Black 1-factors and Dudeney sets

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Abstract

A set of Hamilton cycles in the complete graph K_n is called a Dudeney set if every path of length two lies on exactly one of the cycles. It has been conjectured that there is a Dudeney set for every complete graph. It is known that there exists a Dudeney set for K_n when n is even, but the question is still unsettled when n is odd.

In this paper, we define a black 1-factor in K_{p+1} for an odd prime p, and show that if there exists a black 1-factor in K_{p+1}, then we can construct a Dudeney set for K_{p+2}. We also show that if there is a black 1-factor in K_{p+1}, then 2 is a quadratic residue modulo p. Using this result, we obtain some new Dudeney sets for K_n when n is odd.

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1 Introduction

A Dudeney set for the complete graph K_n is a set of Hamilton cycles with the property that every path of length two (2-path) in K_n lies on exactly one of the cycles. We call the problem of constructing a Dudeney set in K_n for all natural numbers "Dudeney's round table problem". Dudeney posed this problem in 1907 and 1917 in his books; see [2] for a historical summary. A Dudeney set for K_n has been constructed when n is even [4]. In the case that n is odd, a Dudeney set for K_n has been constructed in the following cases:

(1) $n = 2^k + 1$ ($k \geq 1$) [7];

(2) $n = p + 2$ (p is an odd prime and 2 is a primitive root of $GF(p)$) [2];

(3) $n = p + 2$ (p is an odd prime and -2 is a primitive root of $GF(p)$) [3];

(4) $n = p + 2$ (p is an odd prime, 2 is the square of a primitive root of $GF(p)$ and $p \equiv 3 \pmod 4$) [3];

(5) $n = p + 2$ (p is an odd prime, 2 is the square of a primitive root of $GF(p)$, $p \equiv 1 \pmod 4$, 3 is not a quadratic residue modulo p) [5];

(6) $n = p + 2$ (p is an odd prime, -2 is the square of a primitive root of $GF(p)$, and either

(6-1) $p \equiv 1 \pmod 4$ and 3 is not a quadratic residue modulo p [5], or

(6-2) $p \equiv 3 \pmod 4$ [5];

(7) some sporadic cases ($n = 11, 23, 45$ [2]; $27, 29, 35, 37$ [6]).

In this paper, we define a black 1-factor in K_{p+1} for an odd prime p, and we show that if there exists a black 1-factor in K_{p+1}, then we can construct a Dudeney set for K_{p+2}. We also show that if there is a black 1-factor in K_{p+1}, then 2 is a quadratic residue modulo p. Using these results, we obtain some new Dudeney sets for K_n when n is odd.

2 Notation and preliminaries

From now on, let p be an odd prime ≥ 5 and put $n = p + 1$ and $r = (p - 1)/2$. Denote by $K_n = (V_n, E_n)$ the complete graph on n vertices, where $V_n = \{\infty\} \cup \{0, 1, 2, \ldots, p - 1\}$ is the vertex set and E_n is the edge set.

For any integer i $(0 \le i \le p-1)$ define the 1-factor

$$F_i = \{\{\infty, i\}\} \cup \{\{a,b\} \in E_n \mid a, b \neq \infty, a+b \equiv 2i \pmod{p}\}.$$

Let σ be the vertex permutation $(\infty)(0\ 1\ 2\ 3\ \ldots\ p-1)$ and put $\Sigma = \langle\sigma\rangle$. When $\mathcal{C}$ is a set of cycles or circuits in K_n, define $\Sigma\mathcal{C} = \{\sigma^t C \mid 0 \le t \le p-1, C \in \mathcal{C}\}$.

Let H be a subset of $\{1, 2, 3, \ldots, p-1\}$. We call H a half-set mod p if $H \cup (-H) = \{1, 2, 3, \ldots, p-1\}$ and $|H| = (p-1)/2$. It is well-known that for any half-set H mod p, $\Sigma\{F_0 \cup F_i \mid i \in H\}$ is a Dudeney set of K_n.

For any edge $\{a,b\} \in K_n$, define the length $d(a,b)$:

$$d(a,b) = \begin{cases} \min\{p - |b-a|, |b-a|\} & (\text{if } a, b \neq \infty) \\ \infty & (\text{otherwise}), \end{cases}$$

and define the colour $c(a,b)$:

$$c(a,b) = i \quad (\text{if } \{a,b\} \in F_i \quad (0 \le i \le p-1)).$$

Two colours i, j $(0 \le i, j \le p-1)$ are said to be equivalent if $i = j$ or $i + j = p$ and we write $i \sim j$. Note that the lengths of edges are $\infty, 1, 2, \ldots, r$; the colours of edges are $0, 1, 2, \ldots, p-1$; and the non-equivalent colours of edges are $0, 1, 2, \ldots, r$. The 1-factor F_i $(0 \le i \le p-1)$ is the 1-factor which contains all the edges of colour i. In particular, we call the colour 0 white and call F_0 the white 1-factor.

Put $n' = n + 1 = p + 2$ and define the complete graph $K_{n'} = (V_{n'}, E_{n'})$, where $V_{n'} = V_n \cup \{\lambda\}$. Clearly σ induces a permutation of the edges in $V_{n'}$; we will also denote the permutation by $\sigma = (\infty)(\lambda)(0\ 1\ 2\ 3\ \cdots\ p-1)$.

3 Black 1-factors

A 1-factor B is called a black 1-factor in K_n if

(1) $F_0 \cup B$ is a Hamilton cycle.

(2) B has all lengths $\{\infty, 1, 2, \ldots, r\}$.

(3) B has all non-equivalent colours except 0.

Note that all the lengths of B must be unique, but one colour must appear twice. Let B be a black 1-factor in K_n and put $B = \{e_1 = \{x_1, y_1\}, e_2 = \{x_2, y_2\}, \ldots, e_{r+1} = \{x_{r+1}, y_{r+1}\}\}$. Let the colours of the edges be i_1, i_2,

$\ldots, i_{r+1}$, respectively. We may assume $i_r \sim i_{r+1}$. Put $H = \{i_1, i_2, \ldots, i_r\}$; then H is a half-set. So put

$$\mathcal{C} = \{F_0 \cup F_{i_s} \mid 1 \le s \le r\},$$

then $\Sigma\mathcal{C}$ is a Dudeney set of K_n.

If we insert the vertex λ into all the edges in B, we get a set of 2-paths in $K_{n'}$. Denote this set by B^λ; that is

$$B^\lambda = \{(x_s, \lambda, y_s) \mid 1 \le s \le r+1\}.$$

Put

$$\mathcal{B}^\lambda = F_0 \cup B^\lambda,$$

then $\mathcal{B}^\lambda$ is a circuit of $K_{n'}$.

Proposition 3.1 $\Sigma(\mathcal{C} \cup \{\mathcal{B}^\lambda\})$ has every 2-path in $K_{n'}$ exactly once.

Proof. Divide the set of all 2-paths in $K_{n'}$ into 8 classes: (i) (a, b, c), (ii) (a, ∞, b), (iii) (∞, a, b), (iv) (a, λ, b), (v) (λ, a, b), (vi) (λ, ∞, a), (vii) (λ, a, ∞), (viii) (∞, λ, a), where $a, b, c \neq \infty, \lambda$.

(i), (ii), (iii) are also 2-paths in K_n, so they belong to $\Sigma\mathcal{C}$.

(iv) Since B has all lengths, we have $\Sigma B = E_n$. Hence any 2-path (a, λ, b) belongs to $\Sigma\{\mathcal{B}^\lambda\}$.

(v) We have $\{a, b\}^{\sigma^t} \in F_0$ for some t $(0 \le t \le p-1)$ as F_0 has all lengths. So we can assume that $\{a, b\} \in F_0$ without loss of generality. Therefore we see that the 2-paths (λ, a, b) and (λ, b, a) belong to $\mathcal{B}^\lambda$.

(vi), (vii) Similarly, we can assume $a = 0$. Then the 2-paths $(\lambda, \infty, 0)$ and $(\lambda, 0, \infty)$ belong to $\mathcal{B}^\lambda$.

(viii) Similarly, we can assume that $\{\infty, a\}^{\sigma^t} = \{\infty, b\} \in B$ for some t $(0 \le t \le p-1)$. Then the 2-path (∞, λ, b) belongs to $\mathcal{B}^\lambda$.

By counting the number of 2-paths, we see that every 2-path appears only once in $K_{n'}$. □

For any integer s $(1 \le s \le r+1)$, the edge $e_s = \{x_s, y_s\}$ belongs to F_{i_s}. Define for an integer s $(1 \le s \le r)$,

$$F'_{i_s} = F_{i_s} \setminus \{\{x_s, y_s\}\} \cup \{\{x_s, \lambda\}, \{\lambda, y_s\}\}.$$

Put

$$B' = \{\{x_1, y_1\}, \ldots, \{x_r, y_r\}, \{x_{r+1}, \lambda\}, \{\lambda, y_{r+1}\}\}.$$

For a set of cycles or circuits $\mathcal{C}$, let $\pi(\mathcal{C})$ be the set of the 2-paths in $\mathcal{C}$.

Proposition 3.2 $\pi(\{F_0 \cup F'_{i_s} \mid 1 \leq s \leq r\} \cup \{F_0 \cup B'\}) = \pi(\{F_0 \cup F_{i_s} \mid 1 \leq s \leq r\} \cup \{F_0 \cup B^\lambda\})$.

Proof. Since the set of all edges in $(\cup_{1 \leq s \leq r} F_i) \cup B^\lambda$ and the set of all edges in $(\cup_{1 \leq s \leq r} F'_i) \cup B'$ are the same, the proposition holds. □

Theorem 3.1 $\mathcal{H} = \Sigma(\{F_0 \cup F'_{i_s} \mid 1 \leq s \leq r\} \cup \{F_0 \cup B'\})$ is a Dudeney set of $K_{n'}$.

Proof. From Prop. 3.1 and 3.2, $\mathcal{H}$ has every 2-path in $K_{n'}$ exactly once. It is trivial that all elements of $\mathcal{H}$ are Hamilton cycles. Therefore $\mathcal{H}$ is a Dudeney set of $K_{n'}$. □

We have proved that, if there exists a black 1-factor in K_{p+1}, then we can construct a Dudeney set of K_{p+2}.

Example Let $p = 7, n = p + 1 = 8, n' = p + 2 = 9$. Then

$$B = \{\{\infty, 1\}, \{5, 6\}, \{2, 4\}, \{0, 3\}\}$$

is a black 1-factor in K_8. Put $H = \{1, 2, 3\}$. Then we have

$$B^\lambda = \{(\infty, \lambda, 1), (5, \lambda, 6), (2, \lambda, 4), (0, \lambda, 3)\},$$

and

$$\begin{aligned} F'_1 &= \{\{\infty, \lambda\}, \{\lambda, 1\}, \{0, 2\}, \{6, 3\}, \{5, 4\}\} \\ F'_2 &= \{\{\infty, 2\}, \{1, 3\}, \{0, 4\}, \{5, \lambda\}, \{\lambda, 6\}\} \\ F'_3 &= \{\{\infty, 3\}, \{2, \lambda\}, \{\lambda, 4\}, \{1, 5\}, \{0, 6\}\} \\ B' &= \{\{\infty, 1\}, \{5, 6\}, \{2, 4\}, \{0, \lambda\}, \{\lambda, 3\}\}. \end{aligned}$$

Thus we obtain a Dudeney set $\mathcal{H}$ of K_9:

$$\begin{aligned} \mathcal{H} = \Sigma\{&(\infty, 0, 2, 5, 4, 3, 6, 1, \lambda), (\infty, 0, 4, 3, 1, 6, \lambda, 5, 2), \\ &(\infty, 0, 6, 1, 5, 2, \lambda, 4, 3), (\infty, 0, \lambda, 3, 4, 2, 5, 6, 1)\}. \end{aligned}$$

4 Property of black 1-factors

In this section, we will find a necessary condition for the existence of a black 1-factor.

Lemma 4.1 Let p be a prime ≥ 5 and put $r = (p-1)/2$. Then we have $\sum_{i=1}^{r} i^2 \equiv 0 \pmod{p}$.

Proof. It is known that $\sum_{i=1}^{r} i^2 = r(r+1)(2r+1)/6$. Since $p \neq 2, 3$, we have $\sum_{i=1}^{r} i^2 \equiv 2^{-1}3^{-1}r(r+1)(2r+1) \equiv 0 \pmod{p}$. □

Theorem 4.1 If there is a black 1-factor in K_{p+1}, then 2 is a quadratic residue modulo p.

Proof. Let B be a black 1-factor in K_{p+1}:

$$B = \{\{\infty, b_0\}, \{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_r, b_r\}\},$$

where $b_0 \neq 0$. Since B has edges of lengths $\infty, 1, 2, \ldots, r$, we have

$$\sum_{i=1}^{r}(a_i - b_i)^2 \equiv \sum_{i=1}^{r} i^2 \equiv 0 \pmod{p}. \qquad (4.1)$$

Since B has edges of non-equivalent colours $1, 2, \ldots, r, c$ ($c \in \{1, 2, \ldots, r\}$), we have

$$b_0^2 + \sum_{i=1}^{r}(2^{-1}(a_i + b_i))^2 \equiv \sum_{i=1}^{r} i^2 + c^2 \equiv c^2 \pmod{p}$$

from Lemma 4.1, that is,

$$2^2 b_0^2 + \sum_{i=1}^{r}(a_i + b_i)^2 \equiv 2^2 c^2 \pmod{p}. \qquad (4.2)$$

Adding (4.1) and (4.2), we have

$$2^2 b_0^2 + 2\sum_{i=1}^{r}(a_i^2 + b_i^2) \equiv 2^2 c^2 \pmod{p}.$$

Since

$$b_0^2 + \sum_{i=1}^{r}(a_i^2 + b_i^2) \equiv \sum_{i=0}^{p-1} i^2 \equiv 2\sum_{i=1}^{r} i^2 \equiv 0 \pmod{p},$$

we have

$$2b_0^2 \equiv 2^2c^2 \pmod{p}.$$

Therefore 2 is a quadratic residue modulo p. □

It is unsettled whether the converse of the statement of Theorem 4.1 holds or not. No counterexample has been found so far.

5 New Dudeney sets

To obtain a black 1-factor in K_{p+1}, 2 must be a quadratic residue modulo p. We see that there are infinitely many primes such that 2 is a quadratic residue modulo p from the following Lemmas.

Lemma 5.1 [1, Theorem 95 (p. 75)] Let p be an odd prime. Then 2 is a quadratic residue modulo p if and only if $p \equiv \pm 1 \pmod{8}$.

Lemma 5.2 [1, Theorem 15 (p. 13)] Let a be an integer $\neq 0$ and m a positive integer. If a and m are relatively prime, then there are infinitely many primes p such that $p \equiv a \pmod{m}$.

The primes less than 100 such that 2 is a quadratic residue modulo p are as follows:

$$p = 7, 17, 23, 31, 41, 47, 71, 73, 79, 89, 97.$$

We found black 1-factors for all of the above primes except $p = 97$. Therefore we obtain Dudeney sets of $n = p + 2$:

$$n = 9, 19, 25, 33, 43, 49, 73, 75, 81, 91.$$

Referring to the list of cases in the Introduction, $n = 9, 33$ belong to (1), $n = 25, 49, 73, 81$ belong to (3), and $n = 19, 43$ belong to (5). The values $n = 75, 91$ don't belong to any of the cases. Thus we obtain Dudeney sets of K_{75} and K_{91}, for which the existence of Dudeney sets has been in doubt.

We finish by showing black 1-factors for $p = 73, 89$.

$p = 73$:
0–32, 1–∞, 2–40, 3–71, 4–63, 5–57, 6–70, 7–50, 8–28, 9–69, 10–67, 11–42, 12–31, 13–68, 14–47, 15–39, 16–64, 17–29, 18–41, 19–21, 20–66, 22–26, 23–60, 24–25, 27–37, 30–58, 33–72, 34–56, 35–61, 36–65, 38–53, 43–51, 44–55, 45–62, 46–49, 48–54, 52–59.

$p = 89$:
50–35, 54–14, 75–84, 5–19, 70–72, 17–79, 10–44, 45–21, 68–18, 71–48, 41–42, 47–30, 59–1, 88–82, 7–27, 62–67, 22–81, 8–15, 74–66, 23–77, 12–64, 25–63, 26–38, 51–55, 34–16, 73–31, 58–87, 2–69, 20–53, 36–80, 9–52, 37–78, 11–32, 57–0, ∞–65, 24–43, 46–33, 56–28, 61–86, 3–6, 83–4, 85–49, 40–29, 60–76, 13–39.

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