ or $\{(x, i), (y, j)\}$ where $x \neq y$ and without loss of generality we assume $i < j$.

- Select the elements at the first, second and third locations. A typical block will be $\{(x,i),(y,i),(x \circ_1 y,j),(x \circ_2 y,j)\}$, where j = $(i+1) \pmod 3$. From the $(3n+1)$ construction for triple system the set of subblocks $\{(x,i),(y,i),(x \circ_1 y,j)\}$ forms a BIBD$(3n + 1, 3, 2)$.

- Similarly if we select 1^{st},2^{nd},and 4^{th} location elements of each ordered block, a BIBD$(3n+1,3,2)$ will follow.
- Now suppose we select entries from the 1^{st},3^{rd},and 4^{th} locations to form a design with block size $k_1 = 3$, then we have blocks of the type $\{(x,i),(x\circ_1 y,j),(x\circ_2 y,j)\}$. Recall, we assume that the pairs are of type $\{(x,i),(y,i)\}$ $x \neq y$ or $\{(x,i),(y,j)\}$ $x \neq y$ and without loss of generality $i < j$. Suppose when the MOLSs are superimposed the pair (x,y) and pair (y,x) occurs at $(a,b)^{th}$ and $(c,d)^{th}$ locations. Correspondingly there will be two Type 2 blocks $\{(a,i), (b,i)\ (x,j),(y,j)\}$ and $\{(c,i), (d,i)\ (y,j),(x,j)\}$. Hence Type $\{(x,i),(y,i)\}$ $x \neq y$ pairs occur twice within the smaller blocks under consideration.
  For pairs $\{(x,i),(y,j)\}$ $x \neq y$ with $i < j$, observe that by definition of Latin square and due to the facts that Latin squares are MOLS, there exists y_1 and y_2 such that $y_1 \neq y_2$ and $x\circ_1 y_1 = y$ and $x\circ_2 y_2 = y$. Hence we get two Type 2 blocks:
  $$\{(x,i),(y_1,i),(x\circ_1 y_1,j),(x\circ_2 y_1,j)\},$$
  and
  $$\{(x,i),(y_2,i),(x\circ_1 y_2,j),(x\circ_2 y_2,j)\}.$$
  From these blocks we get the two smaller blocks which contain the pair $\{(x,i),(y,j)\}$ $x \neq y$ with $i < j$ twice.
- Similar arguments hold when entries are selected from the 2^{nd}, 3^{rd}, and 4^{th} locations.

□

COROLLARY 3. *The necessary conditions given in Theorem 17 for the existence of* BOBIBD($v = 3n+1,4,12t+4,3,6t+2$) *are sufficient for non-negative integers t except possibly for* $v = 7, 10,$ *and* 19.

THEOREM 22. *The necessary Conditions given in Theorem 17 are sufficient for the existence of a BOBIBD*$(3n+1,4,8,3,4)$.

PROOF. BIBD$(3n+1,4,2)$ exists and BOBIBD$(4,4,4,3,2)$ exists. So take four copies of each block and arrange as BOBIBD$(4,4,4,3,2)$. □

COROLLARY 4. *The necessary conditions given in Theorem 17 are sufficient for BOBIBD*$(3n+1,4,12t+8,3,6t+4)$.

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