Classification of Starters

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Abstract

A *starter* in an odd order abelian group G is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (|G| - 1)/2\}$, for which $\{s_i\} \cup \{t_i\} = G \setminus \{0\}$ and $\{\pm(s_i - t_i)\} = G \setminus \{0\}$. If $s_i + t_i = s_j + t_j$ holds only for $i = j$, then the starter is called a *strong* starter. Only cyclic groups are considered in this work, where starters and strong starters up to order 35 and 37, respectively, are classified using an exact cover algorithm. The results are validated by double counting.

1 Introduction

A *starter* in an odd order abelian group G is a set of unordered pairs $S = \{\{s_i, t_i\} : 1 \leq i \leq (|G| - 1)/2\}$, for which $\{s_i\} \cup \{t_i\} = G \setminus \{0\}$ and $\{\pm(s_i - t_i)\} = G \setminus \{0\}$. If $s_i + t_i = s_j + t_j$ holds only for $i = j$, then the starter is called a *strong* starter.

Stanton and Mullin [6] introduced starters about 40 years ago in an article concerning construction of Room squares. Starters are also useful in constructing other combinatorial structures, such as Room cubes, Howell designs, Kirkman triple systems and Kirkman squares, 1-factorizarions of K_{2n}, and round robin tournaments [1, 4]. They can also be used to construct cyclic packings, which in turn can be used to construct optimal optical orthogonal codes.

Pike and Shalaby [5] computed the number of starters in cyclic groups up to order 31 and the value for $\mathbb{Z}_{33}$ can be found in [1]. Strong starters

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up to $\mathbb{Z}_{27}$ have been enumerated by Kocay, Stinson and Vanstone [4] using an orderly algorithm.

Two starters S_1 and S_2 in an abelian group G are *equivalent*, if there exists $\theta \in \mathrm{Aut}(G)$, such that $\{\theta(x), \theta(y)\} \in S_2$ for all $\{x, y\} \in S_1$. An *automorphism* of a starter S is an element $\theta \in \mathrm{Aut}(G)$, such that $\{\theta(x), \theta(y)\} \in S$ for all $\{x, y\} \in S$.

The automorphisms of a cyclic group of order n are given by [7, p. 157]

$$\theta(x) = mx \pmod{n}, \quad \gcd(m, n) = 1. \tag{1}$$

The number of valid multipliers m is $\phi(n)$.

Example 1 The two strong starters in $\mathbb{Z}_7$ are $S_1 = \{\{1,3\},\{2,6\},\{4,5\}\}$ and $S_2 = \{\{1,5\},\{2,3\},\{4,6\}\}$. These are equivalent; any multiplier $m \in \{3,5,6\}$ will map one starter onto the other.

In this work, an exact cover algorithm is applied to construct and classify starters and strong starters in cyclic groups up to order 35 and 37, respectively. Double counting based on the orbit-stabilizer theorem is utilized to validate the new results. Some incorrect values are detected in the process of verifying old results.

2 Classification

The classification of starters and strong starters consists of two main problems, generating all the starters and rejecting equivalent ones, in order to get the number of inequivalent ones.

Generating all the starters can be considered an instance of the *exact cover* problem. In the exact cover problem, we have a set and a collection of its subsets, and the task is to cover the set with the given subsets so that each element of the set is covered exactly once. An instance of the exact cover problem can be formulated as a system of integer linear equations.

2.1 Exact cover formulation

All nonzero elements of the group $G = \mathbb{Z}_n$ have to be covered exactly once by the elements of the pairs of the starter and also by the difference of the elements in a pair.

Example 2 An exact cover formulation for finding starters in $\mathbb{Z}_5$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_{1,2} \\ x_{1,3} \\ x_{1,4} \\ x_{2,3} \\ x_{2,4} \\ x_{3,4} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \qquad x_{i,j} \in \{0,1\}.$$

The element $x_{i,j}$ takes value 1 if the pair i,j is in the starter, and 0 otherwise. The first and the second row declare that the differences 1 and 4, and 2 and 3, respectively, should be covered exactly once. The last four rows declare that the elements 1, 2, 3 and 4 should occur in exactly one pair.

When searching for starters in the cyclic group $\mathbb{Z}_n$ as in Example 2, the total number of unknowns is $\binom{n-1}{2}$ and the number of equations is $(n-1)/2 + n - 1 = (3n-3)/2$.

The instance of the exact cover problem obtained can be solved with the library `libexact` [2], which is based on Knuth's Dancing links algorithm [3].

When searching for strong starters, the algorithm is modified to reject any partial solutions containing pairs with equal sums.

2.2 Isomorph rejection

Solving the instances as discussed above will output all distinct starters. In order to find the number of inequivalent starters, isomorph rejection has to be carried out for each solution found to this aim. The elements of each pair are multiplied modulo n with the integers m satisfying the conditions $1 < m < n$ and $\gcd(m,n) = 1$, and the starter is rejected if the multiplication results in a starter that is lexicographically smaller than the original one. Otherwise, the starter is accepted, and the size of its automorphism group is stored.

2.3 Validation of results

Double counting is utilized to validate the results. By the orbit-stabilizer theorem, the total number of distinct starters of order n is

$$DS(n) = \sum_{i=1}^{IS(n)} \frac{\phi(n)}{|\mathrm{Aut}(S_i)|},$$

where $IS(n)$ is the number of inequivalent starters of order n, $\phi(n)$ is the order of the group described in (1) acting on starters, and $S_1, S_2, \ldots, S_{IS(n)}$

are the inequivalent starters of order n. The notations $DSS(n)$ and $ISS(n)$ will be used in the discussion of the results for the number of distinct and inequivalent strong starters of order n, respectively.

3 Results

Starters and strong starters are here classified for cyclic groups up to order 35 and 37, respectively. The number of distinct and inequivalent starters and strong starters are presented in Table 1. Some discrepancies in comparison with values obtained by Kocay et al. [4] and surveyed in [1] were encountered. Values differing from the ones presented in previous articles are marked with a dagger. The distinct strong starters in $\mathbb{Z}_7$ and $\mathbb{Z}_{13}$ are presented in Example 1 and Table 2, respectively. In Table 2, the first four and the last four starters form equivalence classes.

Table 1: Distinct and inequivalent starters and strong starters

n	$DS(n)$	$IS(n)$	$DSS(n)$	$ISS(n)$
3	1	1	0	0
5	1	1	0	0
7	3	2	† 2	1
9	9	3	0	0
11	25	5	4	2
13	133	14	† 8	2
15	631	87	32	4
17	3,857	242	224	14
19	25,905	1,453	800	52
21	188,181	15,715	† 6,660	555
23	1,515,283	68,882	27,554	1,257
25	13,376,125	668,812	158,680	7,934
27	128,102,625	7,116,903	† 1,201,626	† 66,757
29	1,317,606,101	47,057,378	9,415,980	336,297
31	14,534,145,947	484,472,150	76,761,968	2,558,894
33	170,922,533,545	8,546,128,509	672,858,900	33,642,945
35	2,138,089,212,789	89,087,066,368	6,015,305,136	250,637,748
37			59,514,095,596	1,653,170,339

The starters and strong starters with nontrivial automorphisms are categorized by the order of automorphism group in Table 3 and Table 4, respectively. Note that there exists always a *patterned* starter $S = \{\{-x, x\} : x \in G \setminus \{0\}\}$, which has an automorphism group of order $\phi(n)$.

Table 2: Distinct strong starters in $\mathbb{Z}_{13}$

{1,10}	{2,8}	{3,4}	{5,7}	{6,11}	{9,12}
{1,10}	{2,7}	{3,4}	{5,11}	{6,8}	{9,12}
{1,4}	{2,8}	{3,12}	{5,7}	{6,11}	{9,10}
{1,4}	{2,7}	{3,12}	{5,11}	{6,8}	{9,10}
{1,7}	{2,12}	{3,8}	{4,5}	{6,10}	{9,11}
{1,2}	{3,6}	{4,11}	{5,9}	{7,12}	{8,10}
{1,11}	{2,4}	{3,7}	{5,10}	{6,12}	{8,9}
{1,6}	{2,9}	{3,5}	{4,8}	{7,10}	{11,12}

Classifying the starters in $\mathbb{Z}_{35}$ took about 23 months of CPU time on an HP CP4000 BL ProLiant supercluster with 2.6 GHz AMD Opteron 64-bit processors. The CPU time as well as the number of starters increases by a factor of about ten when the order of the group increases by 2. Therefore, computing the starters in $\mathbb{Z}_{37}$ would take about 20 years of CPU time on current hardware and with the current implementation.

Note that an overall speed-up could be obtained by carrying out isomorph rejection on certain partial starters as well. However, the order of the automorphism groups of the groups considered are rather small—for example, 36 and 24 for $\mathbb{Z}_{37}$ and $\mathbb{Z}_{39}$, respectively—so, taking the slowdown mentioned in the previous paragraph into account, one further group could then be considered. Unfortunately, by using such an approach, the validation of the results by double counting would not be possible.

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Table 3: Nontrivial automorphisms of starters

$n \backslash \vert Aut(S) \vert$	2	3	4	5	6	7	8	9	10
3	1								
5			1						
7		1			1				
9		1			1				
11				2					1
13		3							
15	4		7				1		
17									
19		14						4	
21	33	10			11				
23									
25				6					
27		117						13	
29						19			
31		836		67					
33	3,511			66					25
35	30,543	367	350		49				

$n \backslash \vert Aut(S) \vert$	11	12	15	16	18	20	22	24	28	30
3										
5										
7										
9										
11										
13		1								
15										
17				1						
19					1					
21		1								
23	5						1			
25						1				
27					1					
29									1	
31			7							1
33						1				
35		16						1		

Table 4: Nontrivial automorphisms of strong starters

$n\backslash\lvert\mathrm{Aut}(S)\rvert$	3	5	7	9	11	15
7	1					
11		2				
13	2					
15						
17						
19	6			4		
21						
23					5	
25						
27						
29			14			
31	192	34				7
33						
35	68					
37	1,500			19		

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