Some Notes on Partitions Inspired by Schur

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Abstract

Let $f_6(n)$ denote the number of partitions of the natural number n into parts co-prime to 6. This function was originally studied by Schur. We derive two explicit formulas for $f_6(n)$, one of them in terms of the partition function $p(n)$. We also derive three recurrences for $f_6(n)$.

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1. Introduction

Let $f_6(n)$ denote number of partitions of a natural number n into parts $\equiv \pm 1 \pmod 6$, that is, parts that are co-prime to 6. In [6], Schur proved that $f_6(n)$ equals the number of partitions of n into parts that differ by at least 3, if consecutive multiples of 3 are omitted. (It is also true that $f_6(n)$ counts the number of partitions of n into distinct parts that are not multiples of 3.) Schur's paper generalized (1) Euler's result that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts; (2) the Rogers-Ramanujan formulas, which equate the number of partitions of n into parts $\equiv \pm 1 \pmod 5$ with the number of partitions of n whose parts differ by at least 2. Bressoud [3] gave a combinatorial proof of Schur's theorem, which is mentioned by Andrews [2]. Indeed, Schur gave a formula (with parameter α) that equates an infinite determinant with an infinite product. The latter can represent the generating function for certain types of partitions, whereas the n-dimensional sub-determinants of the infinite determinant may be computed by means of a second-order recurrence relation.

In this note, we obtain two alternate, explicit formulas for $f_6(n)$. One of these new formulas expresses $f_6(n)$ in terms of the well-known partition functions $q(n)$ and $q_0(n)$. The second formula expresses $f_6(n)$ in terms of $p(n)$, the partition function. In addition, we present three recurrences for $f_6(n)$. We also discuss a few partition functions whose generating functions were explicitly mentioned by Schur, and we apply Schur's formula to obtain generating functions for certain kinds of overpartitions.

2. Preliminaries

Let m, n be natural numbers, with $m \geq 2$, m square-free.

$\omega(j) = j(3j-1)/2$ is the j^{th} pentagonal number, where $j \in Z$

$p(n)$ is the partition function

$q(n)$ is the number of partitions of n into odd parts (or into distinct parts)

$q^*(n)$ is the number of partitions of n into odd parts exceeding 1

$q_0(n)$ is the number of partitions of n into distinct odd parts (or the number of self-conjugate partitions of n)

$q_0^*(n)$ is the number of partitions of n into distinct odd parts exceeding 1

$f_m(n)$ is the number of partitions of n that are co-prime to m

$\Phi_n(z)$ is the n^{th} cyclotomic polynomial

$p_H(n)$ is the number of partitions of n into parts from H, where $H \subset N$

Identities

Let $x, z \in C$, $|x| < 1$, $z \neq 0$.

$$\prod_{n\geq 1}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)} \tag{1}$$

$$\prod_{n\geq 1}(1-x^n)^{-1}=\sum_{n\geq 0}p(n)x^n \tag{2}$$

$$\prod_{n\geq 1}(1+x^n)=\prod_{n\geq 1}(1-x^{2n-1})^{-1}=\sum_{n\geq 0}q(n)x^n \tag{3}$$

$$\prod_{n\geq 1}(1+x^{2n-1})=\sum_{n\geq 0}q_0(n)x^n \tag{4}$$

$$\prod_{n\geq 1}(1+x^n)^{-1}=\sum_{n\geq 0}(-1)^n q_0(n)x^n \tag{5}$$

$$\prod_{n\geq 1}(1+x^{2n+1})=\sum_{n\geq 0}q_0^*(n)x^n \tag{6}$$

$$\prod_{n\geq 1}(1-x^{2n+1})^{-1}=\sum_{n\geq 0}q^*(n)x^n \tag{7}$$

$$\prod_{n\geq 1}(1-x^n)(1-x^n z)(1-x^{n-1}z^{-1})(1-x^{2n-1}z^2)(1-x^{2n-1}z^{-2}) \tag{8}$$

$$=\sum_{n=-\infty}^{\infty}x^{\omega(-n)}(z^{3n}-z^{-3n-1})$$

$$\sum_{n\geq 0}p_H(n)x^n=\prod_{n\in H}(1-x^n)^{-1} \tag{9}$$

$$\prod_{n\geq 1}(1-x^{6n-1})^{-1}(1-x^{6n-5})^{-1}=\sum_{n\geq 0}f_6(n)x^n \tag{10}$$

$$\prod_{n\geq 1}(1-x^{k(2n-1)-l})(1-x^{k(2n-1)+l})(1-x^{2kn})=\sum_{n=-\infty}^{\infty}(-1)^n x^{kn^2+ln} \tag{11}$$

Theorem A If $|t| < 1$ and α is arbitrary, then

$$\begin{vmatrix} 1+\alpha t & t^3-t^4 & 0 & 0 & \cdots \\ -1 & 1+\alpha t^2 & t^4-t^6 & 0 & \cdots \\ 0 & -1 & 1+\alpha t^3 & t^5-t^8 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \prod_{j=1}^{\infty}(1+\alpha t^j + t^{2j+1})$$

Theorem B Let the square-free integer m be the product of r distinct prime factors, with $r \geq 1$. Then

$$\sum_{n\geq 0} f_m(n)z^n = \prod_{n\geq 1}(\Phi_m(z^n))^{(-1)^{r-1}}$$

Remarks: Identities (1) through (4) are well-known, and (5) follows from (3) and (4). Schur specifically mentions the infinite product that appears in (6) (see [6], p. 492), but not the partition function for which it is a generating function. Identity (7) follows easily from (3). Identity (8) is the quintuple product identity. Identity (9) is Theorem 1.1 from [1], and (10) follows from (9). Identity (11) is adapted from (19.9.1) on p. 283 of [4]. Theorem A is Satz 4 in [6]. Theorem B is Theorem 1 in [5]. If $f(n)$ is any partition function, we define $f(0) = 1$, and $f(\alpha) = 0$ if α is not a non-negative integer.

3. The Main Results

Our first result is an explicit formula for $f_6(n)$, the function that is the main object of study of Schur's paper, in terms of $q(n)$ and $q_0(n)$. (Note that parts $\equiv \pm 1 \pmod 6$ are precisely the parts that are co-prime to 6.)

Theorem 1

$$f_6(n) = \sum_{k=0}^{[n/3]}(-1)^k q(n-3k)q_0(k)$$

Proof: If we apply Theorem B with $m = 6$ (and hence $r = 2$), we obtain

$$\sum_{n\geq 0} f_6(n)z^n = \prod_{n\geq 1}(\Phi_6(z^n))^{-1} = \prod_{n\geq 1}(1-z^n+z^{2n})^{-1} =$$

$$\prod_{n\geq 1}\frac{1+z^n}{1+z^{3n}} = (\prod_{n\geq 1}(1+z^n))(\prod_{n\geq 1}(1+z^{3n})^{-1}) =$$

$$(\sum_{n\geq 0} q(n)z^n)(\sum_{n\geq 0}(-1)^n q_0(\frac{n}{3})z^n)$$

The conclusion now follows by invoking (3) and (5) and matching coefficients of like powers of z. ■

Next, we express $f_6(n)$ in terms of $p(n)$.

Theorem 2

$$f_6(n) = \sum_{j=-\infty}^{\infty} p(\frac{n-\omega(j)}{3})$$

Proof: In (8), let $z = e^{\frac{2\pi i}{3}}$, so that $z + z^{-1} = 2\cos\frac{2\pi}{3} = -1 = 2\cos\frac{4\pi}{3} = z^2 + z^{-2}$. This yields

$$\prod_{n\geq 1}(1-x^n)(1+x^n+x^{2n})(1+x^{2n-1}+x^{4n-2}) = \sum_{n=-\infty}^{\infty} x^{\omega(-n)}(\frac{e^{2\pi i n} - e^{-\frac{2\pi i}{3}}e^{2\pi i n}}{1-e^{-\frac{2\pi i}{3}}})$$

which simplifies to

$$\prod_{n\geq 1}(1+x^{2n-1}+x^{4n-2})\prod_{n\geq 1}(1-x^{3n}) = \sum_{n=-\infty}^{\infty} x^{\omega(n)} \qquad (12)$$

Now

$$\sum_{n\geq 0} f_6(n)x^n = \prod_{n\geq 1}(\Phi_6(x^n))^{-1} = \prod_{n\geq 1}(1-x^n+x^{2n})^{-1} =$$

$$\prod_{n\geq 1}\frac{1+x^n+x^{2n}}{1+x^{2n}+x^{4n}} = \prod_{n\geq 1}(1+x^{2n-1}+x^{4n-2})$$

so (12) yields

$$(\sum_{n\geq 0} f_6(n)x^n)(\prod_{n\geq 1}(1-x^{3n})) = \sum_{n=-\infty}^{\infty} x^{\omega(n)} \tag{13}$$

and

$$\sum_{n\geq 0} f_6(n)x^n = (\prod_{n\geq 1}(1-x^{3n})^{-1})(\sum_{n=-\infty}^{\infty} x^{\omega(n)}) \tag{14}$$

Invoking (2), we have

$$\sum_{n\geq 0} f_6(n)x^n = (\sum_{n\geq 0} p(\frac{n}{3})x^n)(\sum_{n=-\infty}^{\infty} x^{\omega(n)})$$

The conclusion now follows by matching coefficients of like powers of x. ■

Remarks: Invoking (1) and matching coefficients of like powers of x in (13), we obtain the following recurrence relation for $f_6(n)$:

$$f_6(n) + \sum_{k\geq 1}(-1)^k(f_6(n-3\omega(k)) + f_6(n-3\omega(-k))) = \begin{cases} 1 \text{ if } n = \omega(\pm r) \\ 0 \text{ otherwise} \end{cases} \tag{15}$$

Note that $f_6(n)$ also counts the number of partitions of n into distinct parts such that (i) no part is a multiple of 4; (ii) $2k+1$ and $4k+2$ do not appear as parts in the same partition.

From Theorem 2, we easily deduce the following lower bound for $f_6(n)$:

Corollary 1 $f_6(n) \geq p([\frac{n}{3}])$

Proof: Theorem 2 implies that

$$f_6(3k) = p(k) + \sum_{j\neq 0} p(\frac{n-\omega(j)}{3}) \quad ;$$

$$f_6(3k+1) = p(k) + \sum_{j\neq 0,1} p(\frac{n-\omega(j)}{3}) \quad ;$$

$$f_6(3k+2) = p(k) + \sum_{j\neq 0,-1} p(\frac{n-\omega(j)}{3})$$

from which the conclusion follows. ■

A different application of (8) produces a recurrence for $f_6(n)$ that is similar to, but not identical with (14).

Theorem 3

$$\sum_{k=-\infty}^{\infty} (f_6(n-\frac{9k^2-3k}{2}) - f_6(n-\frac{9k^2+9k+2}{2})) = \begin{cases} (-1)^r & \text{if } n=\omega(\pm r) \\ 0 & \text{otherwise} \end{cases}$$

Proof: In (8), replace x by x^3 and z by x. This yields

$$\prod_{n\geq 1}(1-^{3n})(1-x^{3n+1})(1-x^{3n-4})(1-x^{6n-1})(1-x^{6n-5}) = \sum_{n=-\infty}^{\infty} x^{3\omega(-n)}(x^{3n}-x^{-3n-1})$$

which we rewrite as

$$(\frac{1-x^{-1}}{1-x})\prod_{n\geq 1}(1-x^{3n})(1-x^{3n-2})(1-x^{3n-1})(1-x^{6n-1})(1-x^{6n-5}) = \sum_{n=-\infty}^{\infty}(x^{\frac{9n^2+9n}{2}}-x^{\frac{9n^2-3n-2}{2}})$$

This implies

$$\prod_{n\geq 1}(1-x^n)\prod_{n\geq 1}(1-x^{6n-1})(1-x^{6n-5}) = -x(\sum_{n=-\infty}^{\infty}(x^{\frac{9n^2+9n}{2}}-x^{\frac{9n^2-3n-2}{2}}))$$

hence

$$\prod_{n\geq 1}(1-x^n) = \prod_{n\geq 1}(1-x^{6n-1})^{-1}(1-x^{6n-5})^{-1}(\sum_{n=-\infty}^{\infty}(x^{\frac{9n^2-3n}{2}}-x^{\frac{9n^2+9n+2}{2}})$$

Invoking (1) and (10), we have

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{\omega(n)} = (\sum_{n\geq 0} f_6(n)x^n)(\sum_{n=-\infty}^{\infty}(x^{\frac{9n^2-3n}{2}}-x^{\frac{9n^2+9n+2}{2}})$$

The conclusion now follows by matching coefficients of like powers of x. ■

A third recurrence for $f_6(n)$ is obtained by using identity (11), as is shown below.

Theorem 4

$$f_6(n)+\sum_{k\geq 1}(-1)^k\{f_6(n-(3k^2-2k))+f_6(n-(3k^2+2k))\} = \begin{cases} (-1)^r & \text{if } n = 6\omega(\pm r) \\ 0 & \text{otherwise} \end{cases}$$

Proof: In (11), let $k = 3$ and $l = 2$. This yields

$$\prod_{n\geq 1}(1-x^{6n-5})(1-x^{6n-1})(1-x^{6n}) = \sum_{n=\infty}^{\infty} x^{3n^2+2n}$$

Therefore

$$\prod_{n\geq 1}(1-x^{6n}) = (\prod_{n\geq 1}(1-x^{6n-5})^{-1}(1-x^{6n-1})^{-1}) \sum_{n=\infty}^{\infty} x^{3n^2+2n}$$

Invoking (10), we have

$$\prod_{n\geq 1}(1-x^{6n}) = (\sum_{n\geq 0} f_6(n)x^n) \sum_{n=\infty}^{\infty} x^{3n^2+2n}$$

The conclusion now follows by invoking (1) and matching coefficients of like powers of x. ■

Table 1 below enumerates $f_6(n)$ in the range $1 \leq n \leq 30$.

n	$f_6(n)$	n	$f_6(n)$
1	1	16	10
2	1	17	12
3	1	18	14
4	1	19	16
5	2	20	18
6	2	21	20
7	3	22	23
8	3	23	26
9	3	24	30
10	4	25	34
11	5	26	38
12	6	27	42
13	7	28	47
14	8	29	53
15	9	30	60

Next, we express $q_0^*(n)$ in terms of $q_0(n)$.

Theorem 5

$$q_0^*(n) = \sum_{k=0}^{n} (-1)^k q_0(n-k)$$

Proof: (6) implies

$$\sum_{n\geq 0} q_0^*(n) z^n = \prod_{n\geq 2} (1 + z^{2n-1}) = (1+z)^{-1} \prod_{n\geq 1} (1 + z^{2n-1})$$

$$= (\sum_{n\geq 0} (-1)^n z^n)(\sum_{n\geq 0} q_0(n) z^n) = \sum_{n\geq 0} (\sum_{k=0}^{n} (-1)^k q_0(n-k)) z^n$$

The conclusion now follows by matching coefficients of like powers of z. ■

We now express $q^*(n)$ in terms of $q(n)$, by means of Theorem 6 below.

Theorem 6 If $n \geq 1$, then $q^*(n) = q(n) - q(n-1)$.

Proof: (7) implies

$$\sum_{n\geq 0} q^*(n)z^n = \prod_{n\geq 1}(1-z^{2n+1})^{-1} = (1-z)\prod_{n\geq 1}(1-z^{2n-1})^{-1}$$

$$= (1-z)\sum_{n\geq 0} q(n)z^n = \sum_{n\geq 0} q(n)z^n - \sum_{n\geq 0} q(n)z^{n+1} =$$

$$1 + \sum_{n\geq 1} q(n)z^n - \sum_{n\geq 1} q(n-1)z^n = 1 + \sum_{n\geq 1}(q(n) - q(n-1))z^n$$

The conclusion now follows by matching coefficients of like powers of z. ■

Remarks: If we define $q(-1) = 0$, then Theorem 6 holds for all $n \geq 0$.

Finally, we interpret some partition functions that correspond to generating functions given by Schur. (See (5), p. 489 of [5].) If we let $\alpha = 1$ in the right member of Theorem A, we obtain

$$\prod_{n\geq 1}(1 + t^n + t^{2n+1}) = \sum_{n\geq 0} f(n)t^n$$

Here $f(n)$ is the number of overpartitions of n into distinct parts such that (i) odd parts exceeding 1 may be overlined; (ii) if $4k+3$ and $2k+1$ appear as parts in the same partition and $4k+3$ is overlined, then so is $2k+1$.

Similarly, if we let $\alpha = t$ in the right member of Theorem A, we obtain

$$\prod_{n\geq 1}(1 + t^{n+1} + t^{2n+1}) = \sum_{n\geq 0} g(n)t^n$$

Here $g(n)$ is the number of overpartitions of n into distinct parts such that (i) odd parts exceeding 1 may be overlined; (ii) if $4k+1$ and $2k+1$ appear as parts in the same partition and $4k+1$ is overlined, then so is $2k+1$.

Next, let $\alpha = x$, $t = x^2$ in the right member of Theorem A to obtain

$$\prod_{n\geq 1}(1 + x^{2n+1} + x^{4n+2}) = \sum_{n\geq 0} b(n)x^n$$

Here $b(n)$ is the number of partitions of n into distinct parts exceeding 2 such that (i) no part is a multiple of 4; (ii) $2k+1$ and $4k+2$ do not appear as parts in the same partition. Note that the generating function for $b(n)$ differs by one factor from the generating function for $f_6(n)$. Theorem 7 below gives an explicit formula for $b(n)$.

Theorem 7

$$b(n) = \sum_{j=0}^{[\frac{n}{3}]} (-1)^j (q(n-3j) - q(n-1-3j)) q_0^*(j)$$

Proof:

$$\sum_{n\geq 0} b(n)x^n = \prod_{n\geq 1}(1+x^{2n+1}+x^{4n+2}) = \prod_{n\geq 1}\frac{1-x^{6n+3}}{1-x^{2n+1}} = \prod_{n\geq 1}(1-x^{6n+3})\prod_{n\geq 1}(1-x^{2n+1})^{-1}$$

$$= (\sum_{n\geq 0} q_0^*(n)x^{3n})(\sum_{n\geq 0}(q(n)-q(n-1))x^n) = (\sum_{n\geq 0} q_0^*(\frac{n}{3})x^n)(\sum_{n\geq 0}(q(n)-q(n-1))x^n)$$

$$= \sum_{n\geq 0}(\sum_{k=0}^{n} q_0^*(\frac{n-k}{3})(q(k)-q(k-1))x^n = \sum_{n\geq 0}(\sum_{j=0}^{[\frac{n}{3}]}(q(n-3j)-q(n-1-3j))q_0^*(j))x^n$$

if we let $n-k=3j$. The conclusion now follows by matching coefficients of like powers of x. ■

4. References

1. G. E. Andrews *The Theory of Partitions* Cambridge University Press (1984)

2. G. E. Andrews *q-Series: their development and application in analysis, combinatorics, physics, and computer algebra* (1986) Amer. Math. Soc.

3. D. M. Bressoud A combinatorial proof of Schur's 1926 partition theorem *Proc. Amer. Math. Soc.* **79** (1980) 338-340

4. G. H. Hardy & E. M. Wright *An Introduction to the Theory of Numbers* (1960) 4th ed. Oxford University Press

5. N. Robbins On partitions and cyclotomic polynomials *Integers* v. 0 (2000) A6

6. I. Schur Zur additiven Zahlentheorie *Sitzungsbericht der Preussichen Akademie der Wissenschaft (1926) Physikalisch-Mathematische Klasse*, 488-495