On the Structures of V_4-magic and $\mathcal{Z}_4$-magic Graphs

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ABSTRACT. This paper settles in the negative the following open question: Are V_4-magic graphs necessarily $\mathcal{Z}_4$-magic? For abelian group A, we examine the properties of A-magic labelings with constant weight 0, called *zero-sum* A-magic, and utilize well-known results on edge-colorings in order to construct (from 3-regular graphs) infinite families that are V_4-magic but not $\mathcal{Z}_4$-magic. Noting that our arguments lead to connected graphs of order $2n$ for all $n \geq 11$ that are V_4-magic and not $\mathcal{Z}_4$-magic, we conclude the paper by investigating the zero-sum integer-magic spectra of graphs, including Cartesian products, and give a conjecture about the zero-sum integer-magic spectra of 3-regular graphs.

1. Introduction.

Unless otherwise indicated, the term *multigraph* shall refer to a graph that is loopless, connected, and (following Harary [8]) possibly simple. If $G = (V, E)$ is a non-simple multigraph with precisely $n \geq 2$ parallel edges incident to distinct vertices x and y, then we may denote those edges by

$$(\{x,y\},1), (\{x,y\},2), (\{x,y\},3), ..., (\{x,y\},n).$$

If there exists precisely one edge in E incident to both x and y, then that edge shall be denoted $\{x, y\}$. Accordingly, with each edge uniquely denoted, we note that E is a set (in contrast to multiset) whether G is simple or not.

Let $(A, +)$ be an abelian group with identity 0 and let $A^* = A - \{0\}$. Let $G = (V, E)$ be a multigraph. Then a *labeling of* G is a function ϕ from E into A^*, and for fixed $e \in E$, $\phi(e)$ is called the *label of* e *under* ϕ.

For fixed $v \in V$, *the weight of* v *under* ϕ, denoted $w_\phi(v)$, is the sum of the labels (under ϕ) of the edges incident to v. It is clear that w_ϕ is a function from

V to A induced by ϕ. In the special case that w_ϕ is a constant function, a name is given to ϕ; particularly, ϕ is an *A-magic labeling of G* if and only if for some $c \in A$, $w_\phi(v) = c$ for every $v \in V$. Such a labeling is said to have an *A-magic value* of c. Correspondingly, *G is A-magic* if and only if there exists an A-magic labeling of G.

The topic of group-magic graphs represents a branch in the more general area of magic graphs, traceable to Sedlàček [24]. Since then, various authors have defined and explored graph labelings, using such terms as edge-magic, vertex-magic, total-magic, semi-magic, pseudomagic, and supermagic. (For definitions of these terms and surveys of related results, see [7] and [29].) Early work in the area of group-magic labelings was conducted by Doob [2, 3, 4] and Stanley [27, 28], the latter of whom studied $\mathcal{Z}$-magic labelings in the context of linear homogeneous diophantine equations. More recent investigations of A-magic labelings include works by D. Combe et al [1], S-M Lee et al [11], R.M. Low and S-M Lee [19, 20], and W.C. Shiu and R.M. Low [25]. Particular focus has been placed upon both $\mathcal{Z}$-magic and $\mathcal{Z}_k$-magic labelings, leading to the invention of the *integer-magic spectra* of graphs. (See [9, 10, 12, 14, 15, 16, 17, 18, 21, 22, 23, 26].) Attention has also been given to V_4-labelings of graphs (for examples, in [13] and [30]), where V_4 is the Klein group $\mathcal{Z}_2 \times \mathcal{Z}_2$, the non-cyclic abelian group of smallest order. This, in turn, has given rise to an open question: if G is V_4-magic, is G necessarily $\mathcal{Z}_4$-magic?

In Section 2 of this paper, we demonstrate that the answer to this question is 'no' through an investigation of specially constructed simple graphs. To that end, we give the definitions of *zero-sum A-magic labelings* and *pendant-extensions of multigraphs* which will link the properties of $\mathcal{Z}_4$-magic and V_4-magic multigraphs to the traditional notions of chromatic index and factorability. In Section 3, we present additional results that arose in our investigation.

We close this section by presenting the two promised definitions as well as a related theorem that will facilitate the discussion.

Definition 1.1. Let G be a multigraph. Then G is *zero-sum A-magic* if and only if there exists an A-magic labeling ϕ of G with A-magic value 0. Such a labeling is called a *zero-sum A-magic labeling of G*. □

We observe that every zero-sum A-magic multigraph is A-magic, but that the converse is not true. For example, the 3-cycle C_3 is easily checked to be $\mathcal{Z}_3$-magic (by letting $\phi(e) = 1$ for all $e \in E(C_3)$), but not zero-sum $\mathcal{Z}_3$-magic.

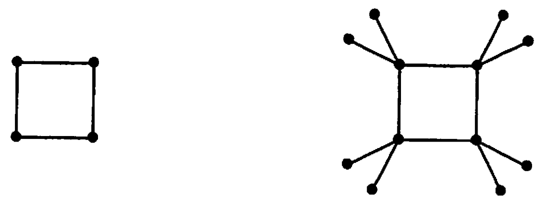
Definition 1.2. Let G be a multigraph and let μ be a positive integer. Then the μ-pendant extension of G is the multigraph μ-$pe(G)$ obtained from G by attaching μ distinct pendant edges to each vertex in $V(G)$. Formally, μ-$pe(G)$ is the multigraph such that

(i) $V(\mu\text{-}pe(G))$ is the union of disjoint sets $V(G)$ and Y, where $Y = \{y_{v,j} | v \in V(G) \text{ and } 1 \le j \le \mu\}$, and

(ii) $E(\mu\text{-}pe(G))$ is the union of disjoint sets $E(G)$ and F, where $F = \{\{v, y_{v,j}\} | v \in$

$V(G)$ and $1 \leq j \leq \mu\}$.
If $\mu = 1$, then 1-$pe(G)$ shall be abbreviated $pe(G)$. □

We note that μ-$pe(G)$ is a special case of $G \circ G^*$, the corona of G with G^*, defined in [6]. Particularly, for integer $\mu \geq 1$, μ-$pe(G)$ is isomorphic to $G \circ K_\mu^c$. For illustration, we display C_4 and 2-$pe(C_4)$ in Figure 1.



C_4 $\qquad$ 2-$pe(C_4)$

C_4 and the 2-pendant extension of C_4

Figure 1.

Theorem 1.3. Let A be an abelian group and let G be a multigraph. Then
(i) G is zero-sum A-magic if and only if $pe(G)$ is A-magic, and
(ii) if the maximum order among the orders of the elements of A is the finite number m_A, then the following are pairwise equivalent:

(a) G is zero-sum A-magic

(b) for every non-negative integer j, $(jm_A + 1)$-$pe(G)$ is A-magic

(c) for some non-negative integer j, $(jm_A + 1)$-$pe(G)$ is A-magic.

Proof: Throughout the proof, we will suppose that μ-$pe(G)$ has respective vertex and edge sets $V(G) \bigcup Y$ and $E(G) \bigcup F$, where Y and F are as defined in Definition 1.2.

Proof of (i) ($\rightarrow$) Suppose there exists a zero-sum A-magic labeling ϕ of G. Let c be an arbitrary element of A^*. Then the following function $\phi' : E(G) \bigcup F \rightarrow A^*$ is easily seen to be an A-magic labeling of $pe(G)$ with A-magic weight c:

$$\phi'(e) = \begin{cases} \phi(e) & \text{if } e \in E(G) \\ c & \text{if } e \in F. \end{cases}$$

($\leftarrow$) Now suppose that for some $c \in A$, there is an A-magic labeling γ of $pe(G)$ with A-magic value c. Since every vertex in $V(G) \bigcup Y$ has weight c under γ, and since every edge in F is incident to precisely one vertex in Y with degree 1, it follows that every edge in F is labeled c under γ (implying $c \neq 0$). Now let γ' be the restriction of γ to $E(G)$. Clearly, γ' is a labeling of G. Moreover, since every vertex in $V(G)$ is incident to precisely one edge in F (and at least one edge not in F), we have $w_{\gamma'}(v) + c = w_\gamma(v)$ for every $v \in V(G)$. But $w_\gamma(v) = c$

by assumption, implying $w_{\gamma'}(v) = 0$. Thus γ' is a zero-sum A-magic labeling of G, giving the result.

Proof of (ii): It is clear that (b) $\rightarrow$ (c). It thus suffices to prove (a) $\rightarrow$ (b) and (c) $\rightarrow$ (a).

We begin with the proof of (a) $\rightarrow$ (b). The argument will be similar to that of (i), but will require the additional assumption that m_A is finite.

Suppose that ϕ is a zero-sum A-magic labeling of G and fix arbitrary integer $j \geq 0$ and $c \in A^*$. We claim that ϕ' is an A-magic labeling of (jm_A+1)-$pe(G)$, where ϕ' is the function from $E(G) \bigcup F$ into A^* such that

$$\phi'(e) = \begin{cases} \phi(e) & \text{if } e \in E(G) \\ c & \text{if } e \in F. \end{cases}$$

To see this, we argue that c is the weight (under ϕ') of every vertex in $V(G) \bigcup Y$. If $y \in Y$, then $w_{\phi'}(y) = c$ because y has degree 1 and is incident to an edge in F. On the other hand, if $v \in V(G)$, then v is incident to precisely the edges in the disjoint union of the set of edges in $E(G)$ incident to v with the set of $jm_A + 1$ pendant edges incident to v. Thus $w_{\phi'}(v) = w_\phi(v) + c(jm_A + 1)$. However, $w_\phi(v) = 0$ since ϕ is a zero-sum A-magic labeling of G. And, $c(jm_A + 1) = c$ since the order of c divides m_A. So $w_{\phi'}(v) = c$.

We now show (c) $\rightarrow$ (a). Suppose that for some $c \in A$ and some non-negative integer j, there is an A-magic labeling γ of (jm_A+1)-$pe(G)$ with A-magic value c. Since every vertex in $V(G) \bigcup Y$ has weight c under γ, and since every edge in F is incident to precisely one vertex in Y with degree 1, it follows that every edge in F is labeled c under γ (implying $c \neq 0$). Now let γ' be the restriction of γ to $E(G)$. Clearly, γ' is a labeling of G. Moreover, since every vertex in $V(G)$ is incident to precisely $jm_A + 1$ edges in F (and at least one edge not in F), we have $w_{\gamma'}(v) + c(jm_A + 1) = w_\gamma(v)$ for every vertex $v \in V(G)$. But $w_\gamma(v) = c$ by assumption. And, as argued above, $c(jm_A + 1) = c$. Therefore $w_{\gamma'}(v) = 0$, implying that γ' is a zero-sum A-magic labeling of G. □

2. Constructing Z_4-magic multigraphs and V_4-magic multigraphs from multigraphs of maximum degree 3.

We shall begin with a consideration of V_4-magic multigraphs, then turn to Z_4-magic multigraphs.

Observation 2.1. Suppose that G is a zero-sum V_4-magic multigraph, and suppose that $v \in V(G)$. If the degree of v is 3, then for every zero-sum V_4-magic labeling ϕ of G, the three edges incident to v receive the distinct labels $(1,0),(0,1),(1,1)$ under ϕ. If the degree of v is 2, then for every zero-sum V_4-magic labeling ϕ of G, the two edges incident to v receive the same label under ϕ. □

Lemma 2.2. Suppose that G is a zero-sum V_4-magic multigraph. Then the subdivision of any edge in G results in a multigraph G' that is zero-sum V_4-magic.

Proof: Let ϕ be a zero-sum V_4-magic labeling of G and let G' be the multigraph that results by subdividing edge e' incident to vertices u and v into edges $\{u,x\}$ and $\{x,v\}$. Then the function ϕ' from $E(G')$ to V_4^* is easily checked to be a zero-sum V_4-magic labeling of G', where

$$\phi'(e) = \begin{cases} \phi(e') & \text{if } e = \{u,x\} \text{ or } \{x,v\} \\ \phi(e) & \text{otherwise.} \end{cases}$$ □

Theorem 2.3. Let G be a 3-regular multigraph. Then G is zero-sum V_4-magic if and only if the chromatic index of G is 3.

Proof: (→) Suppose G is zero-sum V_4-magic and accordingly let ϕ be a zero-sum V_4-magic labeling of G. From Observation 2.1, ϕ is a 3-edge coloring of G.

(←) Suppose that G has chromatic index 3, and let C be a 3-edge coloring of G in which the assigned colors are $(1,0),(0,1)$ and $(1,1)$. Then C is a zero-sum V_4-magic labeling of G since the sum of the 3 distinct colors is 0. □

Corollary 2.4. Let G be a 3-regular multigraph. Then the following are pairwise equivalent:

(i) G has chromatic index 3

(ii) for every non-negative integer j, $(2j+1)$-$pe(G)$ is V_4-magic

(iii) for some non-negative integer j, $(2j+1)$-$pe(G)$ is V_4-magic.

Proof: This follows from Theorem 2.3, $m_{V_4} = 2$, and Theorem 1.3(ii). □

The next corollary follows from Theorem 2.3 and the well-known result that every 3-regular multigraph with a cut-edge has chromatic index 4. (See [5].)

Corollary 2.5. Let G be a 3-regular multigraph with a cut-edge. Then G is not zero-sum V_4-magic. □

We now turn to $\mathcal{Z}_4$-magic graphs.

Observation 2.6. Suppose that G is a zero-sum $\mathcal{Z}_4$-magic multigraph, and suppose $v \in V(G)$. Let E_v denote the set of edges incident to v.

(i) If the degree of v is 3, then for every zero-sum $\mathcal{Z}_4$-magic labeling ϕ of G, either (1) two edges in E_v receive the label 1 under ϕ and one edge in E_v receives the label 2 under ϕ, or (2) two edges in E_v receive the label 3 under ϕ and one edge in E_v receives the label 2 under ϕ.

(ii) If the degree of v is 2, then for every zero-sum $\mathcal{Z}_4$-magic labeling ϕ of G, either (1) the two edges in E_v each receive the label 2 under ϕ, or (2) the two edges in E_v receive the labels 1 and 3 under ϕ. □

Theorem 2.7. Let G be a 3-regular multigraph. Then G is zero-sum $\mathcal{Z}_4$-magic if and only if G has a 2-regular spanning subgraph.

Proof: ($\rightarrow$) Suppose γ is a zero-sum $\mathcal{Z}_4$-magic labeling of G. From Observation 2.6, every vertex in $V(G)$ is incident to one and only one edge with label 2 under γ. Let $M = \{e \in E(G) | \gamma(e) = 2\}$. Then $G-M$ is a 2-regular spanning subgraph of G.

($\leftarrow$) Suppose that H is a 2-regular spanning subgraph of G. Let γ be the mapping from $E(G)$ into $\mathcal{Z}_4^*$ defined by $\gamma(e) = 1$ for $e \in E(H)$, and 2 otherwise. Then γ is a zero-sum $\mathcal{Z}_4$-magic labeling of G. □

Corollary 2.8. Let G be a 3-regular multigraph. Then the following are pairwise equivalent:

(i) G has a 2-regular spanning subgraph

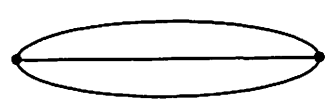
(ii) for every non-negative integer j, $(4j+1)$-$pe(G)$ is $\mathcal{Z}_4$-magic

(iii) for some non-negative integer j, $(4j+1)$-$pe(G)$ is $\mathcal{Z}_4$-magic.

Proof: This follows from Theorem 2.7, $m_{\mathcal{Z}_4} = 4$, and Theorem 1.3(ii). □

Clearly, each multigraph G falls into one of four categories: (1) G is both $\mathcal{Z}_4$-magic and V_4-magic; (2) G is neither $\mathcal{Z}_4$-magic nor V_4-magic; (3) G is $\mathcal{Z}_4$-magic and not V_4-magic; and (4) G is V_4-magic and not $\mathcal{Z}_4$-magic. Let $\mathcal{G}$ be the collection of all (loopless connected) 3-regular multigraphs, and let $\mathcal{G}_s$ be the subset of $\mathcal{G}$ containing the simple (multi)graphs in $\mathcal{G}$. Also, let $pe(\mathcal{G})$ denote the collection of 1-pendant extensions of the multigraphs in $\mathcal{G}$ and let $pe(\mathcal{G}_s)$ be the subset of $pe(\mathcal{G})$ containing the 1-pendant extensions of simple graphs in $\mathcal{G}_s$. We make the following observations.

The multigraph G_1 of smallest order in $\mathcal{G}$ is the order-3 dipole, which has chromatic index 3 and contains a 2-regular spanning subgraph. (See Figure 2.) Thus, by Theorems 2.3 and 2.7, G_1 is both zero-sum $\mathcal{Z}_4$-magic and zero-sum V_4-magic. By Theorem 1.3(i), it therefore follows that $pe(G_1)$ is the multigraph of smallest order in $pe(\mathcal{G})$ that is both $\mathcal{Z}_4$-magic and V_4-magic.



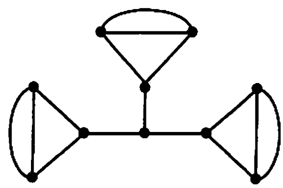
The multigraph G_1 of smallest order in $\mathcal{G}$
that is both zero-sum $\mathcal{Z}_4$-magic and zero-sum V_4-magic.

Figure 2.

Similarly, the simple graph of smallest order in $\mathcal{G}_s$ with chromatic index 3 and a 2-regular spanning subgraph is K_4. Thus, by Theorems 2.3, 2.7 and 1.3(i), it follows that $pe(K_4)$ is the simple graph of smallest order in $pe(\mathcal{G}_s)$ that is both $\mathcal{Z}_4$-magic and V_4-magic. We note that there are infinitely many connected multigraphs in $pe(\mathcal{G})$ that are both $\mathcal{Z}_4$-magic and V_4-magic.

The multigraph G_2 of smallest order in $\mathcal{G}$ with neither chromatic index 3 nor a 2-regular spanning subgraph has order 10 and is displayed in Figure 3.

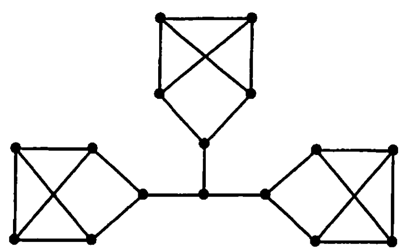
(See [5].) It follows (from Theorems 2.3, 2.7, and 1.3(i)) that $pe(G_2)$ is the multigraph of smallest order in $pe(\mathcal{G})$ that is neither $\mathcal{Z}_4$-magic nor V_4-magic.



The multigraph G_2 of smallest order in $\mathcal{G}$ that is neither zero-sum $\mathcal{Z}_4$-magic nor zero-sum V_4-magic.

Figure 3.

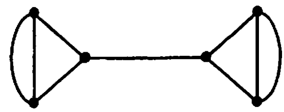
The simple graph G_3 of smallest order in $\mathcal{G}_s$ with neither chromatic index 3 nor a 2-regular spanning subgraph has order 16 and is displayed in Figure 4. (See [5].) Thus $pe(G_3)$ is the graph of smallest order in $pe(\mathcal{G}_s)$ that is neither $\mathcal{Z}_4$-magic nor V_4-magic. Note that there are infinitely many connected multigraphs in $pe(\mathcal{G})$ that are neither $\mathcal{Z}_4$-magic nor V_4-magic.



The simple graph G_3 of smallest order in $\mathcal{G}_s$ that is neither zero-sum $\mathcal{Z}_4$-magic nor zero-sum V_4-magic.

Figure 4.

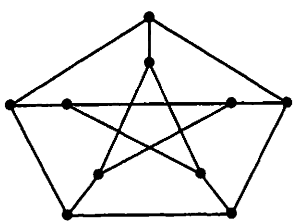
The multigraph G_4 of smallest order in $\mathcal{G}$ that does not have chromatic index 3 yet has a 2-regular spanning subgraph is displayed in Figure 5. Thus $pe(G_4)$ is the multigraph of smallest order in $pe(\mathcal{G})$ that is $\mathcal{Z}_4$-magic and not V_4-magic.



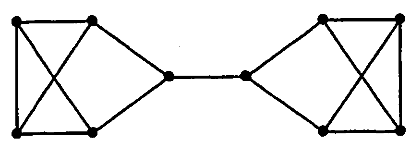
The multigraph G_4 of smallest order in $\mathcal{G}$
that is zero-sum $\mathcal{Z}_4$-magic and not zero-sum V_4-magic.

Figure 5.

The two simple graphs G_5 and G_6 of smallest order in $\mathcal{G}_s$ that do not have chromatic index 3 yet have 2-regular spanning subgraphs are displayed in Figure 6. Thus $pe(G_5)$ and $pe(G_6)$ are the graphs of smallest order in $pe(\mathcal{G}_s)$ that are $\mathcal{Z}_4$-magic and not V_4-magic. There are infinitely many connected multigraphs in $pe(\mathcal{G})$ with this property.



G_5 (the Petersen graph)



G_6

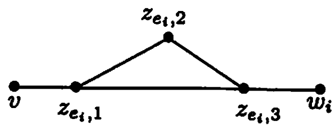
Simple graphs of smallest order in $\mathcal{G}_s$
that are zero-sum $\mathcal{Z}_4$-magic and not zero-sum V_4-magic.

Figure 6.

We note that every zero-sum V_4-magic multigraph in $\mathcal{G}$ is also zero-sum $\mathcal{Z}_4$-magic. To see this, suppose G is zero-sum V_4-magic. Then by Theorem 2.3, G has chromatic index 3. Let C be a 3-edge coloring of G with colors a, b and c, and let $H_{a,b}$ be the subgraph of G whose edges are colored either a or b under C. Then $H_{a,b}$ is a spanning 2-regular subgraph of G, implying by Theorem 2.7 that G is zero-sum $\mathcal{Z}_4$-magic.

In order to establish the existence of multigraphs that are V_4-magic but not $\mathcal{Z}_4$-magic, we expand our discussion from 3-regular multigraphs to multigraphs G such that $2 = \delta(G) < \Delta(G) = 3$.

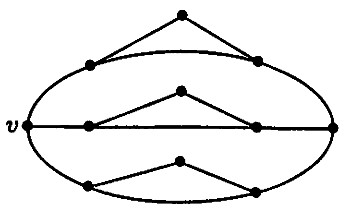
We begin with a construction. Let $G = (V, E)$ be a multigraph with a vertex v of degree 3 incident to distinct edges e_1, e_2, e_3. We assume that besides being incident to v, edge e_i is incident to vertex w_i. (Since we are dealing with multigraphs, the vertices w_1, w_2, w_3 need not be pairwise distinct.) Then $G(v)$ is the graph that results by replacing each edge e_i, $1 \leq i \leq 3$, by the structure in Figure 7.



Structure to replace edges incident to v
in the formation of $G(v)$

Figure 7.

In Figure 8, we illustrate $G_1(v)$ where G_1 is order-3 dipole illustrated in Figure 2.



The multigraph $G_1(v)$

Figure 8.

Theorem 2.9. Let G be a 3-regular multigraph with chromatic index 3 and let $v \in V(G)$. Then $G(v)$ is zero-sum V_4-magic.

Proof. Let H denote the 3-regular multigraph that results by smoothing each of the three vertices of degree 2 in $G(v)$. Since G has chromatic index 3, it is easily checked that H also has chromatic index 3. Thus by Theorem 2.3, H is zero-sum V_4-magic. Lemma 2.2 then implies that $G(v)$ is zero-sum V_4-magic. □

Corollary 2.10. Let G be a 3-regular multigraph with chromatic index 3 and let $v \in V(G)$. Then for every non-negative integer j, $(2j+1)$-$pe(G(v))$ is V_4-magic. Hence, for every non-negative integer j, $(4j+1)$-$pe(G(v))$ is V_4-magic.

Proof: This follows from Theorem 2.9, $m_{V_4} = 2$, and Theorem 1.3(ii). □

We note that the next theorem does not require the 3-regularity of G, but rests solely on the existence of a vertex of degree 3.

Theorem 2.11. Let G be a multigraph with vertex v of degree 3. Then $G(v)$ is not zero-sum $\mathcal{Z}_4$-magic.

Proof: Suppose to the contrary that ϕ is a zero-sum $\mathcal{Z}_4$-magic labeling of $G(v)$. By making appeals to Observation 2.6, we note that since v has degree 3, there exists i such that $\phi(\{v, z_{e_i,1}\}) = 2$. Hence $\phi(\{z_{e_i,1}, z_{e_i,2}\}) = \phi(\{z_{e_i,1}, z_{e_i,3}\}) = a \in \{1,3\}$, implying $\phi(\{z_{e_i,2}, z_{e_i,3}\}) = -a$, a contradiction of Observation 2.6

with respect to the labels of the edges incident to $z_{e_i,3}$. □

Corollary 2.12. Let G be a 3-regular multigraph and let $v \in V(G)$. Then for every non-negative integer j, $(4j+1)$-$pe\big(G(v)\big)$ is not $\mathcal{Z}_4$-magic.

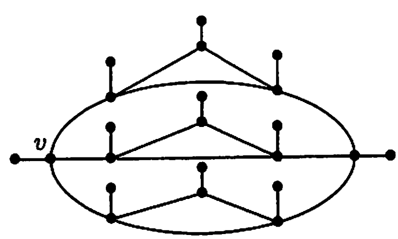
Proof: This follows from Theorem 2.11, $m_{\mathcal{Z}_4} = 4$, and Theorem 1.3(ii). □

By corollaries 2.10 and 2.12, we are now able to conclude the existence of infinitely many multigraphs that are V_4-magic but not $\mathcal{Z}_4$-magic.

Theorem 2.13. Let G be a 3-regular multigraph with chromatic index 3 and let $v \in V(G)$. Then for every non-negative integer j, $(4j+1)$-$pe\big(G(v)\big)$ is V_4-magic but not $\mathcal{Z}_4$-magic. □

As noted earlier, the 3-regular multigraph in $\mathcal{G}$ with smallest order is G_1 (given in Figure 2) with chromatic index 3. So by Theorem 2.13 with $j = 0$, $pe\big(G_1(v)\big)$ (given below in Figure 9) is V_4-magic but not $\mathcal{Z}_4$-magic. We believe that among multigraphs with these magic properties, $pe\big(G_1(v)\big)$ is unique with smallest order. Moreover, by Lemma 2.2 and theorems 1.3(ii), 2.9, and 2.11, performing k subdivisions of any edge of G_1 will result in a (simple) multigraph G_1^* such that $G_1^*(v)$ has a 1-pendant extension with order $2(11+k)$ that is V_4-magic but not $\mathcal{Z}_4$-magic. We thus have the following:

Theorem 2.14. For all $n \geq 11$, there exists a simple connected graph of order $2n$ that is V_4-magic but not $\mathcal{Z}_4$-magic. □



The multigraph $pe\big(G_1(v)\big)$

Figure 9.

We observe that although $G_1(v)$ is zero-sum V_4-magic and not zero-sum $\mathcal{Z}_4$-magic, it is the case that $G_1(v)$ is V_4-magic (since $G_1(v)$ is zero-sum V_4-magic) and $\mathcal{Z}_4$-magic (since it is easy to construct a labeling of $G_1(v)$ with $\mathcal{Z}_4$-magic value 2).

To this point, our search for multigraphs that are zero-sum V_4-magic but not zero-sum $\mathcal{Z}_4$-magic has focussed upon the structure of connected loopless multigraphs with vertices of degree 2 or 3. Within the scope of this paragraph, let us expand the definition of multigraph to include those structures with loops, and let us assume that G is a connected multigraph such that $2 = \delta(G) <$

$\Delta(G) = 3$, the order of G is at least 2, and G has at least one loop. Then it is easily checked that the (perhaps multiple) subdivision of at least one of those loops results in a multigraph G' that is not zero-sum V_4-magic. We thus see that if G' is zero-sum V_4-magic with order at least 2 and $2 \le \delta(G') \le \Delta(G') = 3$, the smoothing of any of its vertices of degree 2 cannot result in a multigraph with a loop.

3. Additional comments and results.

In the process of developing the ideas that led to the constructions in Section 2, we discovered other related results that may be of interest.

First, we recall that for fixed positive integer μ, μ-$pe(G)$ is the multigraph obtained from G by attaching μ pendant edges to each vertex of G. This definition can be generalized in such a way that the number of pendant edges attached to vertex v is j_v, resulting in a multigraph G' in which not all vertices have the same number of attached pendant edges. We note that Theorem 1.3 can be generalized accordingly; for example, it is easy to see that if A is an abelian group with elements of maximum finite order m_A, and if $j_v = k_v m_A + 1$ for some non-negative integer k_v, then G is zero-sum A-magic if and only if G' is A-magic.

Second, Corollary 2.5 indicates that no 3-regular multigraph G with a cut-edge is zero-sum V_4-magic, and is proved using the fact that the chromatic index of G is necessarily 4. We present a stronger result on multigraphs with cut-edges for $A = \oplus_{i=1}^{n} \mathcal{Z}_2$ (that is, the direct sum of n copies of $\mathcal{Z}_2$) for each $n \ge 1$.

Theorem 3.1. Let G be a connected multigraph with a cut-edge $e^* = \{x, y\}$. Then for all positive integer n, G is not zero-sum $\oplus_{i=1}^{n} \mathcal{Z}_2$-magic.

Proof: If G is isomorphic to K_2, then the result is clear. Thus, we may assume that the multigraph $G - \{e^*\}$ has two components G_x and G_y where, with no loss of generality, G_x contains vertex x and has order at least 2. We proceed by contradiction, assuming that for some positive integer n, there exists a zero-sum $\oplus_{i=1}^{n} \mathcal{Z}_2$-magic labeling ϕ of G. Let E_1 denote the set of edges in $E(G_x)$ incident to x, and let $E_2 = E(G_x) - E_1$. Noting that $w_\phi(v) = 0$ for every $v \in V(G)$, we consider $\sum_{v \in V(G_x) - \{x\}} w_\phi(v)$. This sum, clearly 0, is equal to $\sum_{e \in E_1} \phi(e) + \sum_{e \in E_2} 2\phi(e)$. Since $2z = 0$ for every $z \in \oplus_{i=1}^{n} \mathcal{Z}_2$, it follows that $\sum_{e \in E_2} 2\phi(e) = 0$, implying that $\sum_{e \in E_1} \phi(e) = 0$. Therefore, since $w_\phi(x) = 0$ and $w_\phi(x) = \phi(e^*) + \sum_{e \in E_1} \phi(e)$, it must be the case that $\phi(e^*) = 0$, contradicting the definition of labeling. □

Third, as noted in the Introduction, many papers on the topic of group-magic labelings have concerned themselves with the integer-magic spectrum of a multigraph G, defined to be the subset $im(G)$ of $\mathcal{N}$ such that $1 \in im(G)$ if and only if G is $\mathcal{Z}$-magic and $k \ge 2 \in im(G)$ if and only if G is $\mathcal{Z}_k$-magic. Analogously, we define the zero-sum integer-magic spectrum of G to be $zim(G)$ (a subset of $\mathcal{N}$), where $1 \in zim(G)$ if and only if G is zero-sum $\mathcal{Z}$-magic and $k \ge 2 \in zim(G)$ if and only if G is zero-sum $\mathcal{Z}_k$-magic. We note that for any multigraph G, $2 \in zim(G)$ if and only if every vertex in $V(G)$ has even degree. Additionally, it can be easily checked that for $2\mathcal{N}$ equal to the set of positive even integers,

$$zim(C_n) = \begin{cases} \mathcal{N} & \text{if } n \text{ is even} \\ 2\mathcal{N} & \text{if } n \text{ is odd.} \end{cases}$$

More generally, for arbitrary nontrivial abelian group A, and for any multigraph G that has both even size and an Euler tour, we observe that G is zero-sum A-magic. (For any $a \in A^*$, assign labels a and $-a$ in alternating fashion along the edges of the tour.) Therefore, $zim(G) = \mathcal{N}$. It now follows that

Theorem 3.2. For any integer $k \geq 1$, every $4k$-regular multigraph has a zero-sum integer-magic spectrum equal to $\mathcal{N}$. □

Now suppose that G is a 3-regular multigraph. We have already observed above that G cannot be zero-sum $\mathcal{Z}_2$-magic. However, if G is zero-sum $\mathcal{Z}_4$-magic, then by Theorem 2.7, G has a 2-regular spanning subgraph H. For fixed $k \geq 3$, we form a zero-sum $\mathcal{Z}_k$-magic labeling ϕ of G by assigning 1 to each edge in H and $k-2$ to each edge of $G-H$. Analogously, we form a zero-sum $\mathcal{Z}$-magic labeling by assigning 1 to each edge in H and -2 to each edge of $G-H$. Thus we have

Theorem 3.3. Let G be a 3-regular multigraph. If G is zero-sum $\mathcal{Z}_4$-magic, then $zim(G) = \mathcal{N} - \{2\}$. □

Since the simple graph G_3 in Figure 4 is not zero-sum $\mathcal{Z}_4$-magic and clearly not zero-sum $\mathcal{Z}_2$-magic, we explored $zim(G_3)$, discovering that $zim(G_3) = \mathcal{N} - \{2, 4\}$. To see this, we first note that all 3-regular multigraphs G are zero-sum $\mathcal{Z}_3$-magic with the assignment of 1 to each edge in G; hence $3 \in zim(G_3)$. For $k \geq 5$, we form a zero-sum $\mathcal{Z}_k$-magic labeling of G_3 by assigning

: 1 to each edge of the left and right 5-cycles;
: 4 to the vertical cut-edge and each of the two diagonals in the (middle) 5-cycle that is incident to that vertical cut edge
: $k-2$ to all other edges.

Thus $k \in zim(G_3)$ for $k \geq 5$. Finally, by replacing $k-2$ with -2 in the above labeling, we have a zero-sum $\mathcal{Z}$-magic labeling of G_3, giving $1 \in zim(G_3)$.

A similar argument shows that $zim(G_2)$ (see Figure 3) is also $\mathcal{N} - \{2, 4\}$. For $k \geq 5$ or $k = 0$, we assign the label 4 to the vertical cut-edge, the label 1 to the diagonal and vertical edges of the left and right triangles and the two "multi"-edges of the middle triangle, and $k-2$ to the remaining edges. We conjecture that for any 3-regular multigraph G, $zim(G)$ is either $\mathcal{N} - \{2\}$ or $\mathcal{N} - \{2, 4\}$.

We now close this section with some results on zero-sum A-magic labelings of Cartesian products of multigraphs. Recalling that the Cartesian product of multigraphs G_1 and G_2, denoted $G_1 \Box G_2$, is a graph with $|V(G_1)|$ row copies of G_2 and $|V(G_2)|$ column copies of G_1, we note the following theorems, the first of which is given without proof:

Theorem 3.4. For $i = 1, 2$, if G_i is a zero-sum A_i-magic multigraph, then

(i) $G_1 \Box G_2$ is zero-sum $A_1 \times A_2$-magic

(ii) if $A_1 = A_2 = A$, then $G_1 \Box G_2$ is zero-sum A-magic

(iii) $zim(G_1) \bigcap zim(G_2) \subseteq zim(G_1 \Box G_2)$. $\Box$

Theorem 3.5. Let Q_n denote the n-dimensional hypercube. For $n \geq 1$,

$$zim(Q_n) = \begin{cases} \phi & \text{if } n = 1 \\ \mathcal{N} & \text{if } n \text{ is even} \\ \mathcal{N} - \{2\} & \text{if } n \geq 3 \text{ is odd.} \end{cases}$$

Proof: Proceeding by induction on the evens, we see that the claim is clearly true for $n = 2$ since Q_2 is C_4. Now suppose that k is an even integer such that $zim(Q_k) = \mathcal{N}$. Since Q_{k+2} is $Q_k \Box Q_2$, the results follows from Theorem 3.4(ii). Thus $zim(Q_n) = \mathcal{N}$ if n is even.

Suppose n is odd. The claim is clearly true if $n = 1$. Moreover, by Theorem 2.7, Q_3 is zero-sum $\mathcal{Z}_4$-magic. Thus the claim is true for $n = 3$ by Theorem 3.3. Now suppose n is an arbitrary odd integer at least 5. Then Q_n is $Q_{n-3} \Box Q_3$, where $n - 3$ is an even integer at least 2. Since $zim(Q_3) = \mathcal{N} - \{2\}$ and $zim(Q_{n-3}) = \mathcal{N}$ as previously shown, it follows from Theorem 3.4(iii) that $\mathcal{N} - \{2\} \subseteq zim(Q_n)$. But 2 is not in the zero-sum integer-magic spectrum of any odd regular graph, so the result follows for odd n. $\Box$

It is clear that if a multigraph G is zero-sum A-magic for all abelian groups A, then $zim(G) = \mathcal{N}$. To consider the converse, let A be an arbitrary abelian group with non-identity element a and suppose that $zim(G) = \mathcal{N}$. Then the subgroup of A generated by a, denoted $< a >$, is either isomorphic to $\mathcal{Z}$ or isomorphic to $\mathcal{Z}_k$ for some integer $k \geq 2$. Thus, G is zero-sum $< a >$-magic, implying that G is zero-sum A-magic. We have therefore shown

Theorem 3.6. Let G be a multigraph. Then $zim(G) = \mathcal{N}$ if and only if G is zero-sum A-magic for every non-trivial abelian group A. $\Box$

Theorem 3.6 now leads to the following corollary.

Corollary 3.7. Let Q_n denote the n-dimensional hypercube.

(i) If $n \geq 2$ is even, then Q_n is zero-sum A-magic for every non-trivial abelian group A.

(ii) If $n \geq 3$ is odd, then Q_n is zero-sum A-magic for every non-trivial abelian group A except A isomorphic to $\mathcal{Z}_2$.

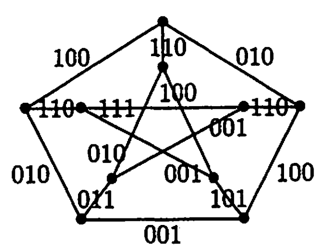
Proof: (i) This follows from Theorems 3.5 (for even n) and 3.6.

(ii) By Theorem 3.5 for odd $n \geq 3$, Q_n is not zero-sum $\mathcal{Z}_2$-magic. Thus, we suppose to the contrary that for some odd $n_0 \geq 3$ and some non-trivial abelian group A_0 not isomorphic to $\mathcal{Z}_2$, Q_{n_0} is not zero-sum A_0-magic. Then for every non-trivial subgroup H of A_0, Q_{n_0} is not zero-sum H-magic. Thus, every non-identity element a in A_0^* has order 2, implying that A_0 is isomorphic to $\oplus_{i=1}^m \mathcal{Z}_2$ for some positive integer $m \geq 2$. So, since V_4 is a non-trivial subgroup of $\oplus_{i=1}^m \mathcal{Z}_2$, Q_{n_0} is not zero-sum V_4-magic. But Q_3 is zero-sum V_4-magic by Theorem 2.3, giving $n_0 \geq 5$. And, Q_2 is zero-sum V_4-magic by inspection. Thus, by Theorem 3.4(ii) and an inductive argument, Q_n is zero-sum V_4-magic for all odd $n \geq 3$, a contradiction of the conclusion that Q_{n_0} is not zero-sum V_4-magic. □

We close with a result on the Petersen graph.

Theorem 3.8. The Petersen graph is zero-sum A-magic for every non-trivial abelian group A except for A isomorphic to $\mathcal{Z}_2$ or V_4.

Proof: Let P denote the Petersen graph. In the discussion surrounding Figure 6, we have established that P is zero-sum $\mathcal{Z}_4$-magic and not zero-sum V_4-magic. Thus by Theorem 3.3, $zim(P) = \mathcal{N} - \{2\}$, implying that P is not zero-sum $\mathcal{Z}_2$-magic and P is zero-sum A-magic for any non-trivial abelian group A with an element of order k, $3 \leq k \leq \infty$. We now claim that except for A isomorphic to $\mathcal{Z}_2$ or V_4, P is zero-sum A-magic for any non-trivial abelian group with non-identity elements each of order 2. Since any such group is isomorphic to $\oplus_{i=1}^n \mathcal{Z}_2$ for some $n \geq 3$, it suffices to show that P is zero-sum $\oplus_{i=1}^3 \mathcal{Z}_2$-magic. We present such a labeling in Figure 10. □



A zero-sum $\oplus_{i=1}^3 \mathcal{Z}_2$-magic labeling of the Petersen graph
Figure 10.

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