A technique for constructing magic labelings of 2−regular graphs

Dan McQuillan
Department of Mathematics, Norwich University
Northfield Vermont 05663, USA
E-mail address: dmcquill@norwich.edu

Abstract

A new technique is given for constructing a vertex-magic total labeling, and hence an edge-magic total labeling, for certain finite simple 2−regular graphs. Let C_r denote the cycle of length r. Let n be an odd positive integer with $n = 2m + 1$. Let k_i denote an integer such that $k_i \geq 3$, for $i = 1, 2, \dots l$, and write nC_{k_i} to mean the disjoint union of n copies of C_{k_i}. Let G be the disjoint union $G \cong C_{k_1} \cup \cdots \cup C_{k_l}$. Let $I = \{1, 2, \dots l\}$ and let J be any subset of I. Finally let $G_J = \left(\bigcup_{i \in J} nC_{k_i}\right) \cup \left(\bigcup_{i \in I-J} C_{nk_i}\right)$, where all unions are disjoint unions. It is shown that if G has a vertex-magic total labeling (VMTL) with a magic constant of h then G_J has VMTLs with magic constants $6m(k_1 + k_2 + \cdots + k_l) + h$ and $nh - 3m$. In particular, if G has a strong VMTL then G_J also has a strong VMTL.

Keywords: Graph; Labeling; Vertex-magic; Strong vertex-magic ; Edge-magic.

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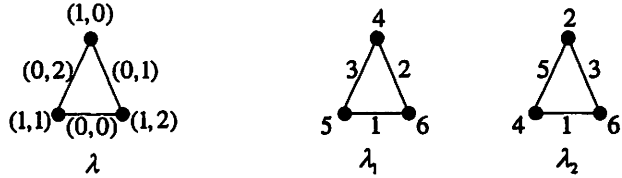
Throughout this paper, $G = (V, E)$ will denote a 2−regular finite simple graph with vertex-set V and edge-set E. Let Γ be an Abelian group. Given any map $\lambda : V \cup E \to \Gamma$ the λ−weight $wt_\lambda(v)$ of a vertex v is the sum $\lambda(v) + \sum \lambda(e)$, where the sum ranges over all edges e incident with v. We say that λ is *magic* if $wt_\lambda(v) = h$, where h is a constant (called the *magic constant*) that does not depend on the choice of vertex v. If λ is injective, we say that it is a Γ−labeling of G. In this paper, Γ will either be $\mathbb{Z}$ or $\mathbb{Z} \times \mathbb{Z}$. Throughout the paper, $p_i : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ will denote the cononical projection defined by $p_1(x, y) = x$ and $p_2(x, y) = y$. A magic $\mathbb{Z}$−labeling with range $\{1, 2, \dots, |V| + |E|\}$ is called a *vertex-magic total labeling*, or VMTL (see [2] or [5]).

Let $\gamma : V \cup E \to \{1, 2, \dots, |V| + |E|\}$ be a bijection. The γ−weight of an edge e is the sum $\gamma(e) + \gamma(v) + \gamma(v')$, where v and v' are the ends of e. We say that γ is an *edge-magic total labeling* if the γ−weight of each edge is a constant. It is easy to see how to convert a VMTL of a 2−regular graph into an edge-magic



A magic $\mathbb{Z}\times\mathbb{Z}-$labeling and two corresponding VMTLs

Figure 1

total labeling of the same graph (see also [3]) and so we will henceforth no longer directly discuss edge-magic total labelings.

If t is a positive integer, then A_t will denote the set $\{0, 1, \ldots, t-1\}$. Furthermore, n will denote an odd integer such that $n \geq 3$ and we write $n = 2m + 1$. The proof of the following is an easy exercise.

Proposition 1 *Let t_1 and t_2 be positive integers. Let λ be a magic $\mathbb{Z}\times\mathbb{Z}-$labeling of G with range $A_{t_1} \times A_{t_2}$ and magic constant (h_1, h_2). Let x be a vertex or edge of G. Then:*

1. *The map $\lambda_1 : V \cup E \to \mathbb{Z}$ defined by $\lambda_1(x) = t_2 p_1(x) + p_2(x) + 1$ is a VMTL of G with magic constant $t_2 h_1 + h_2 + 3$.*
2. *The map $\lambda_2 : V \cup E \to \mathbb{Z}$ defined by $\lambda_2(x) = t_1 p_2(x) + p_1(x) + 1$ is a VMTL of G with magic constant $t_1 h_2 + h_1 + 3$.*

An example of a magic $\mathbb{Z} \times \mathbb{Z}-$labeling λ of C_3 with range $A_2 \times A_3$ and corresponding VMTLs λ_1 and λ_2 is shown in Fig. 1.

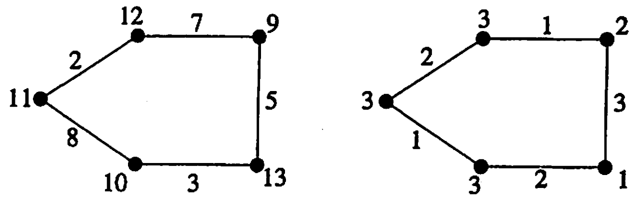
Let $[a_{i,j}]$ denote the $3 \times n$ matrix:

$$[a_{i,j}] = \begin{bmatrix} 0 & 1 & \cdots & m & m+1 & m+2 & \cdots & 2m \\ 2m & 2m-2 & \cdots & 0 & 2m-1 & 2m-3 & \cdots & 1 \\ m & m+1 & \cdots & 2m & 0 & 1 & \cdots & m-1 \end{bmatrix}$$

It has the property that each row consists of a permutation of $0, 1, \ldots, n-1$ and each column sum is $3m$. This example is due to Kotzig [1].

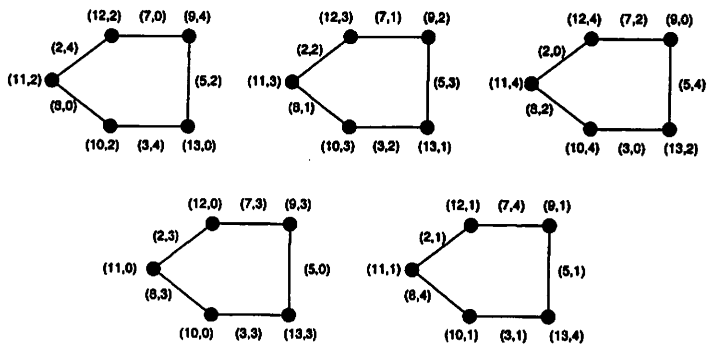
We will write nG to denote the disjoint union of n copies of G.

Lemma 2 *Let f be a magic $\mathbb{Z}-$labeling of G with range B and magic constant h. Then there is a magic $\mathbb{Z} \times \mathbb{Z}-$labeling of nG with range $B \times A_n$ and magic constant $(h, 3m)$.*



A magic $\mathbb{Z}$ – labeling f and a coloring η for C_5

Figure 2



A magic $\mathbb{Z}\times\mathbb{Z}$ – labeling of $5C_5$

Figure 3

The following is a modification of the proof of Wallis' theorem in [4].

Proof. By Vizing's theorem G can be properly edge-colored with the 3 colors 1, 2 and 3. Let χ be such a coloring. At each vertex v, there will be exactly one of the three colors not represented. Define $\eta(v)$ to be that color, and let η agree with χ on the edges.

Now let $G_1 \cup G_2 \cup \ldots \cup G_n$ denote a disjoint union where for each j, G is isomorphic to G_j. Let $x \in V \cup E$ and let x_j be the corresponding object in G_j. One checks that $\lambda(x_j) = (f(x), a_{\eta(x),j})$ is the required labeling. ∎

An example of a magic $\mathbb{Z}$–labeling of a 5–cycle and a 3–coloring η for $n = 5$ is shown in Fig. 2. The corresponding magic $\mathbb{Z} \times \mathbb{Z}$–labeling for $5C_5$ is shown in Fig. 3.

Lemma 3 *Let f be a magic $\mathbb{Z}$–labeling of the k–cycle C_k with range B and magic constant h. Then there is a magic $\mathbb{Z} \times \mathbb{Z}$–labeling f^* of C_{nk} with range $B \times A_n$ and magic constant $(h, 3m)$.*

This lemma is proven in detail for the special case where $B = A_{2k}$ in [3]. That proof does not use the assumption that $B = A_{2k}$ and so it works just as well here. We provide only a summarized version of that proof, as well as an example with $k = 4$ and $n = 5$ shown in Fig. 4 and Fig. 5.

Notation 4 *[3] Let $v_0, v_1, \cdots, v_{2m}$ be the vertices of the cycle C_n of length n, where for convenience we set $v_j = v_i$ whenever $j \equiv i \bmod (2m+1)$. In particular, $v_{2m+1} = v_0$. Let the edges be denoted by (v_i, v_{i+1}). Then $\alpha : V \cup E \to \mathbb{Z}$ will denote the following map*

1. $\alpha(v_j) = j$ *for* $0 \leq i \leq 2m$,
2. $\alpha(v_{2i-1}, v_{2i}) = m - i$, *for* $0 \leq i \leq m$,
3. $\alpha(v_{2i}, v_{2i+1}) = 2m - i$, *for* $0 \leq i \leq m$.

Proof. (adapted from [3]): Let the vertex-set of C_k be denoted by $\{u_0, \cdots, u_{k-1}\}$, and for convenience we set $u_j = u_i$ whenever $j \equiv i \bmod k$. In particular, $u_k = u_0$. Thus the edge-set of C_k will be denoted by $\{(u_i, u_{i+1})\}$ for $0 \leq i \leq k-1$. Finally, let $w_0, w_1, \ldots, w_{nk-1}$ be the vertices of C_{nk} with edges (w_i, w_{i+1}), and for convenience we set $w_{nk+j} = w_j$, for $j = 0, 1, \ldots, n-1$. Let x be an integer, $0 \leq x \leq nk - 1$. By the division algorithm, x can be written uniquely in the form $x = qn + r$ where $0 \leq q \leq k-1$ and $0 \leq r \leq n - 1 = 2m$. We define f^* componentwise, as follows. Let $p_1(f^*(w_x)) = f(u_q)$, and for the edges, set

$$p_1\left(f^*\left(w_x, w_{x+1}\right)\right) = \begin{cases} f\left(u_q, u_{q+1}\right) & \text{if } r \text{ is even} \\ f\left(u_{q-1}, u_q\right) & \text{if } r \text{ is odd} \end{cases}$$

Finally, for the second component:
$p_2(f^*(w_x)) = \alpha(v_r)$ and $p_2(f^*(w_x, w_{x+1})) = \alpha(v_r, v_{r+1})$ ■

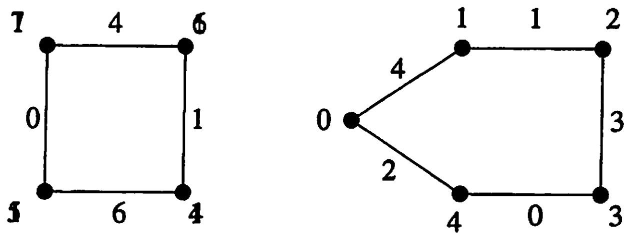
A magic $\mathbb{Z}$−labeling of C_4, as well as α for $n = 5$ are shown in Fig. 4. The corresponding magic $\mathbb{Z} \times \mathbb{Z}$−labeling for C_{20} is shown in Fig. 5.

Notation 5 *Let G be the disjoint union of l cycles $G \cong C_{k_1} \cup \cdots \cup C_{k_l}$, and let $I = \{1, 2, \ldots, l\}$. Let J be a subset of I, and let*

$$G_J = \left(\bigcup_{i \in J} nC_{k_i}\right) \cup \left(\bigcup_{i \in I-J} C_{nk_i}\right)$$

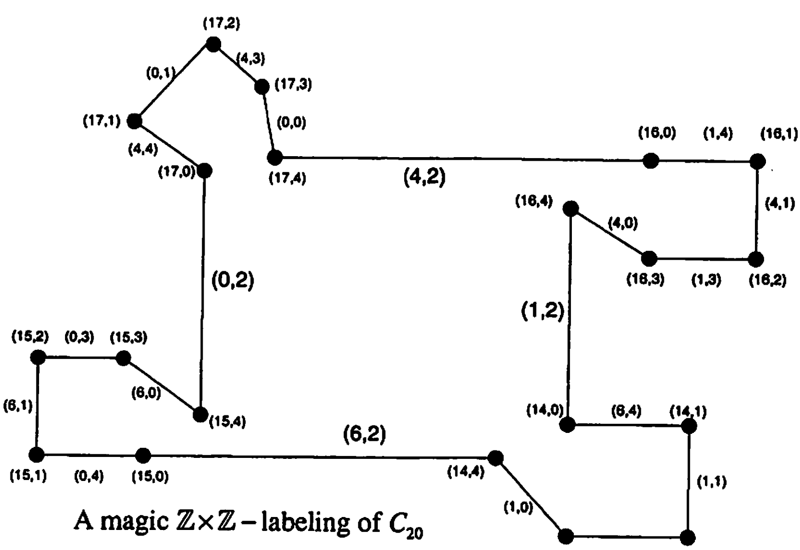
where all unions are disjoint unions.

Theorem 6 *Assume that G has a magic $\mathbb{Z}$−labeling with a range of B and a magic constant of h. Then for any subset J of I, the graph G_J has a magic $\mathbb{Z} \times \mathbb{Z}$−labeling with a range of $B \times A_n$ and a magic constant of $(h, 3m)$.*



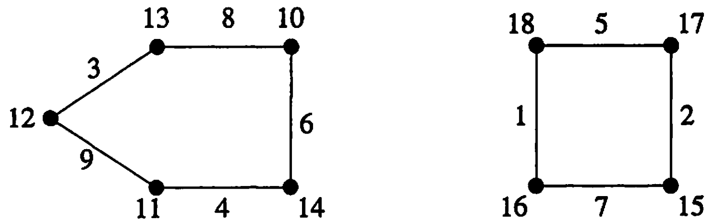
A magic $\mathbb{Z}$ – labeling of C_4 and $\alpha$for $n = 5$

Figure 4



A magic $\mathbb{Z}\times\mathbb{Z}$ – labeling of C_{20}

Figure 5



A strong VMTL for $C_5 \cup C_4$

Figure 6

Proof. This follows immediately from $|I-J|$ applications of the previous lemma as well as an application of lemma 2 (if $J \neq \phi$). ■

Theorem 7 *Assume that G has a VMTL σ with a magic constant of h. Then for any subset J of I, the graph G_J has VMTLs with magic constants of $nh-3m$ and $6|V|m+h$.*

Proof. Let $\delta : V \cup E \to \{0, 1, \ldots, |V| + |E| - 1\}$ be defined by $\delta(x) = \sigma(x) - 1$ for each $x \in V \cup E$. Observe that δ is a magic $\mathbb{Z}$–labeling with range $A_{2|V|}$ and magic constant $h' = h - 3$. By the previous theorem, there is a magic $\mathbb{Z} \times \mathbb{Z}$–labeling λ with a range of $A_{2|V|} \times A_n$ and a magic constant of $(h', 3m)$. By proposition 1 part 1, there is a VMTL λ_1 of G_J with a magic constant of $nh' + 3m + 3 = nh - 3m$. By proposition 1 part 2, there is a VMTL λ_2 of G_J with a magic constant of $2|V|(3m) + h' + 3 = 6|V|m + h$. ■

A VMTL of $C_5 \cup C_4$ is shown in Fig. 6. If one follows the proof of the previous theorem through on that example, one obtains the figures 2,3,4 and 5.

A *strong VMTL* of a graph is a VMTL such that the smallest available labels are used on the edges (see Fig. 6). Thus the VMTL σ of G is strong if the labels $1, 2, \ldots, |V|$ are used on the edges. Following through the proof of the previous theorem, this would mean that δ labels the edges with labels $0, 1, \ldots, |V|-1$ and therefore, λ would label the edges of G_J with the $n|V|$ labels $\{(x, y)|x \in A_{|V|}, y \in A_n\}$. Thus, λ_1 would label the edges with $1, 2, \ldots, n|V|$. We have:

Corollary 8 *Assume that G has a strong VMTL. Then for any subset J of I, the graph G_J has a strong VMTL.*

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