Rhombicuboctahedron Designs

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Abstract

We prove that the complete graph K_v can be decomposed into rhombicuboctahedra if and only if $v \equiv 1$ or $33 \pmod{96}$.

1 Introduction

The spectrum of integers v for which the complete graph K_v can be decomposed into copies of the graph of one of the Platonic solids is determined for the tetrahedron, octahedron and cube but only partial results are available for the icosahedron and dodecahedron. The current state of knowledge, see also [1], appears to be as follows.

1. Tetrahedron designs are equivalent to Steiner systems $S(2,4,v)$. The necessary and sufficient condition is $v \equiv 1$ or $4 \pmod{12}$, [9].

2. Octahedron designs are equivalent to Steiner triple systems $S(2,3,v)$ which can be decomposed into Pasch configurations. The necessary and sufficient condition is $v \equiv 1$ or $9 \pmod{24}$, $v \neq 9$, [8], [2].

3. Cube designs exist if and only if $v \equiv 1$ or $16 \pmod{24}$, [10], [4].

4. A necessary condition for the existence of icosahedron designs is $v \equiv$ 1, 16, 21 or 36 (mod 60). They are known to exist for $v \equiv 1$ (mod 60), [3], and for $v = 16$, [1].

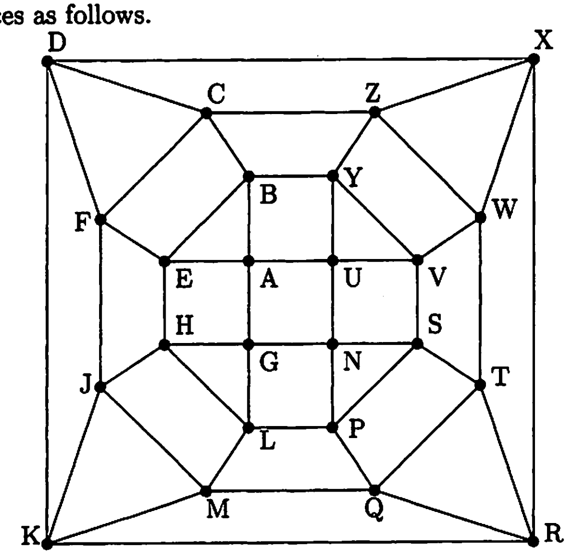
5. A necessary condition for the existence of dodecahedron designs is $v \equiv$ 1, 16, 25 or 40 (mod 60). They are known to exist for $v \equiv 1 \pmod{60}$, [3], and for $v = 25, 40$ and 76, [1]. There is no dodecahedron design of order $v = 16$, [1].

A natural extension of the above is to consider decompositions into the Archimedean graphs, of which there are two infinite families (the prisms and antiprisms), as well as thirteen further examples. However, the only one which seems to have received any attention is the cuboctahedron and in [7] it was shown that such designs exist if and only if $v \equiv 1$ or 33 (mod 48). Our investigations suggest that, although decomposition results for some of the other Archimedean graphs can be obtained, determining the existence spectrum completely is not easy. However, in the case of the rhombicuboctahedron we have been successful in obtaining the entire spectrum. It is easy to show that the admissibility condition for a rhombicuboctahedron design on v points is $v \equiv 1$ or 33 (mod 96). We prove the existence of such systems for all these orders of v. This paper therefore can be regarded as a companion to [7]. In short, we prove the following theorem.

Theorem 1.1 *Rhombicuboctahedron designs exist if and only if* $v \equiv 1$ *or* 33 (mod 96).

2 Construction

We first present rhombicuboctahedron designs of orders 33 and 97, both of which were obtained by a computer search assuming appropriate cyclic automorphisms. The rhombicuboctahedron has 24 vertices, 48 edges and 26 faces, and we will represent them by ordered 24-tuples $(A, B, C, D, E, F, G, H, J, K, L, M, N, P, Q, R, S, T, U, V, W, X, Y, Z)$ where the co-ordinates represent vertices as follows.



Lemma 2.1 *There exists a rhombicuboctahedron design of order* 33.
Proof. Let the vertex set of the complete graph be $Z_{11} \times Z_3$. The decomposition consists of the rhombicuboctahedra
((0,0),(0,1),(0,2),(1,0),(1,1),(2,0),(1,2),(2,2),(4,1),(6,0),(6,1),(9,2),(6,2),(9,0),(7,0),(3,0),(3,2),(10,1),(4,2),(8,0),(2,1),(7,1),(8,2),(9,1)) under the action of the mapping $(i,j) \mapsto (i+1,j) \pmod{11}$. □

Lemma 2.2 *There exists a rhombicuboctahedron design of order* 97.
Proof. Let the vertex set of the complete graph be Z_{97}. The decomposition consists of the rhombicuboctahedra (0,4,1,2,6,11,7,18,3,16,34,51,25,5,27,59,55,96,71,94,52,21,37,73) under the action of the mapping $i \mapsto i+1 \pmod{97}$. □

In general our method of proof uses a standard technique (Wilson's fundamental construction). For this we need the concept of a *group divisible design* (GDD). Recall therefore that a 3-GDD of type u^t is an ordered triple (V, G, B) where V is a base set of cardinality $v = tu$, G is a partition of V into t subsets of cardinality u called *groups* and B is a family of subsets of cardinality 3 called *blocks* which collectively have the property that every pair of elements from different groups occurs in precisely one block but no pair of elements from the same group occurs at all. We will also need 3-GDDs of type $u^t w^1$. These are defined analogously, with the base set V being of cardinality $v = tu + w$ and the partition G being into t subsets of cardinality u and one set of cardinality w.

Two of the main ingredients which we will need in applying Wilson's fundamental construction are given in the above lemmas, and the third is a decomposition of the complete tripartite graph $K_{16,16,16}$ into 16 rhombicuboctahedra. We present this next.

Lemma 2.3 *There exists a decomposition of the complete tripartite graph $K_{16,16,16}$ into* 16 *rhombicuboctahedra.*
Proof. Let the three partitions of $K_{16,16,16}$ be $\{(i,0) : 0 \le i \le 15\}$, $\{(i,1) : 0 \le i \le 15\}$ and $\{(i,2) : 0 \le i \le 15\}$. The decomposition consists of the rhombicuboctahedra ((0,0),(0,1),(0,2),(1,0),(1,2),(2,1), (2,2),(4,0),(7,2),(10,1),(15,1),(13,0),(9,1),(11,0),(3,2),(3,0),(15,2),(8,1),(5,2),(13,1),(14,0),(10,2),(10,0),(6,1))
under the action of the mapping $(i,j) \mapsto (i+1,j) \pmod{16}$.
We will refer to this design as a rhombicuboctahedron-GDD of type 16^3.
□

We are now in a position to present the main results.

Lemma 2.4 *There exists a rhombicuboctahedron design of order* $v = 96t + 1,\ t \geq 3$.

Proof. There exists a 3-GDD of type $6^t, t \geq 3$, [11], see also [6]. This is called the *master* GDD. Replace each element of the base set V by 16 elements (i.e. inflate by a factor 16) and adjoin a further element, ∞. On every inflated group of the 3-GDD, together with the element ∞, place the rhombicuboctahedron design of order 97 from Lemma 2.2. Further, replace each block of the master GDD by the rhombicuboctahedron-GDD of type 16^3 from Lemma 2.3, called the *slave* GDD. □

Lemma 2.5 *There exists a rhombicuboctahedron design of order* $v = 96t + 33,\ t \geq 3$.

Proof. There exists a 3-GDD of type $6^t 2^1, t \geq 3$, [5], see also [6]. As in the previous lemma, replace each element of the base set V by 16 elements and adjoin a further element, ∞. Again on every inflated group of the 3-GDD, together with the element ∞, place the rhombicuboctahedron design of order 97 from Lemma 2.2 or, in the case of the inflated group of cardinality 32, the rhombicuboctahedron design of order 33 from Lemma 2.1. Finally replace each block of the master GDD by the slave rhombicuboctahedron GDD of type 16^3 from Lemma 2.3. □

The above just leaves the orders $v = 129$, 193 and 225. Rhombicuboctahedron designs for these orders can be constructed using respectively as the master GDD, a 3-GDD of type 2^4, 2^6 and $2^4 6^1$, [11], [5], and proceeding as in the proofs of Lemmas 2.4 and 2.5 using the rhombicuboctahedron design of order 33 from Lemma 2.1, the rhombicuboctahedron design of order 97 from Lemma 2.2 and the rhombicuboctahedron GDD of type 16^3 from Lemma 2.3. We state this formally.

Lemma 2.6 *There exist rhombicuboctahedron designs of orders* 129, 193 *and* 225.

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