Ramsey Multiplicities of Graphs with Five Vertices[†]

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Abstract. The Ramsey multiplicity $M(G)$ of a graph G is defined to be the smallest number of monochromatic copies of G in any two-coloring of edges of $K_{R(G)}$, where $R(G)$ is the smallest integer n such that every graph on n vertices either contains G or its complement contains G. With the help of computer algorithms, we obtain the exact values of Ramsey multiplicities for most of isolate-free graphs on five vertices, and establish upper bounds for a few others.

1 Introduction

In this note, we only consider graphs without multiple edges or loops. For a graph G, the *complement* of G is denoted by G^c. The set of all non-isomorphic graphs with n vertices is denoted by $\mathcal{G}_n$. Any two-coloring of the edges of K_n containing k monochromatic copies of G is called a (G, n, k)-coloring. For graphs G and H, the number of copies of H in G is denoted by $g(G, H)$. Please refer to [1] for more notation of graph theory.

For a graph G, the *Ramsey number* $R(G)$ is the smallest integer n such that every graph on n vertices either contains G or its complement contains G. The *Ramsey multiplicity* $M(G)$ of a graph G is defined to

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be the smallest number of monochromatic copies of G in any two-coloring of the edges of $K_{R(G)}$. Let $\mathcal{M}(G)$ denote the set of graphs $\{F \mid F \in \mathcal{G}_{R(G)}, \text{ and } g(F,G) + g(F^c,G) = M(G)\}$.

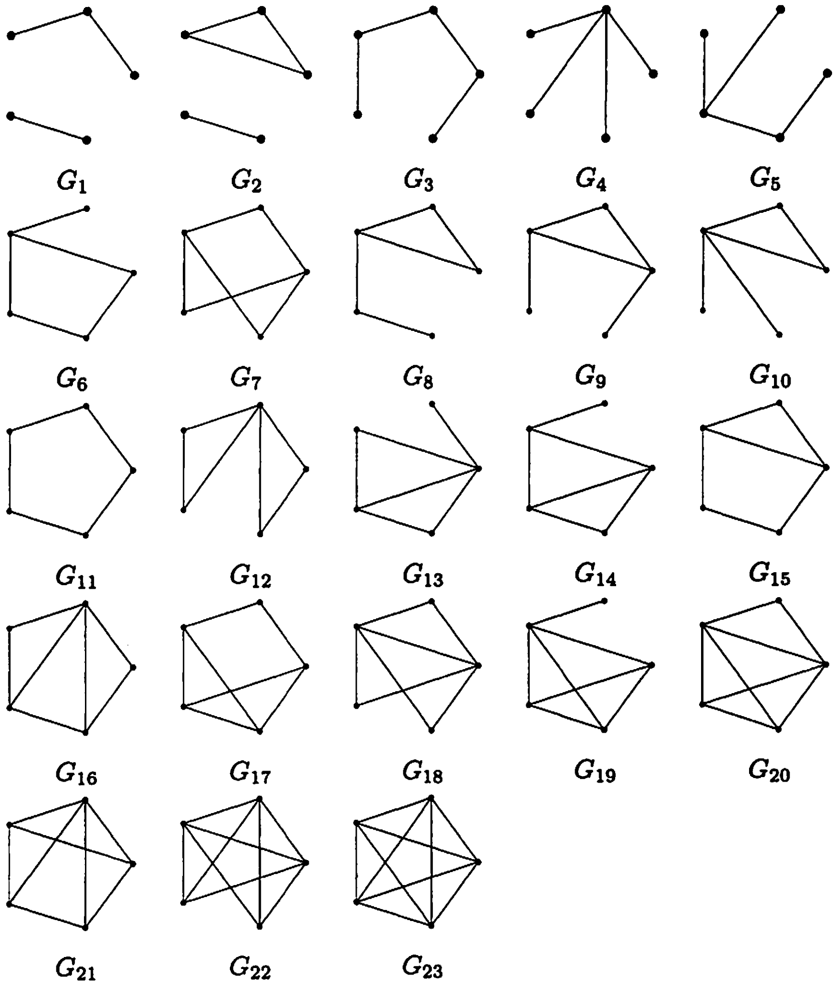


Figure 1: All 23 isolate-free graphs on 5 vertices

In 1974, Harary and Prins [4] initiated research on Ramsey multiplicity, and determined $M(G)$ for star graphs and all graphs G of order four or less, except for K_4 and $K_4 - e$. Later, Schwenk (cited in [3]) obtained the value of $M(K_4 - e)$, Piwakowski and Radziszowski showed that $M(K_4) = 9$

[7].

In total, there are 23 isolate-free graphs of order five, which are labeled $G_1, G_2, \cdots, G_{23}$ as in [2], see Figure 1. The exact values of Ramsey numbers are shown in Table 1 for all isolate-free graphs G of order five except the graph K_5 ($43 \le R(K_5) \le 49$) [5]. The graph G_4 in Figure 1 is a star, the Ramsey multiplicities of which is determined by Harary and Prins [4]. In this note, we obtain some Ramsey multiplicities or upper bounds for isolate-free graphs of order five except G_4 and G_{23}.

Table 1: The exact values of $R(G)$ for isolate-free graphs G of order five

G	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8
$R(G)$	6	7	6	7	6	6	10	9
G	G_9	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}
$R(G)$	9	9	9	9	10	10	9	10
G	G_{17}	G_{18}	G_{19}	G_{20}	G_{21}	G_{22}	G_{23}	
$R(G)$	10	14	18	18	15	22	43-49	

2 Computation of Ramsey multiplicity

By the definition of Ramsey multiplicity, for a given graph G, we have

$$M(G) = \min\{g(F, G) + g(F^c, G) | F \in \mathcal{G}_{R(G)}\} \tag{1}$$

The computation of Ramsey multiplicity is based on the formula (1). First we generate all graphs with $R(G)$ vertices, where $R(G)$ is the Ramsey number of graph G; then we compute the number of copies of G in F (i.e., $g(F, G)$) and the number of copies of G in F^c (i.e., $g(F^c, G)$) for each graph with $R(G)$ vertices; finally the minimal value of $g(F, G) + g(F^c, G)$ is calculated. The program *nauty* is used to generate all graphs on $R(G)$ vertices, by which the number of vertices can reach 11 (even 12) in a reasonable time [6]. In this paper, we use *nauty* only to generate all graphs with at most 10 vertices. A program is developed to compute $g(F, G)$ and $g(F^c, G)$), which is available from the first author by email. In general, we obtain the exact values of $M(G_i)$ for $1 \le i \le 17, i \ne 4$ by the above programs, which are presented in Table 2, where G_M is a graph such that $g(G_M, G) + g(G_M^c, G) = M(G)$, i.e., a graph that belongs to $\mathcal{M}(G)$. The statistics for $\mathcal{M}(G)$ is also given in Table 2. For each graph G_i ($1 \le i \le 17$), a corresponding graph $F_j \in \mathcal{M}(G_i)$ ($1 \le j \le 14$) is listed in Figure 2.

We test the programs on some graphs with four vertices and obtain the same known exact values. For example, it was proved that $M(G_4) = 1$

in [4]. By the above programs, we find the graph F_4, which shows that $M(G_4) \le 1$. So we also obtain $M(G_4) = 1$.

Example 1 *Computation of* $M(G_{11})$.

Let $G = G_{11}$, then $R(G) = 9$. The set of non-isomorphic graphs $\mathcal{G}_9$ was generated by program *nauty*. For each graph $F \in \mathcal{G}_9$, the number $g(F, G) + g(F^c, G)$ is computed. Then we have $M(G) = \min\{g(F, G) + g(F^c, G) | F \in \mathcal{G}_{R(G)}\} = 12$. The graph F_8 shown in Figure 2 contains exactly no copies of G_{11}, and F_8^c contains exactly 12 copies of G_{11}.

Table 2: The Ramsey multiplicity for isolate-free graphs G of order five

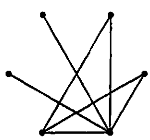
G	G_1	G_2	G_3	G_4	G_5	G_6	G_7	G_8	G_9
G_M	F_1	F_2	F_3	F_4	F_5	F_5	F_6	F_7	F_7
$g(G_M, G)$	8	0	6	0	6	2	0	0	0
$g(G_M^c, G)$	13	10	12	1	10	2	9	60	60
$\lvert\mathcal{M}(G)\rvert$	6	2	4	8	2	2	2	2	2
$M(G)$	21	10	18	1[4]	16	4	9	60	60

G	G_{10}	G_{11}	G_{12}	G_{13}	G_{14}	G_{15}	G_{16}	G_{17}
G_M	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}	F_{14}
$g(G_M, G)$	0	0	0	20	37	16	8	6
$g(G_M^c, G)$	30	12	2	24	38	16	0	0
$\lvert\mathcal{M}(G)\rvert$	2	4	3	2	2	3	6	2
$M(G)$	30	12	2	44	75	32	8	6

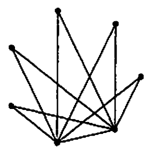
G	G_{18}	G_{19}	G_{20}	G_{21}	G_{22}
G_M	F_{15}	F_{16}	F_{16}	F_{17}	F_{18}
$g(G_M, G)$	10	44	18	1	50
$g(G_M^c, G)$	18	156	78	5	38
$M(G)$	≤ 28	≤ 200	≤ 96	≤ 6	≤ 88

3 Upper bound for Ramsey multiplicity

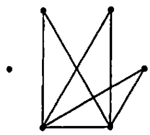
It is difficult to obtain the exact values of Ramsey multiplicities for graphs with larger Ramsey numbers by the above method. The upper bounds for Ramsey multiplicities for the graphs G_i $(18 \le i \le 22)$ listed in Figure 1 are obtained by a simulated annealing algorithm. The results are shown in Table 2 and the corresponding graphs G_M are listed in Figure 2.



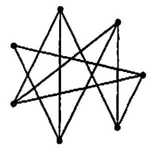
F_1



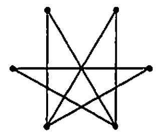
F_2



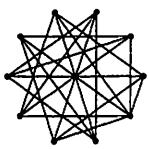
F_3



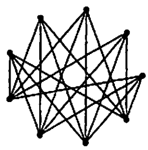
F_4



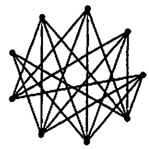
F_5



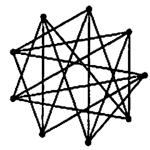
F_6



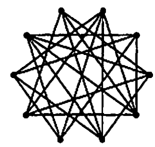
F_7



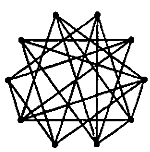
F_8



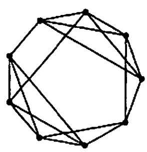
F_9



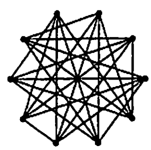
F_{10}



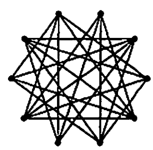
F_{11}



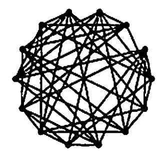
F_{12}



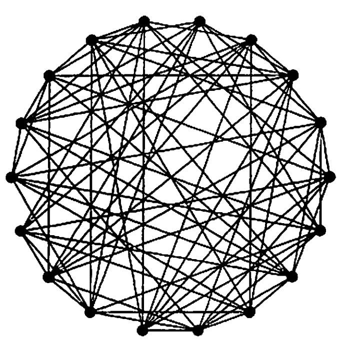
F_{13}



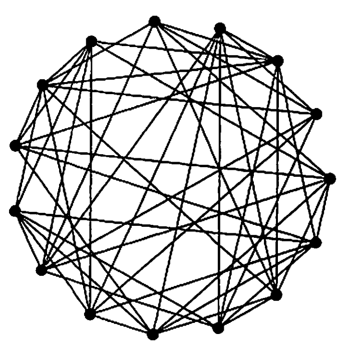
F_{14}



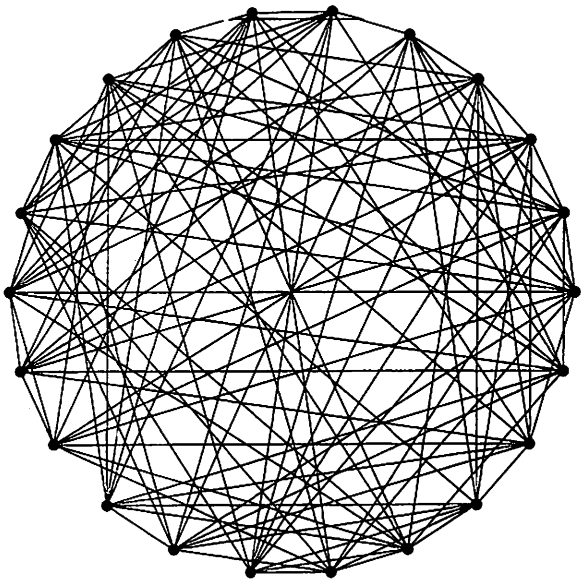
F_{15}



F_{16}



F_{17}



F_{18}

Figure 2: $(H, R(H), M(H))$-colorings

Acknowledgements

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