Sparseness of 4-cycle systems

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Abstract

An avoidance problem of configurations in 4-cycle systems is investigated by generalizing the notion of sparseness, which is originally from Erdős' r-sparse conjecture on Steiner triple systems. A 4-cycle system of order v, $4\mathrm{CS}(v)$, is said to be r-sparse if for every integer j satisfying $2 \leq j \leq r$ it contains no configurations consisting of j 4-cycles whose union contains precisely $j+3$ vertices. If an r-sparse $4\mathrm{CS}(v)$ is also free from copies of a configuration on two 4-cycles sharing a diagonal, called the double-diamond, we say it is strictly r-sparse. In this paper, we show that for every admissible order v there exists a strictly 4-sparse $4\mathrm{CS}(v)$. We also prove that for any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists a strictly r-sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles.

Keywords: 4-Cycle system, Configuration, Avoidance, r-Sparse

1 Introduction

A *4-cycle system* of *order* v, denoted by $4\mathrm{CS}(v)$, is an ordered pair (V, C), where $V = V(K_v)$, the vertex set of the complete graph K_v, and C is a collection of edge-disjoint cycles of length four whose edges partition the edge set of the complete graph. It is well-known that a necessary and sufficient condition for the existence of a $4\mathrm{CS}(v)$ is that $v \equiv 1 \pmod 8$ (see, for example, Rodger [14]). Such orders are said to be *admissible*. Following the usual terminology of cycle systems, we call a cycle of length four a *4-cycle*.

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A 4-cycle system is a natural generalization of the classical combinatorial design called a Steiner triple system, briefly STS, since an STS is just an edge-disjoint decomposition of a complete graph into triangles. A Steiner triple system of order v exists if and only if $v \equiv 1,3 \pmod 6$. In other words, the set of all admissible orders of an STS consists of all the positive integers $v \equiv 1,3 \pmod 6$.

As is the case with Steiner triple systems, various properties which may appear in a 4-cycle system have also been studied (see, for example, Mishima and Fu [13] and references therein). Such properties of cycle systems have also been investigated as a special graph design (see, for example, Jimbo and Kuriki [11]). Among many characteristics of STSs, the numbers of occurrences of particular substructures have been of interest to various areas (see Colbourn and Rosa [3]). In the current paper, we consider an extreme case for 4CSs, namely, avoidance of particular configurations. We first recall a long-standing conjecture on STSs posed by Erdős.

A *(k,l)-configuration* in an STS is a set of l triangles whose union contains precisely k vertices. In 1973, Erdős [4] conjectured that for every integer $r \geq 4$, there exists $v_0(r)$ such that if $v > v_0(r)$ and if v is admissible, then there exists a Steiner triple system of order v with the property that it contains no $(j+2,j)$-configurations for any j satisfying $2 \leq j \leq r$. Such an STS is said to be *r-sparse*. Many results on the r-sparse conjecture and related problems have been since developed. In particular, after major progress due to Ling et al. [12] and earlier development found in their references, the simplest case when $r = 4$, as it is sometimes called the anti-Pasch conjecture, was eventually settled in the affirmative by Grannell et al. [10].

Theorem 1.1 (Grannell, Griggs and Whitehead [10]) *There exists a 4-sparse Steiner triple system of order v if and only if $v \equiv 1,3 \pmod 6$ and $v \neq 7, 13$.*

As far as the authors are aware, the r-sparse conjecture for $r \geq 5$ is still unsettled. In fact, no 7-sparse STS is realized for $v > 3$. Very recent results on sparseness and related problems are found in a series of papers: Forbes et al. [5], Wolf [15, 16] and the first author [6, 7, 8, 9]. For general background on configurations and sparseness in triple systems, the interested reader is referred to Colbourn and Rosa [3].

With regard to 4-cycle systems, the relating result is due to Bryant et al. [1], who investigated the numbers of occurrences of configurations consisting of two 4-cycles. They presented a formula for the number of occurrences of such configurations and studied avoidance and maximizing problems.

Our primary focus in the current paper is on existence of 4-cycle systems which are "sparse" in the sense that they do not contain configurations that consist of many 4-cycles on a small number of vertices in relative terms. In this sense, for a given integer $w \leq v$ the "densest" configurations on w vertices in a 4CS(v) are ones that contain as many 4-cycles as possible. In terms of combinatorial design theory, such a configuration is said to be a maximum 4-cycle packing of order w. More formally, a *4-cycle packing* of *order* w is an ordered pair $(W, \mathcal{D})$ such that $|W| = w$ and $\mathcal{D}$ is a set of 4-cycles sharing no common edges, where vertices of a 4-cycle in $\mathcal{D}$ are elements of W. A 4-cycle packing is said to be *maximum* if no other 4-cycle packing of the same order contains a larger number of 4-cycles. Obviously, if w is admissible, a maximum 4-cycle packing of order w is just a 4CS(w).

The term (k,l)-configuration will also be used for substructures in 4CSs and is defined as a set of l 4-cycles on precisely k vertices where no pair of distinct 4-cycles share the same edge. We denote the set of vertices in a configuration $\mathcal{A}$ by $V(\mathcal{A})$. Two configurations $\mathcal{A}$ and $\mathcal{B}$ are said to be *isomorphic*, denoted as $\mathcal{A} \cong \mathcal{B}$, if there exists a bijection $\phi : V(\mathcal{A}) \to V(\mathcal{B})$ such that for each 4-cycle $C \in \mathcal{A}$, the image $\phi(C)$ is a 4-cycle in $\mathcal{B}$.

In the case of STSs, sparseness is measured by lack of $(j+2, j)$-configurations; one of reasons may be that they are possibly avoidable and form the essential portions of dense configurations (see Forbes, Grannell and Griggs [5]). Based on the following proposition and subsequent observation on $(j+3, j)$-configurations, we propose an avoidance problem similar to the r-sparse conjecture on STSs.

Proposition 1 *For any positive integers j and d, any $(j+3, j+d)$-configuration in a* 4CS *contains a $(j+3, j)$-configuration as a substructure.*

Proof. If a $(j+3, j+d)$-configuration contains a 4-cycle, say C, in which each vertex is also contained in another 4-cycle, then by discarding C we obtain a $(j+3, j+d-1)$-configuration. We prove that for any positive integer d a $(j+3, j+d)$-configuration contains such a 4-cycle. Suppose to the contrary that each 4-cycle in a given $(j+3, j+d)$-configuration $\mathcal{A}$ has at least one vertex appearing in no other 4-cycles. If $d \geq 4$, the total number of vertices exceeds $j+3$, a contradiction. Hence, we have $d = 1, 2$ or 3. However, by counting the total number of vertices, it is easy to see that each case yields a contradiction. □

Proposition 1 says that any denser configuration on $j+3$ vertices, including a 4CS or a maximum packing, contains a $(j+3, j)$-configuration as its substructure. On the other hand, for $j = 2$ and $1 \leq e \leq 3$ every nontrivial 4CS(v) contains $(j+3+e, j)$-configurations (see Bryant et al. [1]). However, as we will see in the next section, we can construct a 4CS containing no $(j+3, j)$-configurations for any j satisfying $2 \leq j \leq 4$. Therefore, it may be natural to ask the following question similar to Erdős' conjecture:

Problem 1.2 *Does there exist for every integer $r \geq 3$ a constant number $v_0(r)$ such that if $v > v_0(r)$ and v is admissible, then there exists a* 4CS(v) *containing no $(j+3,j)$-configurations for any j satisfying $2 \leq j \leq r$?*

Remark. While for any positive integers e and j every nontrivial STS on a sufficiently large number of vertices contains a $(j+2+e,j)$-configuration, we do not know in general the behavior of $(j+3+e,j)$-configurations except for $j=2$. We briefly discuss in Section 3 the maximum number of 4-cycles of a 4-cycle packing avoiding $(j+3,j)$-configurations.

Following the terminology of STSs, we say that a 4CS is *r-sparse* if it contains no $(j+3,j)$-configuration for any j satisfying $2 \leq j \leq r$. Every r-sparse 4CS is also $(r-1)$-sparse for $r \geq 3$. Since no $(5,2)$-configuration can appear in a 4CS, every 4CS is 2-sparse. Up to isomorphism, there are two kinds of $(6,3)$-configuration described by three 4-cycles (a,b,c,d), (a,e,c,f) and (b,e,f,d), and (a,b,c,d), (a,e,c,f) and (b,e,d,f) respectively. A routine argument proves that any $(7,4)$-configuration is isomorphic and can be described by four 4-cycles (a,b,c,d), (a,e,b,f), (c,f,d,g) and (a,c,e,g). Hence, a 4CS is 3-sparse if it lacks the two types of $(6,3)$-configuration, and it is 4-sparse if it also avoids the unique type of $(7,4)$-configuration simultaneously.

Our results presented in the next section give resolution for the existence problem of a 4-sparse 4CS(v).

Theorem 1.3 *There exists a 4-sparse* 4CS(v) *if and only if* $v \equiv 1 \pmod 8$.

Up to isomorphism, there are four possible configurations formed by two 4-cycles in a 4CS, the numbers of vertices ranging from six to eight. While there are two kinds of $(6,2)$-configuration, both $(7,2)$- and $(8,2)$-configurations are unique. A $(6,2)$-configuration sharing a common diagonal, described by two 4-cycles (a,b,c,d) and (a,e,c,f), is called the *double-diamond* configuration. A 4-cycle system is said to be *D-avoiding* if it contains no double-diamond configurations.

Bryant et al. [1] showed that for every admissible order v there exists a D-avoiding 4CS(v).

Theorem 1.4 (Bryant et al.) [1] *There exists a D-avoiding* 4CS(v) *for all* $v \equiv 1 \pmod 8$.

Since a double-diamond configuration appears in both types of $(6,3)$-configuration, every D-avoiding 4CS is 3-sparse but the converse does not hold. In fact, for every small admissible order v one can easily find a 3-sparse 4CS(v) which is not D-avoiding. On the other hand, Bryant et al. [1] showed that the other type of

$(6,2)$-configuration appears constantly depending only on the order v, that is, the number of occurrences is unique between 4CSs of the same order. Considering these facts, we say that a 4CS is *strictly* r-sparse if it is both r-sparse and D-avoiding.

In Section 2, we give a proof of existence of a strictly 4-sparse 4CS(v) for every admissible order v.

Theorem 1.5 *There exists a strictly 4-sparse* 4CS(v) *if and only if* $v \equiv 1 \pmod 8$.

We also study in Section 3 the maximum number of 4-cycles of a 4-cycle packing avoiding $(j+3,j)$-configurations.

Let $ex(v,r)$ be the maximum number of 4-cycles of a 4-cycle packing of order v containing neither double-diamond configurations nor $(j+3,j)$-configurations for every $2 \leq j \leq r$. By probabilistic methods, we prove that for any positive integer $r \geq 2$ the maximum number $ex(v,r) = O(v^2)$.

2 Strictly 4-sparse 4-cycle systems

In this section, we present a proof of Theorem 1.5. Obviously, the proof also verifies Theorem 1.3. To show Theorem 1.5, we first prove two lemmas.

A *jointed-diamond* configuration in a 4CS is a $(7,3)$-configuration described by three 4-cycles (a,b,c,d), (a,e,b,g) and (c,f,d,g); the 4-cycle (a,b,c,d) is referred to as a *joint 4-cycle*. Every $(7,4)$-configuration contains a jointed-diamond configuration as its substructure.

Lemma 2.1 *Let q be a prime power satisfying $q \equiv 1 \pmod 8$ and not a power of three. Then there exists a strictly* 4*-sparse* 4CS(q).

Proof. Let q be a prime power satisfying $q \equiv 1 \pmod 8$ and not a power of three. Let χ be a multiplicative character of order four of GF(q) such that $\chi(x)$ has possible values $1, -1, i, -i$ for $x \neq 0$. Then there exists a 4-cycle $(0,x,x-1,x^2)$, $x \in \mathrm{GF}(q)$, such that $\chi(x^2) = -1$, $\chi((x^2-x+1)^2) = -1$, and $\chi(x(x^2-x+1))) = 1$ (see Bryant et al. [1]). Considering these conditions, we have either $\chi(x) = i$, $\chi(x^2-x+1) = -i$, and $\chi(x(x-1)) = i \cdot \chi(x-1)$, or $\chi(x) = -i$, $\chi(x^2-x+1) = i$, and $\chi(x(x-1)) = -i \cdot \chi(x-1)$. Also, since $q \equiv 1 \pmod 8$, we have $\chi(-1) = 1$. Let α be a primitive element of GF(q) and V the set of all elements of GF(q). Define a set C of 4-cycles as $\{y, x\cdot\alpha^{4n}+y, (x-1)\cdot\alpha^{4n}+y, x^2\cdot\alpha^{4n}+y : y \in \mathrm{GF}(q), 0 \leq n \leq \frac{q-1}{8} - 1\}$. Then (V,C) forms a D-avoiding 4CS(q). In fact, C is developed from the 4-cycle $(0,x,x-1,x^2)$ by the group $G = \{z \mapsto z\cdot\alpha^{4n} +$

$y : y,z \in \mathrm{GF}(q), 0 \leq n \leq \frac{q-1}{8} - 1\}$. To prove that (V,C) is strictly 4-sparse, it suffices to show that (V,C) contains no jointed-diamond configurations. Suppose to the contrary that it contains a jointed-diamond configuration J described by three 4-cycles (a,b,c,d), (a,e,b,g) and (c,f,d,g). Since every 4-cycle in C can be obtained from $(0,x,x-1,x^2)$ by the group G, considering the joint 4-cycle (a,b,c,d), we have $\chi(a-b) = -\chi(c-d)$. However, since the edges $\{a,b\}$ and $\{c,d\}$ lie in diagonals of (a,e,b,g) and (c,f,d,g) respectively, we have $\chi(a-b) = \chi(c-d)$, $i \cdot \chi(c-d)$ or $-i \cdot \chi(c-d)$, a contradiction. The proof is complete. □

Lemma 2.2 *There exists a strictly* 4*-sparse* 4CS(9).

Proof. Let $V = \{0,1,2,\ldots,8\}$ be the set of elements of the cyclic group Z_9. Define a set C of 4-cycles as $\{(0+a,1+a,8+a,5+a) : a \in Z_9\}$. The pair (V,C) forms a 4CS(9) under the transitive action of Z_9 on the vertex set V. Since C has only one 4-cycle orbit, (V,C) is D-avoiding, and hence it is 3-sparse.

Suppose to the contrary that (V,C) is not 4-sparse and contains a jointed-diamond. Take a representative, say $C = (0,1,8,5)$, of the 4-cycle orbit. The two differences of the vertices in a diagonal of C are ± 1 and ∓ 4 respectively. Hence, the joint 4-cycle in a jointed-diamond lying in C has the form (a,b,c,d), where the differences $a-b$ and $c-d$ are each 1, -1, 4 or -4. However, considering the four differences of the adjacent vertices in C, this is a contradiction. □

We now return to the proof of Theorem 1.5. The proof employs a special decomposition of the complete graph into smaller complete graphs.

A *group divisible design* with *index* one is a triple $(V,\mathcal{G},\mathcal{B})$, where

(i) V is a finite set of elements called points,

(ii) $\mathcal{G}$ is a family of subsets of V, called *groups*, which partition V,

(iii) $\mathcal{B}$ is a collection of subsets of V, called *blocks*, such that every pair of points from distinct groups occurs in exactly one blocks,

(iv) $|G \cap B| \leq 1$ for all $G \in \mathcal{G}$ and $B \in \mathcal{B}$.

When all blocks are of the same size k and the number of groups of size n_i is t_i, one refers to the design as a k-GDD of *type* $n_0^{t_0} n_1^{t_1} \cdots n_{g-1}^{t_{g-1}}$, where $t_0 + t_1 + \cdots + t_{g-1} = |\mathcal{G}|$. We need 4-GDDs and the required types are of 12^t $(t \geq 4)$, 4^{3t+1} $(t \geq 1)$, 8^{3t+1} $(t \geq 1)$, and $2^{3t}5^1$ $(t \geq 3)$. For their existence, we refer the reader to Colbourn and Dinitz [2].

Proof of Theorem 1.5. A strictly 4-sparse 4CS(v) is necessarily D-avoiding. We follow a part of the proof of existence of a D-avoiding 4CS(v) by Bryant et al. [1] and consider four cases:

Case (1) : $v \equiv 1 \pmod{24}$. Lemma 2.1 gives a strictly 4-sparse 4CS(v) for $v \leq 73$ and $v \equiv 1 \pmod{24}$. We consider the case $v > 73$. Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type 12^t for $t \geq 4$. For each group $G \in \mathcal{G}$, take $(G \times \{0,1\}) \cup \{\infty\}$ by replacing each point by two new points and adding a new point ∞. Let $\mathcal{H}_G$ be a copy of the strictly 4-sparse 4CS(25) given in Lemma 2.1 on $(G \times \{0,1\}) \cup \{\infty\}$. For each block $B = \{a,b,c,d\} \in \mathcal{B}$, construct a 4-cycle decomposition C_B of a copy of $K_{2,2,2,2}$ on $B \times \{0,1\}$ by developing a 4-cycle $((a,0),(b,0),(c,1),(d,0))$ under the group $\langle (d)(a\ b\ c) \rangle \times Z_2$. Let $W = (V \times \{0,1\}) \cup \{\infty\}$ and $\mathcal{D} = (\bigcup_{G \in \mathcal{G}} \mathcal{H}_G) \cup (\bigcup_{B \in \mathcal{B}} C_B)$. Then $(W, \mathcal{D})$ forms a 4CS($24t+1$). Since no pair of 4-cycles in $\mathcal{D}$ shares a common diagonal, $(W, \mathcal{D})$ is D-avoiding.

It remains to establish that the 4CS contains no (7,4)-configuration. Suppose to the contrary that $(W, \mathcal{D})$ contains a (7,4)-configuration. Then it contains a jointed-diamond configuration J. If the joint 4-cycle in J lies in $\mathcal{H}_G$, the other two 4-cycles in J are also in $\mathcal{H}_G$. Since $\mathcal{H}_G$ is a copy of a strictly 4-sparse 4CS(25), this is a contradiction. If the joint 4-cycle in J lies in C_B, again the other two 4-cycles in J are in C_B. A routine argument proves that C_B contains no jointed-diamond configuration.

Case (2) : $v \equiv 9 \pmod{24}$. Lemma 2.2 gives a strictly 4-sparse 4CS(9). Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type 4^{3t+1} for $t \geq 1$. As in *Case* (1), construct a 4CS($24t+9$) on $(V \times \{0,1\}) \cup \{\infty\}$ by placing a copy of the strictly 4-sparse 4CS(9) given in Lemma 2.2 and decomposing $K_{2,2,2,2}$s into 4-cycles. By following the argument in *Case* (1), the resulting 4CS($24t+9$) is strictly 4-sparse.

Case (3) : $v \equiv 17 \pmod{48}$. Employing the strictly 4-sparse 4CS(17) constructed in Lemma 2.1 and a 4-GDD of type 8^{3t+1} for $t \geq 1$, we obtain the required strictly 4-sparse 4CSs by the same technique as in *Case* (1).

Case (4) : $v \equiv 41 \pmod{48}$. Lemma 2.1 gives a strictly 4-sparse 4CS(v) for $v \leq 137$ and $v \equiv 41 \pmod{48}$. We consider the case $v > 137$. Take a 4-GDD $(V, \mathcal{B}, \mathcal{G})$ of type $2^{3t}5^1$ for $t \geq 3$. For each block $B = \{a,b,c,d\} \in \mathcal{B}$, replace each point in B by four new points and define $A_i = \{i\} \times \{0,1,2,3\}$ for $i \in B$. The points and lines of an affine space over GF(2^2) of dimension 2 form a 4-GDD of type 4^4. For each $B \in \mathcal{B}$, place a 4-GDD of type 4^4 on $B \times \{0,1,2,3\}$ such that the set of groups is $\{A_i : i \in B\}$ and let C_B be the resulting blocks of the 4-GDD on $B \times \{0,1,2,3\}$. For each C_B, $B \in \mathcal{B}$, construct a 4-cycle decomposition $\mathcal{D}_{C_B}$ of a copy of $K_{2,2,2,2}$ on $C_B \times \{0,1\}$ by developing a 4-cycle $((a,i,0),(b,j,0),(c,k,1),(d,l,0))$ under the group $\langle ((d,l))((a,i)\ (b,j)\ (c,k)) \rangle \times$

Z_2. For each group $G \in \mathcal{G}$, take $(G \times \{0,1,2,3\} \times \{0,1\}) \cup \{\infty\}$ and let $\mathcal{H}_G$ be a copy of either the strictly 4-sparse 4CS(17) or 4CS(41) given in Lemma 2.1 on $(G \times \{0,1,\ldots,7\}) \cup \{\infty\}$ according to the group size $|G|$, that is, place a copy of the 4CS(17) if $|G| = 2$, otherwise put a copy of the 4CS(41). Let $W = (V \times \{0,1,2,3\} \times \{0,1\}) \cup \{\infty\}$ and $\mathcal{E} = (\bigcup_{G \in \mathcal{G}} \mathcal{H}_G) \cup (\bigcup_{B \in \mathcal{B}} \mathcal{D}_{C_B})$. It is straightforward to see that $(W, \mathcal{E})$ forms a 4CS$(48t+41)$. The same argument as in *Case* (1) proves that $(W, \mathcal{E})$ is strictly 4-sparse. □

3 r-Sparse 4-cycle packing

In this section, we consider the maximum number of 4-cycles in a 4-cycle packing of order v avoiding $(j+3, j)$-configurations. As with a 4CS, a 4-cycle packing is said to be *r-sparse* if it contains no $(j+3, j)$-configuration for any j satisfying $2 \leq j \leq r$. Also if it is r-sparse and D-avoiding, we say that it is *strictly* r-sparse. We prove that for any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists a strictly r-sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles. It is notable that a resolution for the analogous problem to the r-sparse conjecture on STSs would prove that $c \sim \frac{1}{8}$.

Let $\mathcal{F}$ be a set of configurations of 4-cycles and $ex(v, \mathcal{F})$ the largest positive integer n such that there exists a set C of n 4-cycles on a finite set V of cardinality v having property that C contains no configuration which is isomorphic to a member $F \in \mathcal{F}$.

Theorem 3.1 *For any positive integer $r \geq 2$ and sufficiently large integer v there exists a constant number c such that there exists an r-sparse 4-cycle packing of order v with $c \cdot v^2$ 4-cycles.*

Proof. Let V be a finite set of cardinality v. Define $\mathcal{F}'$ as the set of all nonisomorphic $(j+3, j)$-configurations for $2 \leq j \leq r$ and $\mathcal{F}''$ as the set of all nonisomorphic (4,2)- and (6,2)-configurations. Let $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$. It is easy to see that if $ex(v, \mathcal{F}) \geq c \cdot v^2$ for some constant c, then the assertion of Theorem 3.1 follows.

Pick uniformly at random 4-cycles from V with probability $p = \frac{c'}{v^2}$, independently of the others, where c' satisfies $0 < c' < \frac{1}{44}$. Let b_C be a random variable counting the number of configurations isomorphic to C in the resulting set of 4-cycles and $E(b_C)$ its expected value. Then

$$
\begin{aligned}
E\left(\sum_{C\simeq F\in\mathcal{F}} b_C\right) &\le \binom{v}{4}\cdot\binom{3\cdot\binom{4}{4}}{2}\cdot p^2+\binom{v}{6}\cdot\binom{3\cdot\binom{6}{4}}{2}\cdot p^2\\
&\quad+\sum_{j=2}^{r}\binom{v}{j+3}\cdot\binom{3\cdot\binom{j+3}{4}}{j}\cdot p^j\\
&\le \left[\binom{v}{4}\cdot\binom{3}{2}+\binom{v}{6}\cdot\binom{45}{2}\right]p^2\\
&\quad+\sum_{j=2}^{r}\left(\frac{e\cdot v}{j+3}\right)^{j+3}\cdot\left(\frac{e\cdot(j+3)^3}{8}\right)^{j}\cdot p^j\\
&= \frac{11\cdot c'^2}{8}\cdot v^2+f(v),
\end{aligned}
$$

where $f(v)=O(v)$. By Markov's Inequality,

$$
P\left(\sum_{C\simeq F\in\mathcal{F}} b_C \ge 2\cdot E\left(\sum_{C\simeq F\in\mathcal{F}} b_C\right)\right)\le\frac{1}{2}.
$$

Hence,

$$
P\left(\sum_{C\simeq F\in\mathcal{F}} b_C \le \frac{11\cdot c'^2}{4}\cdot v^2+2\cdot f(v)\right)\ge\frac{1}{2}.
$$

Let t be a random variable counting the number of 4-cycles and $E(t)$ its expected value. Then

$$
E(t)=p\cdot 3\cdot\binom{v}{4}=\frac{c'}{8}\cdot v^2-g(v),
$$

where $g(v)=O(v)$. Since t is a binomial random variable, we have for sufficiently large v

$$
P\left(t<\frac{E(t)}{2}\right) \;<\; e^{-\frac{E(t)}{8}}<\frac{1}{2}.
$$

Hence, if v is sufficiently large, then we have, with positive probability, a set $\mathcal{S}$ of 4-cycles with the property that $|\mathcal{S}|>\frac{E(t)}{2}$ and the number of configurations in $\mathcal{S}$ which are isomorphic to a member of $\mathcal{F}$ is at most $\frac{11\cdot c'^2}{4}\cdot v^2+2\cdot f(v)$. Since $f(v),\ g(v)=O(v)$, by deleting a 4-cycle from each configuration isomorphic to a member of $\mathcal{F}$, we have

$$
ex(v,\mathcal{F})\ge\frac{c'(1-44\cdot c')}{16}\cdot v^2-h(v),
$$

where $h(v)=O(v)$. The proof is complete. □

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